18.217 Combinatorial Theory
Prof. Alex Postnikov

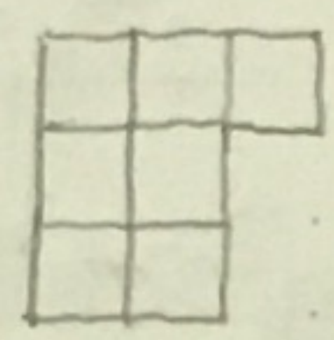
Course webpage: math.mit.edu/18.219/

Young Tableaux

Recommended Textbooks:
- Enumerative Combinatorics vol 2 by Richard Stanley (especially chapter 7)
- Young Tableaux, Fulton
- The Symmetric Group, Sagan

Young Diagrams

E.x. $7 = 3+2+2$ ← partition of 7
$\lambda = (3,2,2) \vdash 7$

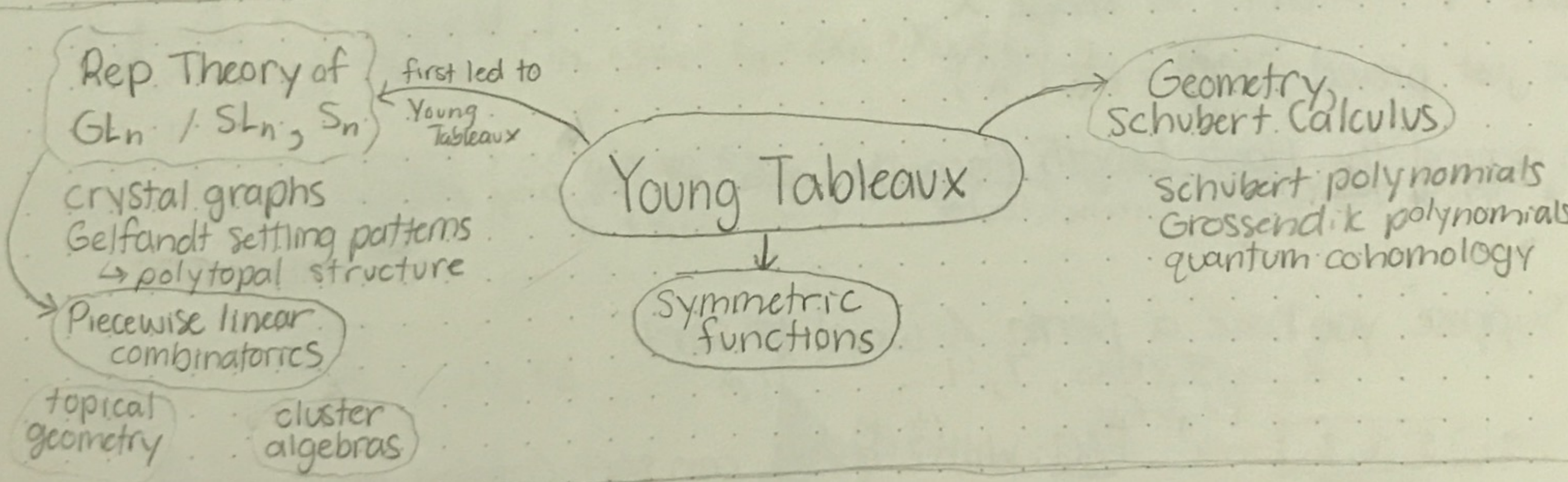
YD: 

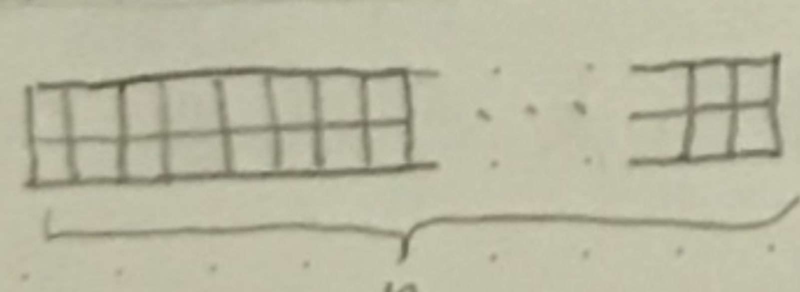
Young Tableaux: Put numbers in boxes of Young Diagram

Standard Young Tableau (SYT): Put numbers in boxes so they increase across rows & columns.

E.x. T =

1	2	4
3	6	
5	7	



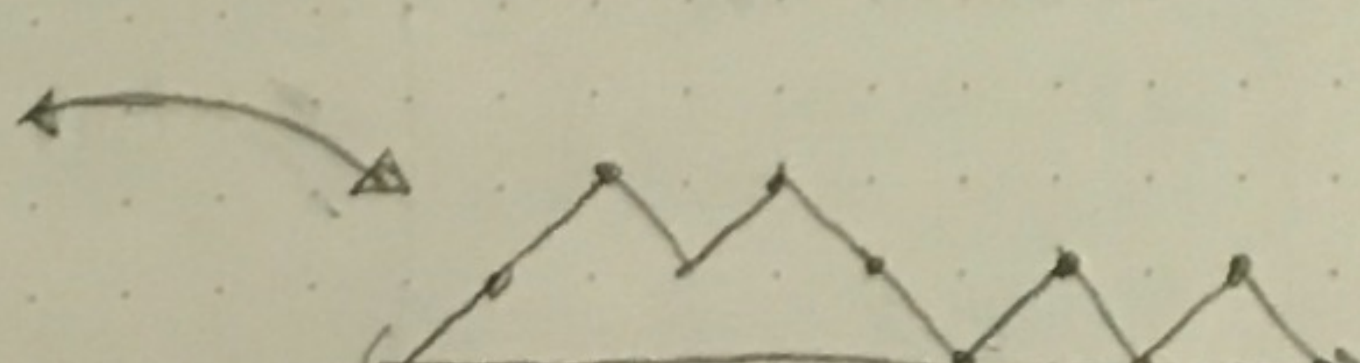
E.x. $\lambda = (n,n) =$ 

Thrm: # of SYT's of shape (n,n) = Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$
Proof: Easy bijection w/ Dyck paths.

