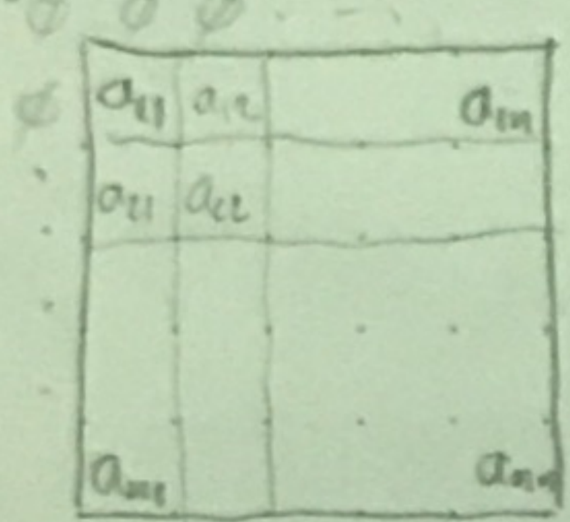
18.217 LECTURE 6

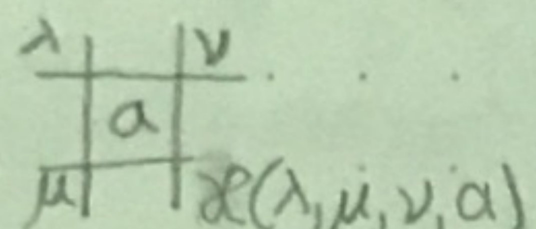
<u>Last time:</u> RSK: $\left\{ \underbrace{\boxed{a_{ij} \geq 0}}_{n\times n} \right\} \longrightarrow \{(P,Q)\}$

$\updownarrow s$ Gelf. Tsetlin patterns

$\left\{ \geq \underbrace{\boxed{RPP's}}_{n\times n} \right\}$

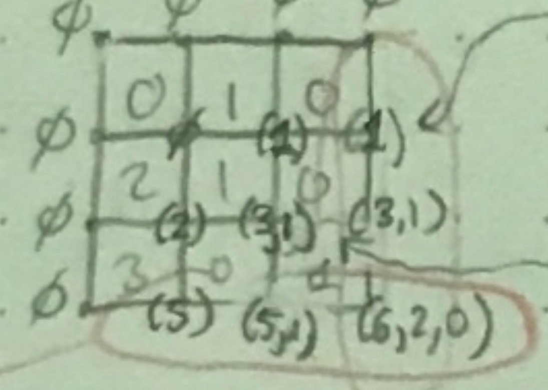
gen. growth diagrams





(toggle rule)

Ex. $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \rightarrow$



? Toggle $(0,0,\ldots)$, $(0,0,\ldots)$ and $(1,0,\ldots)$
$\Leftrightarrow (1,0,\ldots)$

? Toggle $(0,0,\ldots)$, $(1,0,\ldots)$, $(2,0,\ldots)$

2 0 0 0
0 0 0 0
1 0 0 0 add 2 to largest part

GT pattern

$\begin{matrix} 6 & & 2 & & 0 \\ & 5 & & 2 & \\ & & 5 & & \end{matrix} \longleftrightarrow$ P =

1	1	1	1	1	3
2	2				

Column GT pattern: $\begin{matrix} 6 & & 2 & & 0 \\ & 3 & & 1 & \\ & & 1 & & \end{matrix} \longleftrightarrow$ Q =

1	2	2	3	3	3
2	3				

$A \rightsquigarrow (P,Q)$ of shape λ.

<u>Thrm</u> Greene's Thrm   Lets you find λ non-recursively

$A = (a_{ij}) \rightsquigarrow$ multiset of pairs $\binom{i}{j}$ repeated a_{ij} times.

"generalized perm" $\begin{pmatrix} i_1 \leq i_2 \leq \cdots \leq i_N \\ j_1 \ \ j_2 \ \cdots \ \ j_N \end{pmatrix}$