E.x. $n=5$ $\quad$ i^{th} step goes up if i in 1^{st} row, down if i in 2^{nd} row

T =

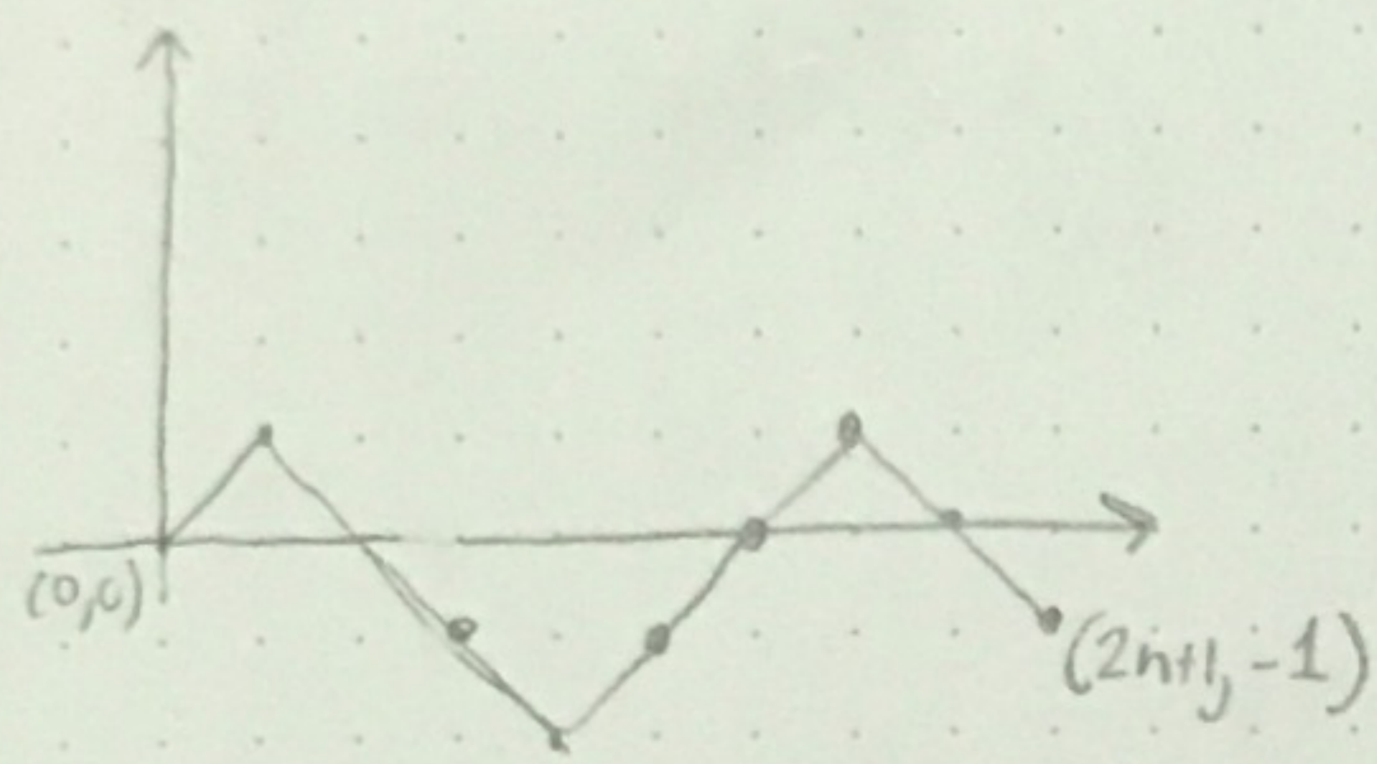
1	2	4	7	9
3	5	6	8	10



Proof that $C_n = \frac{1}{n+1}\binom{2n}{n}$ using Cyclic Shift method on Dyck paths

$\frac{1}{(n+1)}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$

$\binom{2n+1}{n}$ = # of all latice paths w/ n "up" & n-1 "down" steps



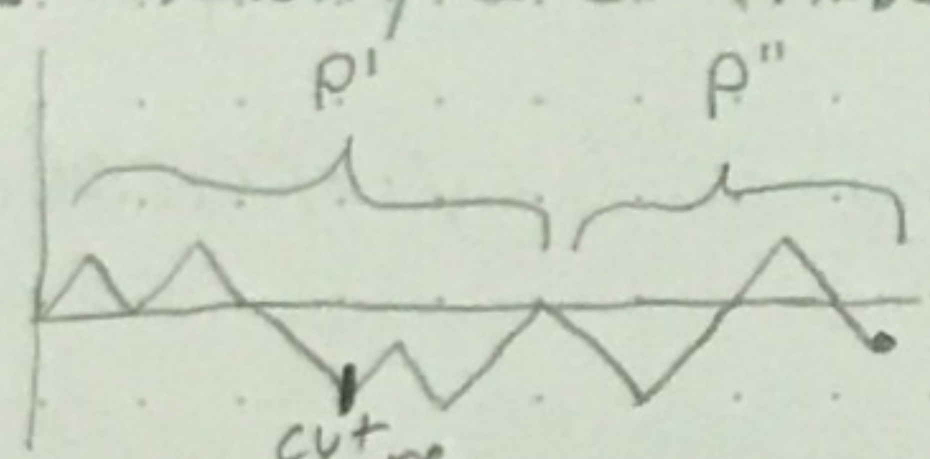
Claim: Exactly $\frac{1}{2n+1}$ of these paths stay above the x-axis

Proof: Write each path as e.x. + - - - + + + - -
Group together all paths whose words are cyclic shifts of each other
→ 2n+1 cyclic shifts total
⇒ Each disjoint gp. has 2n+1 elements

Claim#1 No repititions among 2n+1 cyclic shifts.
Proof: gcd(n, 2n+1) = 1
Our sequence would have to be periodic, but it can't be in any nontrivial way.

Claim#2: Exactly 1 of these shifts gives a Dyck path
Proof:

E.x. ρ =



$p = p' \circ p''$
$\tilde{p} = p'' \circ p'$

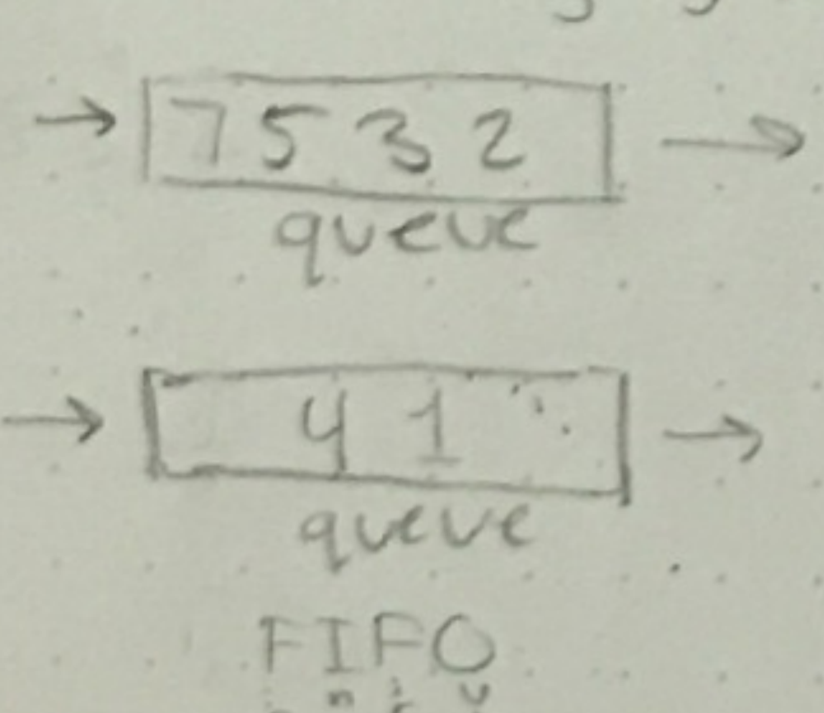
To get Dyck path, cut P at first minimum point

Def: f^λ = # SYT's of shape λ

We just proved $f^{(n,n)} = \frac{1}{n+1}\binom{2n}{n}$

In general the Hook Length Formula gives f^λ for any λ
A proof by Wolfe-Greene-Hooke

Suppose you have a perm & want to sort
2, 3, 5, 1, 6, 7, 4

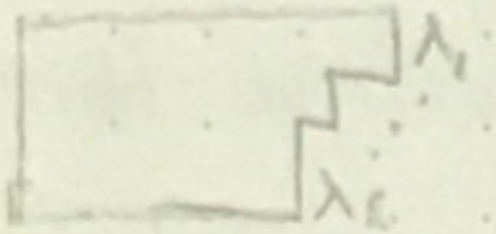


FACT: With 2 queues, can sort Catalan many perms

Q: What about with k queues

18.217 LECTURE 2

9/9/22

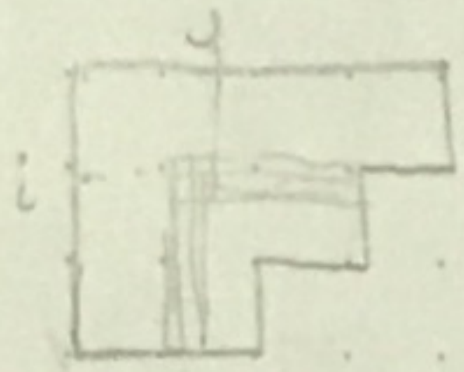
<u>Last time</u>: $\lambda = (\lambda_1, \dots, \lambda_\ell) =$ 

$\lambda_1 \geq \dots \geq \lambda_\ell$

$f^\lambda = \#$ SYT of shape λ

$f^{(n,n)} = C_n = \frac{1}{n+1}\binom{2n}{n}$

~~Hook length formula~~



$(i,j) \in \lambda$

h_{ij} = hook length of i,j

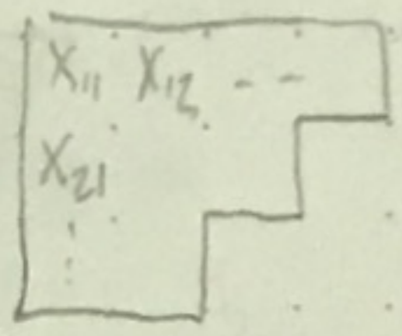
$\lambda \vdash n$

<u>Thrm</u>: $\boxed{f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{ij}}}$

<u>Ex.</u> $\lambda = \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$ $\qquad f^{(3,2)} = \frac{5!}{1\cdot 1\cdot 2\cdot 3\cdot 4} = 5$

<u>Polytopal Proof</u>:

Define 2 convex polytopes in $\mathbb{R}^n$

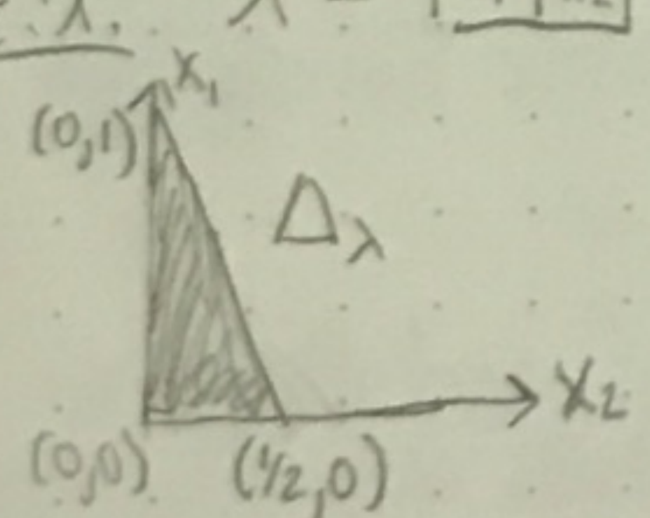
$\Delta_\lambda := \left\{ (x_{ij})_{(i,j)\in\lambda} \;\middle|\; \begin{array}{l} x_{ij} \geq 0 \\ \sum h_{ij} x_{ij} \leq 1 \end{array} \right\}$  h_{ij} is hook length

$P_\lambda := \left\{ (y_{ij})_{(i,j)\in\lambda} \;\middle|\; \begin{array}{ll} y_{ij} \geq 0 & y_{ij} \leq y_{i+1,j} \\ y_{ij} \leq y_{i,j+1} & \sum y_{ij} \leq 1 \end{array} \right\}$

<u>Ex.</u> $\Delta_\lambda = \left\{ \begin{array}{|c|c|c|}\hline x_{11} & x_{12} & x_{13} \\ \hline x_{21} & x_{22} \\ \cline{1-2}\end{array} \;\middle|\; \begin{array}{l} x_{ij} \geq 0 \\ 4x_{11} + 3x_{12} + x_{13} + 2x_{21} + x_{22} \leq 1 \end{array} \right\}$

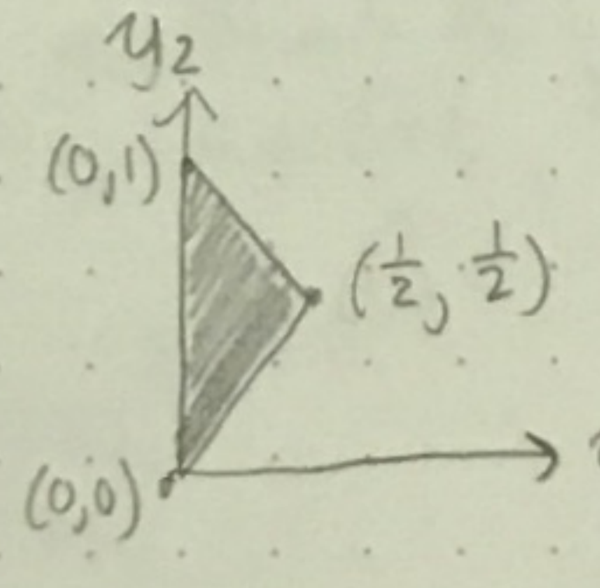
$P_\lambda = \left\{ \begin{array}{|c|c|c|}\hline y_{11} & y_{12} & y_{13} \\ \hline y_{21} & y_{22} \\ \cline{1-2}\end{array} \;\middle|\; \begin{array}{c} 0 \leq y_{11} \leq y_{12} \leq y_{13} \\ \wedge\!| \qquad \wedge\!| \\ y_{21} \leq y_{22} \end{array}, \; \sum y_{ij} \leq 1 \right\}$