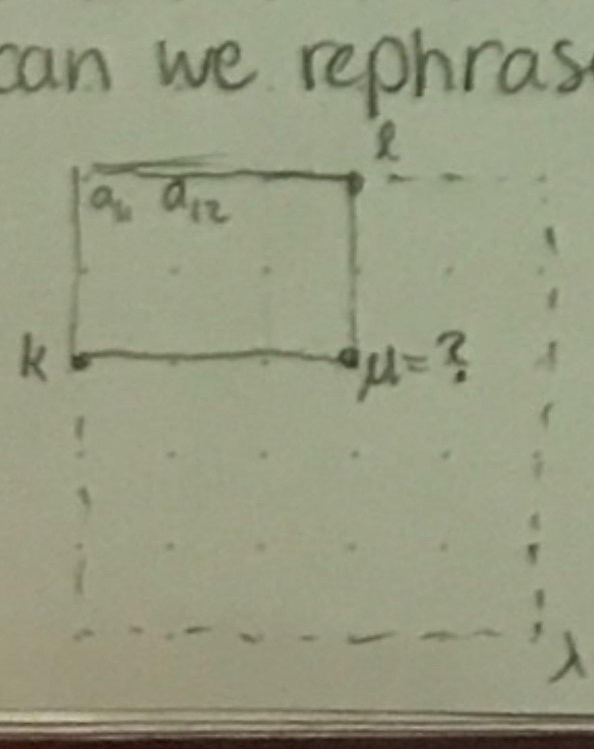
Then λ_1 = size of maximal weakly increasing subseq. in $j_1 \ldots j_n$

$\lambda_1 + \lambda_2$ = max cardinality subset of $\{1,\ldots,N\}$ that can be covered by 2 weakly increasing subseq.

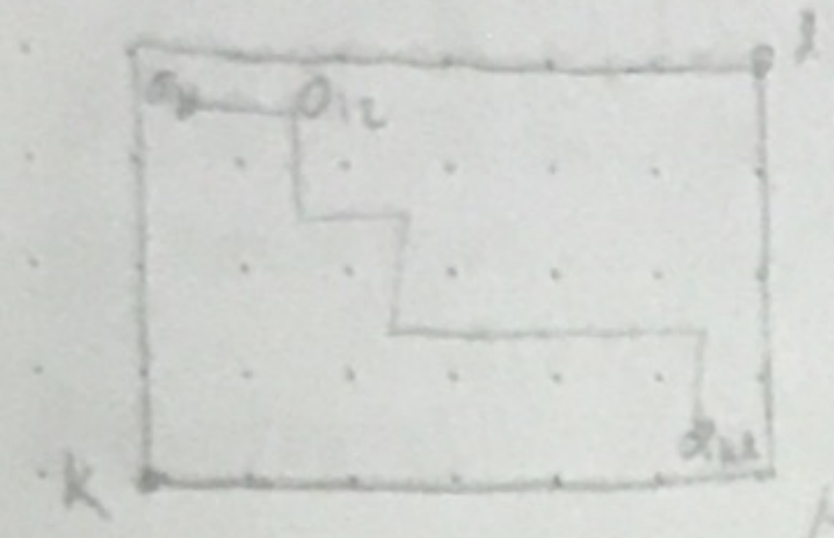
(Note: Finding max seq. Then finding longest incr. subseq. from what is left may not work)

$\lambda_1 + \ldots + \lambda_3$ = ... that can be covered with k weakly increasing subseq.

But how can we rephrase to see λ in terms of A (instead of permutation) non-recursively



$A^{k,\ell} = k \times \ell$ submatrix of A

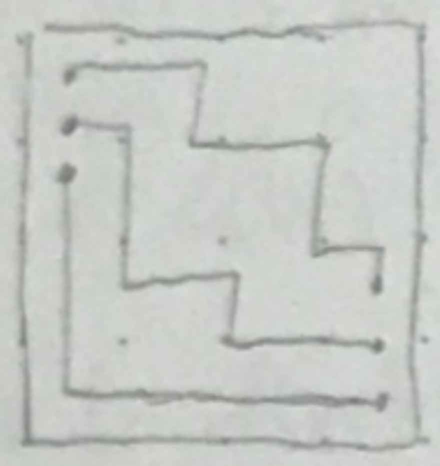


μ_1 = max P lattice path from $(1,1)$ to (k,l)
where value given by
$\sum_{(i,j)\in P} a_{ij}$

$\mu = (\mu_1, \mu_2, \dots)$

Thrm: $\mu = (\mu_1, \dots, \mu_r)$. $r = \min(k,l)$

where $\mu_1 + \mu_2 + \dots + \mu_s$ = max $S \subset [k]\times[l]$ covered by s non-crossing lattice paths connecting pts $(1,1), (2,1), (3,1)$ with $(k, l-s+1), \dots (k,l)$



Where max means largest value of $\sum_{(i,j)\in S} a_{ij}$

Exercise: Prove this.