<u>Ex.</u> $\lambda = \begin{array}{|c|c|}\hline x_1 & x_2 \\ \hline\end{array}$



$2x_1 + x_2 \leq 1$

Area 1/4 for both



$0 \leq y_1 \leq y_2$
$y_1 + y_2 \leq 1$

$(x_1, x_2) \mapsto (y_1, y_2) = (x_1, x_1 + x_2)$

$\mathrm{Vol}(\Delta_\lambda) = \frac{1}{\prod_{(i,j)\in\lambda} h_{ij}} \mathrm{Vol}\left((x_{ij}) \;\middle|\; \begin{array}{l} x_{ij} \geq 0 \\ \sum x_{ij} \leq 1 \end{array}\right) = \frac{1}{\prod_{(i,j)\in\lambda} h_{ij}} \frac{1}{n!}$

(Hook lengths just scale along 1 dimension)

$\mathrm{Vol}(P_\lambda) = f^\lambda \cdot \mathrm{Vol}\left\{(y_1, \dots, y_n) \;\middle|\; \begin{array}{l} 0 \leq y_1 \leq \dots \leq y_n \\ y_1 + \dots + y_n \leq 1 \end{array}\right\}$

$= f^\lambda \frac{1}{n!} \mathrm{Vol}\left\{(y_1, \dots, y_n) \;\middle|\; \begin{array}{l} y_i \geq 0 \\ \sum y_i \leq 1 \end{array}\right\} = f^\lambda \left(\frac{1}{n!}\right)^2$

$HLF \iff \mathrm{Vol}(\Delta_\lambda) = \mathrm{Vol}\, P_\lambda$

__Thm__ $\exists$ continuous bijective piecewise-linear, volume-preserving map, $\varphi_\lambda : \Delta_\lambda \xrightarrow{\sim} P_\lambda$

We'll construct a more general map

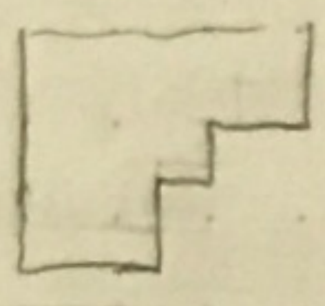
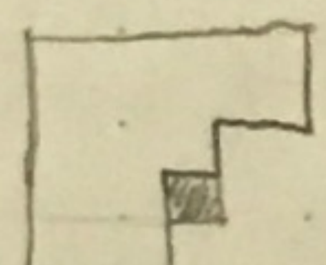
$$\varphi_\lambda : \{(x_{ij})_{(i,j)\in\lambda} \mid x_{ij} \geq 0\} \longrightarrow \left\{(y_{ij})_{(i,j)\in\lambda} \;\middle|\; \begin{array}{l} y_{ij} \geq 0 \\ \text{weakly increasing} \\ \text{in rows \& columns} \end{array}\right\}$$

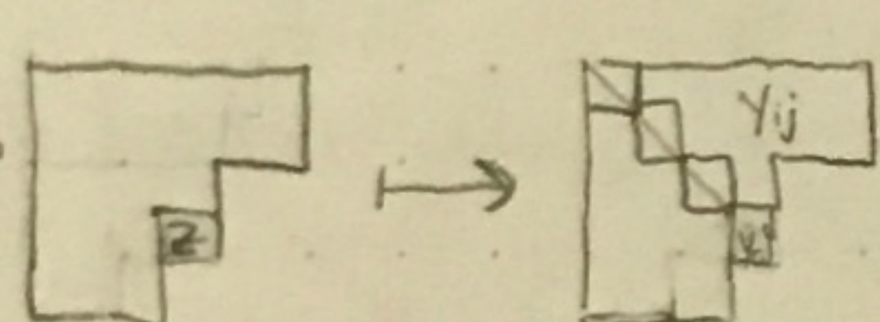
s.t. $\sum h_{ij} x_{ij} = \sum y_{ij}$ ← for $y_{ij} = \varphi(x_{ij})$

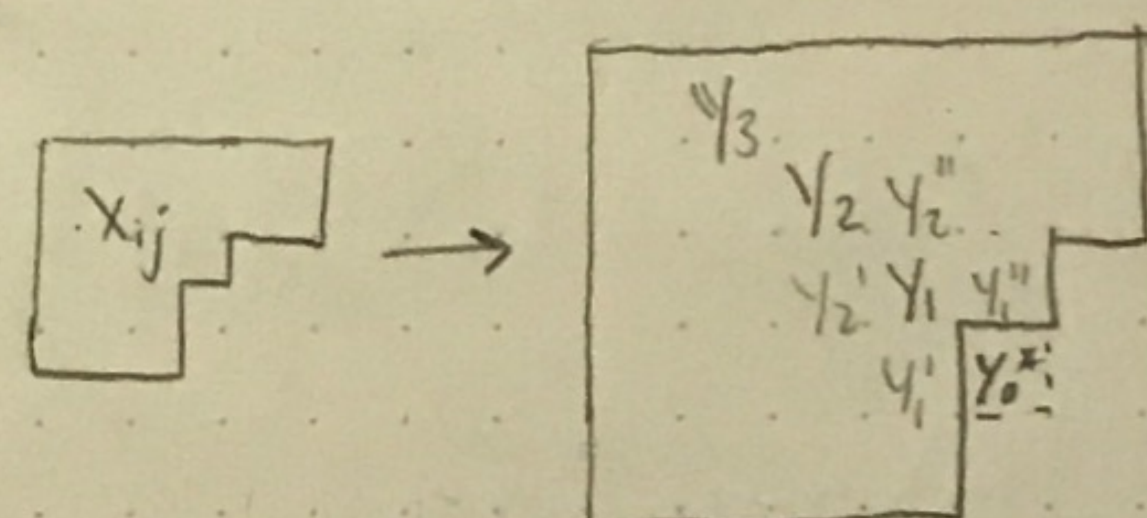
Construction of φ_λ by induction on $|\lambda|$

__Base__ $\lambda = \emptyset \quad \Delta_\emptyset = P_\emptyset = \{\cdot\}$

__Step__: Assume we have already constructed φ_λ. Now add a box to λ

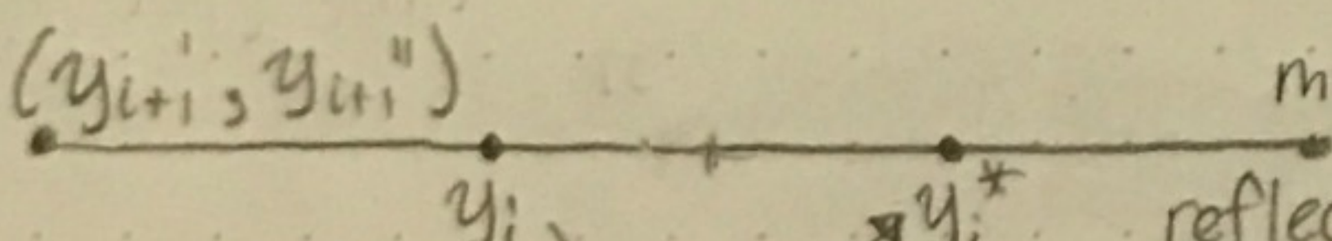
$\lambda =$  $\mu =$  $= \lambda \cup \{1 \text{ box}\}$

$\varphi_\mu :$ 

$\varphi_\lambda :$ 

$$\begin{array}{ccccccc} y_1' & \geq & & \geq y_2' & & \geq y_3' \\ & & y_1 & & \geq y_2 & & \\ & \geq & & \geq & & & \\ y_1'' & & \geq y_2'' & \geq & & \geq y_3'' \end{array}$$

If y_i', y_i'' outside of shape, assume to be 0.