Rational Algebras		Tropical semi-alg	
operations:		operations:	
"$\cdot$"	times	"+"	plus
"/"	divide	"−"	minus
"+"	plus	"max"	maximum

tropicalize $\rightarrow$ / $\leftarrow$ detropicalize

$\min(a,b) := -\max(a,b)$

The above thrm could be detropicalized to something about minors of determinants.
but more on that some other time maybe

Schur Polynomials $\longleftrightarrow$ Schur functions $\leftarrow$ infinitely many terms

$S_\lambda(x_1, \dots, x_n)$ $S_\lambda(x_1, x_2, \dots)$

$\in$

$\mathbb{Z}[x_1, \dots, x_n]^{S_n}$ = symmetric polys in n vars

def: 4 different formulas of S_λ:

1. Combinatorial definition $S_\lambda(x_1, \dots, x_n) = \sum_{\substack{T \text{ a SSYT} \\ \text{w/ entries in } n}} x^{\text{weight}(T)}$
2. Classical definition (Weyl character formula)
   $S_\lambda = \det(\cdots)/\det(\cdots)$
3. S_λ = certain Schubert polynomial
4. Coming later

18.217 LECTURE 7 9/21/22

Schur Polynomials / functions

$S_\lambda(x_1,\dots,x_n) \in \Lambda_n = \mathbb{C}[x_1,\dots,x_n]^{S_n}$

$S_\lambda(x_1, x_2, \dots) \in \Lambda := \text{"}\lim_{n\to\infty}\text{"}\, \Lambda_n$
$= \{$set of formal power series in $x_1, x_2, \dots$ of bounded degree, invariant under perms of $x_1, x_2, \dots\}$

<u>Combinatorial def</u>: $S_\lambda(x_1,\dots,x_n) = S_\lambda^{comb}(x_1,\dots,x_n) = \sum_{\substack{T \text{ SSYT} \\ \text{of shape } \lambda \\ \text{filled w/ entries} \\ \text{in } \{1,\dots,n\}}} x^{weight(T)}$

where $x^{weigh(T)} = \prod_{i=1}^{n} x_i^{(\#\text{ of } i\text{'s in } T)}$

<u>E.x.</u> $S_\square(x_1,\dots,x_n) = x_1+\dots+x_n$

(RSK) $\Rightarrow$ Cauchy formula

<u>Cauchy formula</u>: $\sum_{\substack{\lambda \text{ perm} \\ \text{w/ at most} \\ \min(m,n) \text{ parts}}} S_\lambda(x_1,\dots,x_m)\cdot S_\lambda(y_1,\dots,y_n) = \prod_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \frac{1}{1-x_iy_j}$

$LHS = \sum_{\substack{\text{pairs } P,Q \\ \text{of SSYT}}} x^{weight(P)}\, y^{weight(Q)}$

$RHS = \prod_{i,j} \sum_{a_{ij}\geq 0} (x_iy_j)^{a_{ij}} = \sum_{\substack{m\times n \text{ matrix} \\ A=(a_{ij})}} x^{\text{columns of } A}\, y^{\text{rows of } A}$

And then these are equal by RSK

<u>Classical def of S_λ</u>:

$\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n_{\geq 0}$, $\quad a_\alpha := \det\begin{pmatrix} x_1^{\alpha_1} & \cdots & x_n^{\alpha_1} \\ \vdots & & \vdots \\ x_1^{\alpha_n} & \cdots & x_n^{\alpha_n}\end{pmatrix}$

WLOG assume that $\alpha = (\alpha_1 > \alpha_2 > \dots > \alpha_n)$.

called ρ in Lie Theory $\rightarrow \delta = (n-1, n-2, \dots, 1, 0)$

$a = \lambda + \delta$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$

$a_\delta = \prod_{1\leq i\leq j\leq n} (x_i - x_j)$ (Vandermonde)

<u>Def</u>: $S_\lambda^{class}(x_1,\dots,x_n) = \frac{a_{\lambda+\delta}}{a_\delta}$ (a symmetric poly since numerator & denominator both antisymmetric)

<u>E.x.</u> $n=2$ $\lambda = (1,0) = \square$ $\quad \lambda+\delta = (2,0)$

$S_\lambda^{class}(x_1,x_2) = \frac{\begin{vmatrix} x_1^2 & x_2^2 \\ 1 & 1\end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ 1 & 1\end{vmatrix}} = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2$