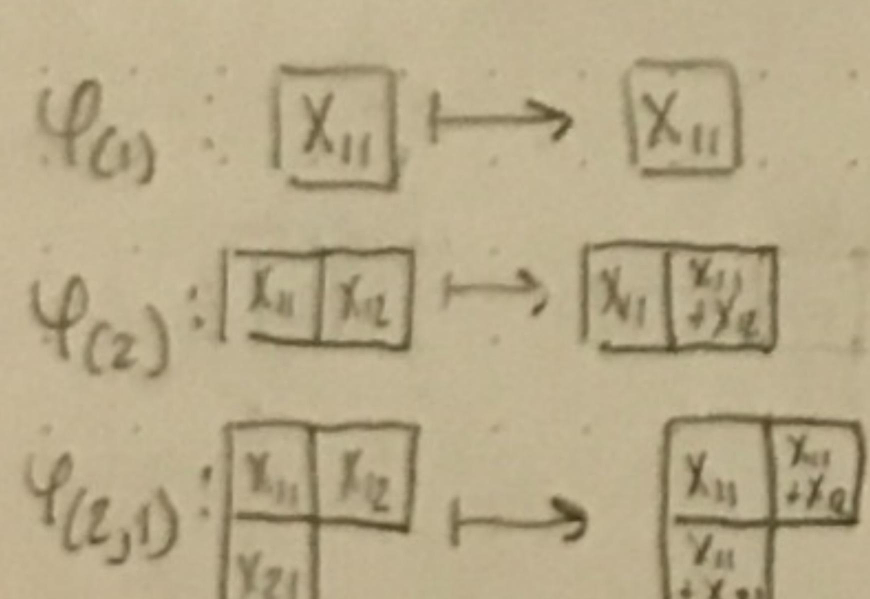
reflect across center of interval

$y_i^* = \max(y_{i+1}', y_{i+1}'') + \min(y_i', y_i'') - y_i$ ← this operation is called a __toggle__
$i \geq 1$

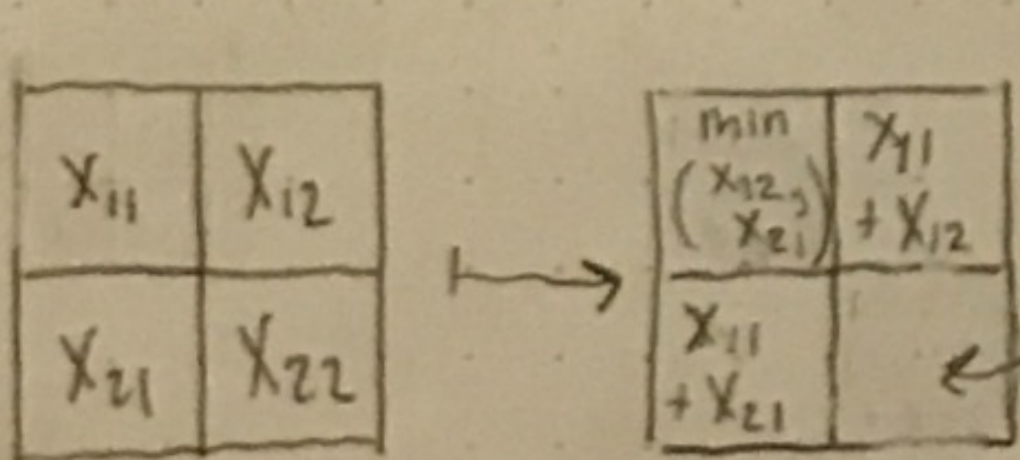
$y_0^* = \max(y_1', y_1'') + z$

*Toggles are volume conserving

__Ex.__



Construction well defined for any order of adding boxes b/c if adding boxes a & b could be in any order, their diagonals are sufficiently far apart to not interfere w/ each other

$\varphi_{(2,2)}$ 

$x_{11} + \max(x_{12}, x_{21}) + x_{22}$

$\sum y_{ij} = x_{12} + x_{21} + x_{11} + x_{12} + x_{11} + x_{22} = 3x_{11} + 2x_{12} + 2x_{21} + x_{22}$