<u>Thrm</u>: $S_\lambda^{comb} = S_\lambda^{class}$

Symmetric group S_n

generators: $s_1, s_2, \dots, s_n$ $\qquad s_i = (i, i+1)$ — adjacent transpositions

Relations: 1.) $s_i^2 = id$

2.) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

3.) $s_i s_j = s_j s_i$ when $|i-j| \geq 2$

Can write permutations $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$

def: This is a reduced decomposition of w if ℓ is as small as possible
$\ell = \ell(w)$ is the length of w

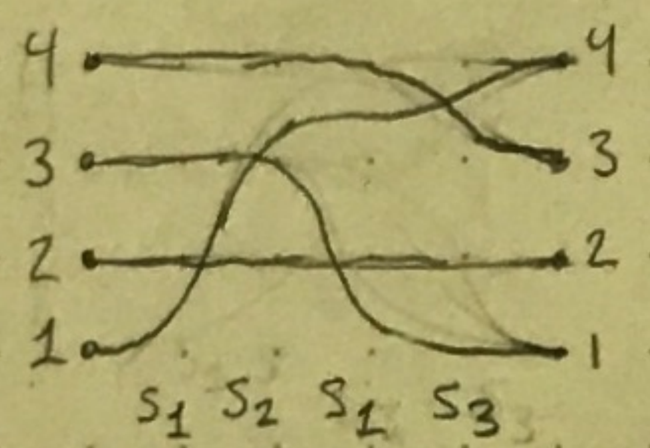
Lemma 1: $\ell(w) = \#$ of inversions in $w := \#\left\{(i,j) \,\middle|\, \begin{matrix} 1 \leq i < j \leq n \\ w(i) > w(j) \end{matrix}\right\}$

Lemma 2: Any 2 reduced decompositions of w can be obtained from each other by a sequence of moves (2) and (3).

Note: This is true in the more general setting of coxeter groups as well

Wiring Diagrams

E.x. $w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$



$w = s_3 s_1 s_2 s_1$

decomposition is reduced iff its wiring diagram has no double crossings

Can see lemma 1 & 2 from wiring diagram

Def: Divided difference operators are operators $\partial_1, \partial_2, \dots, \partial_{n-1}$ acting on $\mathbb{C}[x_1, \dots, x_n]$ by

$$\partial_i : f(x_1, \dots, x_n) \longmapsto \frac{1}{x_i - x_{i+1}} (1 - s_i)(f) = \frac{f(x_1, \dots, x_i, x_{i+1}, \dots, x_n)\ f(x_1, \dots, x_{i+1}, x_i, \dots}{x_i - x_{i+1}}$$

Lemma: Operators satisfy relations

(1)' $\partial_i^2 = 0$

(2) $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

(3) $\partial_i \partial_j = \partial_j \partial_i$ when $|i-j| \geq 2$

Notation: For $w = s_{i_1} \cdots s_{i_\ell}$ reduced decomp, $\partial_w := \partial_{i_1} \cdots \partial_{i_\ell}$ (depends only on w)

$w_0 := \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$, the longest perm in S_n

Schur as Schubert Polynomials

Def: $S_\lambda^{schub} := \partial_{w_0}(x^{\lambda+\delta})$ $\qquad \lambda = (\lambda_1, \dots, \lambda_n)$, $\delta = (n-1, n-3, \dots, 0)$

Thrm: $S_\lambda^{schub}(x_1, \dots, x_n) = S_\lambda^{class}(x_1, \dots, x_n)$

Thrm: $\partial_{w_0} = \frac{1}{\prod_{1 \le i < j \le n} (x_i - x_j)} \left(\sum_{w \in S_n} (-1)^{\ell(w)} w \right)$

Ex. $n=3$

$$\partial_1 \partial_2 \partial_1 = \frac{1}{x_1 - x_2} (1 - s_1) \frac{1}{(x_2 - x_3)} (1 - s_3) \frac{1}{(x_1 - x_2)} (1 - s_1)$$

$$= \frac{1}{(x_1 - x_2)(x_2 - x_3)(x_1 - x_2)} (1 - s_1 - s_2 + s_1 s_2 + s_2 s_1 - s_1 s_2 s_1)$$

$$\partial_{w_0}(x^{\lambda+\delta}) = \frac{1}{\text{Vand}} \cdot \underbrace{\sum_{w \in S_n} (-1)^{\ell(w)} w(x^{\lambda+\delta})}_{a_{\lambda+\delta}}$$