18.217 LECTURE 3

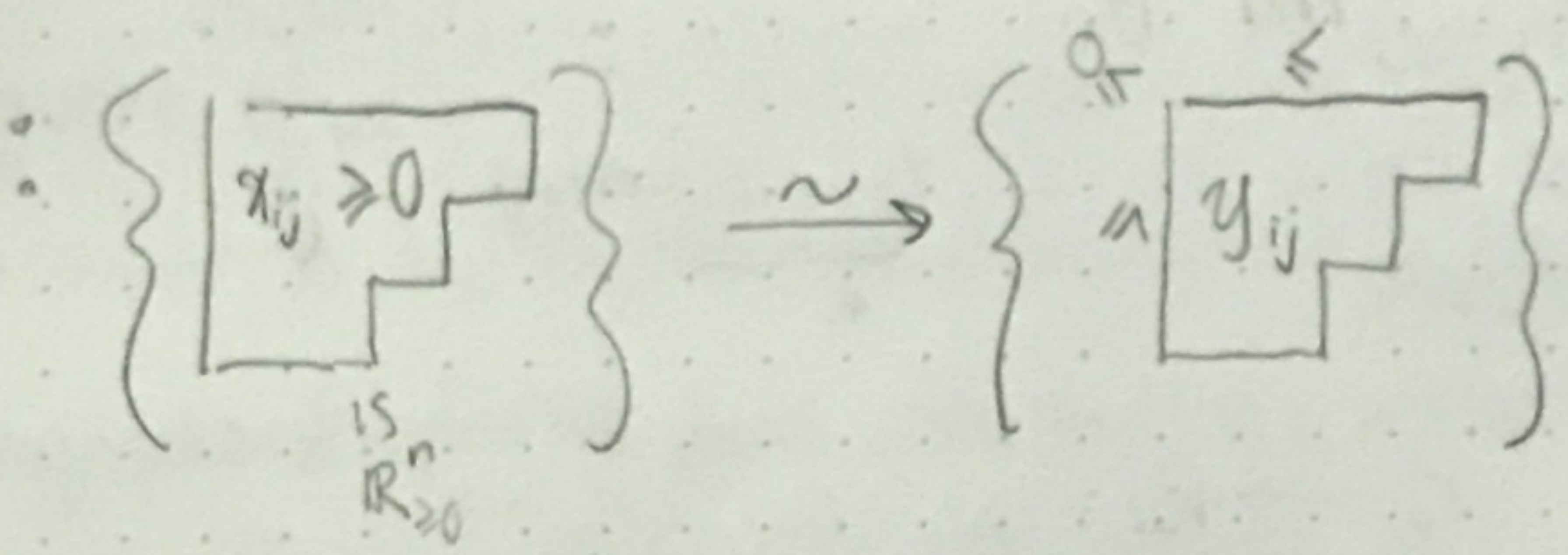
9/12/22

Last time: hook length formula $f^\lambda = \frac{n!}{\pi h_{ij}}$

$$\Updownarrow$$

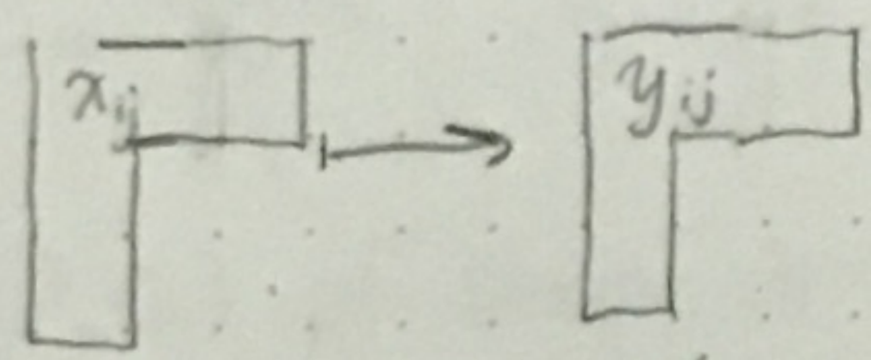
$$\mathrm{Vol}\,\Delta_\lambda = \mathrm{Vol}\,P_\lambda \qquad \lambda \vdash n$$

Map: φ_λ :



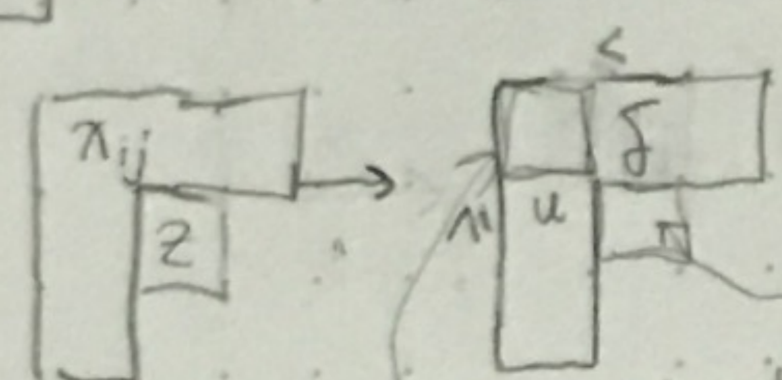
Construction of φ_λ by adding boxes to λ

φ_λ :



See last lecture for toggle & more explanation/examples

$\varphi_{\lambda \cup \{box\}}$:



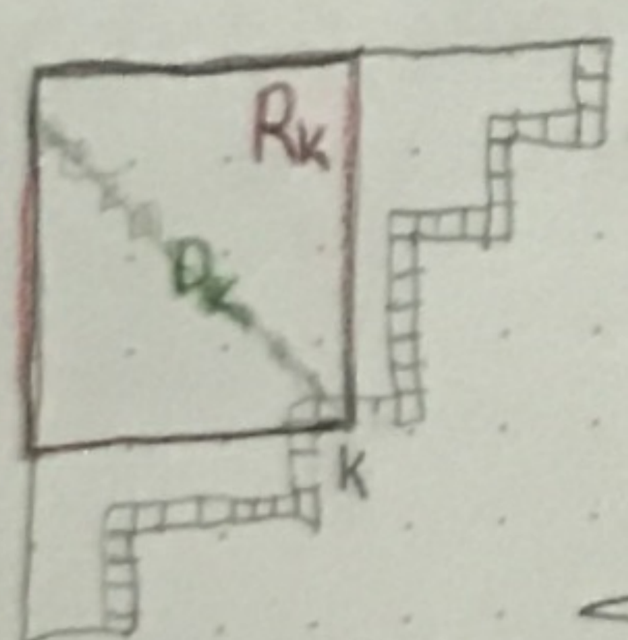
$\max(u,\delta)+z$

x toggle all entries on this diagonal

Lemma 1: If $\varphi_\lambda : (x_{ij}) \mapsto (y_{ij})$ then $\sum_{(i,j)\in\lambda} h_{ij} x_{ij} = \sum_{(i,j)\in\lambda} y_{ij}$

k^{th} rectangle R_K

k^{th} diagonal D_K



Lemma 2: $\sum_{(i,j)\in R_K} x_{ij} = \sum_{(i,j)\in D_K} y_{ij}$

Ex. $\begin{bmatrix} x & y \\ z & t \end{bmatrix} \rightarrow \begin{bmatrix} \min(y,z) & x+y \\ x+z & x+t+\max(y,z) \end{bmatrix}$

$r_1 = x+z \qquad d_1 = x+z$

$r_2 = x+y+z+t \qquad d_2 = x+y+z+t$

Claim: Lemma 1 $\Rightarrow$ Lemma 2

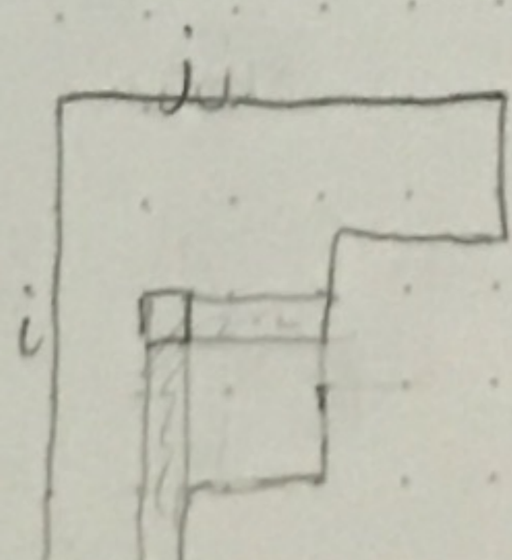
Proof: $r_1, \dots, r_\ell$ rect. sum of x_{ij}

$\qquad\; \shortparallel \qquad\;\; \shortparallel$

$\qquad d_1, \dots, d_\ell$ diag. sum of y_{ij}

$d_1 + \dots + d_\ell = \sum y_{ij}$

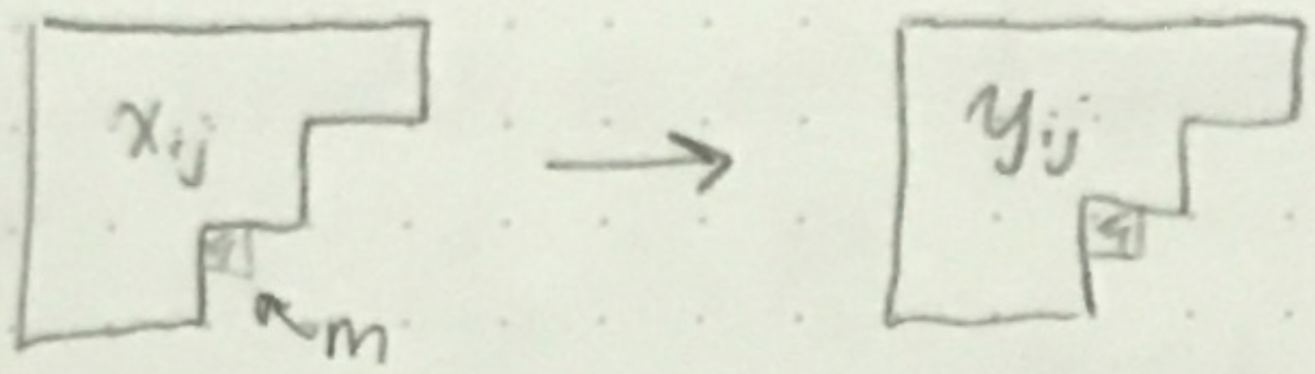
$r_1 + \dots + r_\ell = \sum ?\, x_{ij}$



hook ribbon controls how many extra times the box gets counted

Proof of Lemma 2

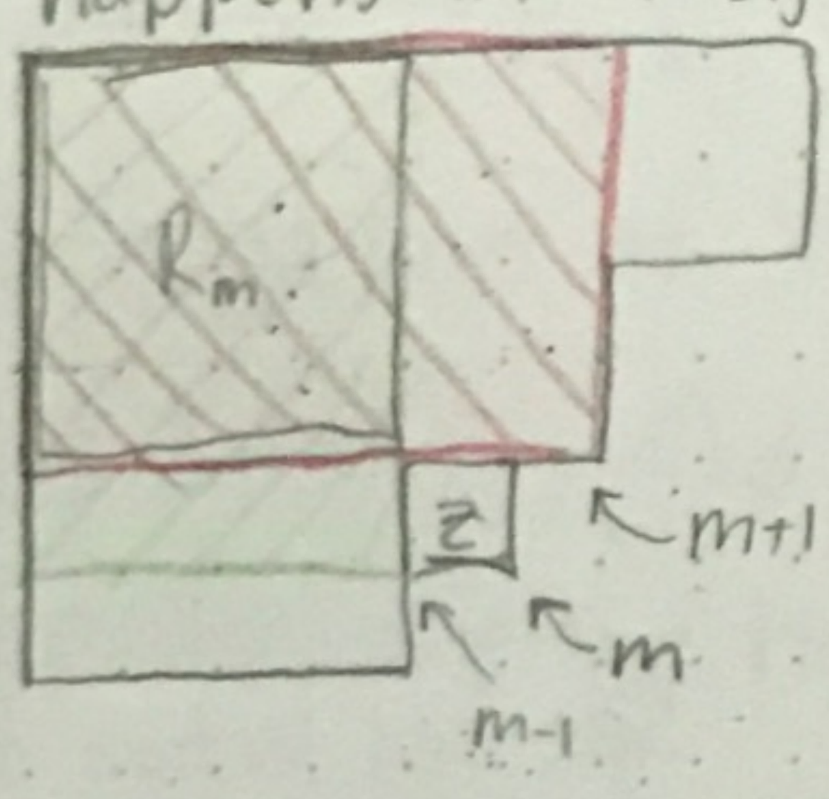
φ_λ :



Assume $r_1, \ldots, r_\ell = m_1, \ldots, m_\ell$ by inductive hypothesis

When we add new box, only r_m & d_m change

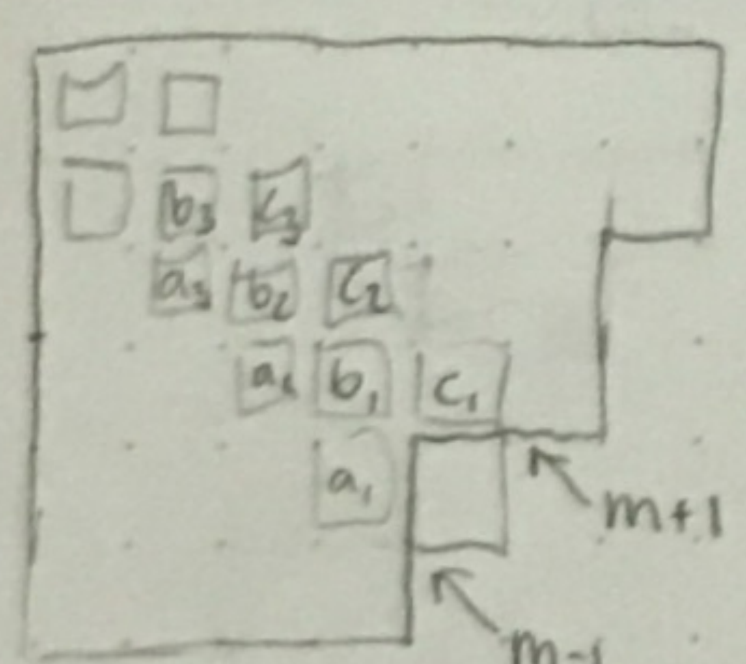
What happens in $(x)_{ij}$ diagram:



Only r_m contains new box, so only r_m changes

$r_m \rightsquigarrow \tilde{r}_m = r_{m-1} + r_{m+1} - r_m + z$

What happens in $(y)_{ij}$ diagram:



$d_{m-1} = a_1 + a_2 + \ldots$

$d_m = b_1 + b_2 + \ldots$

$d_{m+1} = c_1 + c_2 + \ldots$

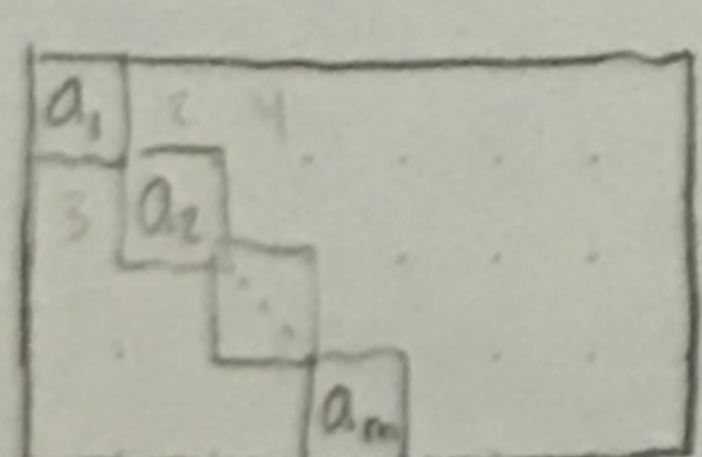
$$d_m \rightsquigarrow \tilde{d}_m = (\max(a_1, c_1) + z) + (\max(a_2, c_2) + \min(a_1, c_1) - b_1) + (\max(a_3, c_3) + \min(a_2, c_2) - b_2) + \ldots$$

$$= (a_1 + a_2 + \ldots) + (c_1 + c_2 + \ldots) - (b_1 + b_2 + \ldots) + z$$

$$= d_{m-1} + d_{m+1} - d_m + z$$

Exercise (on Pset): $\lambda = m \times n$, $m \leq n$

$T =$



• SYT

$a_1, \ldots, a_m$ entries on main diaganol of T

$wt(T) := \prod_{i=1}^{m} \frac{1}{i^{a_{i+1} - a_i}}$ $\quad (a_{m+1} := m \cdot n + 1)$

Using Lemma 2, Prove that $\boxed{\sum_{T \in SYT(m \times n)} wt(T) = 1}$

Ex.

1	2	3
4	5	6

$\frac{1}{1^4 \cdot 2^2}$

1	2	4
3	5	6

1	3	4
2	5	6

1	2	5
3	4	6

$\frac{1}{1^3 \cdot 2^3}$

1	3	5
2	4	6

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$

Thrm: $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$

Rep theory of S_n

Facts: • Irreps V_λ of S_n / $\mathbb{C}$ correspond to $\lambda \vdash n$

• $f^\lambda = \dim V_\lambda$

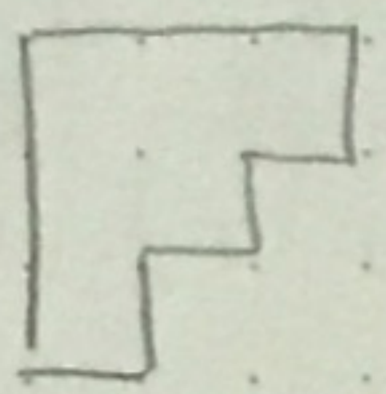
$\Rightarrow$ Then square of dim of each irrep gives the order of S_n

but we also want to see this combinatorially w/ a bijection

Combinatorial proof of Thrm

uses Robinson-Schensted correspondence

$$S_n \xrightarrow[\text{bij}]{\sim} \left\{ (P,Q) \;\middle|\; \begin{array}{l} P, Q \text{ are SYT of} \\ \text{the same shape } \lambda \vdash n \end{array} \right\}$$

$w \mapsto (P, Q)$ of shape $\lambda =$ 

$w^{-1} \mapsto (Q, P)$

first row is size of maximal increasing subseq. in w

first column is size of maximal decreasing subseq. in w

Next time: will present a construction of the correspondence (which makes these properties clear)

18.217 LECTURE 4

9/14/22

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Robinson Schensted Correspondence

$$\underset{\underset{S_n}{\cap}}{w} \longmapsto (P, Q) \qquad \text{2 SYT of the same shape } \lambda \vdash n$$

Schensted insertion algorithm:

$\underset{\text{intermediate SYT}}{T} \xleftarrow{\text{want to insert}} a$

0.) set $i = 1$, $x := a$
1.) If all entries in i^{th} row are $< x$ (or if i^{th} row is empty), then add new box in the end of i^{th} row filled w/ x & stop.
2.) Otherwise, find the leftmost entry y in the i^{th} row s.t. $y > x$. Replace y with x then set $i := i+1$, $x := y$ & go to step 1.

E.x.

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 9 \\ \hline 2 & 7 \\ \cline{1-2} 6 \\ \cline{1-1} 10 \\ \cline{1-1} \end{array} \qquad a = 3 \longrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 9 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 10 \\ \cline{1-1} \end{array}$$

RS correspondence

$w = w_1 \dots w_n$

$P = ((\emptyset \leftarrow w_1) \leftarrow w_2) \dots \leftarrow w_n$ (do Schensted insertion alg. for each w_i in order)

Q: When we insert w_i and add a new box, we simultaneously add a box in same position to Q filled with i.

E.x. $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 5 & 2 \end{pmatrix} \begin{matrix} \rightsquigarrow Q \\ \rightsquigarrow P \end{matrix}$

$$P: \ \emptyset \quad \begin{array}{|c|} \hline 3 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} = P$$

$$Q: \ \emptyset \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} = Q$$

Claim: We can backtrack the entire algorithm to recover our word from P & Q
(You can check yourself)

Observe: First row will be size of longest increasing subsequence
First column will be size of longest decreasing subsequence.
→ sum of first k rows is maximal # of entries that can be covered by k increasing subsequences
(and similarly for columns)
$w = (P, Q) \Leftrightarrow w^{-1}(Q, P)$, but this isn't obvious from this construction

Robinson-Schenstedd-Knuth corr. (RSK)
for semi-standard Young Tableaux (SSYT)
← weekly increasing in rows, strictly increasing in columns

E.x. $\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&2&4\\ \hline 2&2&4&5&7\\ \cline{1-5} 4&5&5&7\\ \cline{1-4}\end{array}$ shape $\lambda = (6,5,4)$

(content?)
<u>weight</u>: $\beta = (3,4,0,3,3,0,2)$ $\beta_i = \#$ of i's in tableau T

<u>RSK</u>: $\left\{\begin{array}{l} n\times n \text{ matrices, } A \\ \text{w/ entries in } \mathbb{Z}_{\geq 0} \end{array}\right\} \xleftrightarrow[\text{bij}]{\text{RSK}} \left\{\begin{array}{l} (P,Q) \text{ SSYT of same shape} \\ \text{w/ column sums } \alpha_1,\dots,\alpha_n, \text{ row sums } \beta_1,\dots,\beta_n \end{array} \text{ s.t. } \begin{array}{l} \text{weight}(P)=\alpha \\ \text{weight}(Q)=\beta \end{array}\right\}$

Fix n.
$\alpha = (\alpha_1,\dots,\alpha_n)$
$\beta = (\beta_1,\dots,\beta_n)$
$\alpha_1+\dots+\alpha_n = \beta_1+\dots+\beta_n$

shape is partition of all entries in A $= \alpha_1+\dots+\alpha_n$

<u>E.x.</u> $n=3$ $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \to \left(\begin{array}{c|c|c} & \cdot & \\ \hline \cdot\,\cdot & \cdot & \\ \hline \cdot\,\cdot\,\cdot & & \cdot \end{array}\right)$

a dot in position $(i,j) \to \binom{i}{j}$. Go in lexicographic order

$\binom{1}{2}\binom{1}{1}\binom{2}{1}\binom{2}{2}\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{3} \rightsquigarrow$ gives $\begin{pmatrix} 1&1&2&2&3&3&3&3 \\ 2&1&1&2&1&1&1&3 \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_N \\ w_1 & \cdots & w_N \end{pmatrix}$

Now use RS correspondence, being careful about $\leq$ vs $<$ for insertion b/c they matter now

P: $\emptyset$ $\begin{array}{|c|}\hline 2\\ \hline\end{array}$ $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \dashrightarrow \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2\\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&3\\ \hline 2&2\\ \cline{1-2}\end{array}$

Q: $\emptyset$ $\begin{array}{|c|}\hline 1\\ \hline\end{array}$ $\begin{array}{|c|}\hline 1\\ \hline 1\\ \hline\end{array} \dashrightarrow \begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 1\\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&3&3\\ \hline 1&3\\ \cline{1-2}\end{array}$

$A \longmapsto (P,Q)$
$A^T \longmapsto (Q,P)$
This property not clear from construction though

We need to be able to invert this. But how? Which 3 do we start with?
Answer: start with rightmost box of Q containing maximal entry