18.217 LECTURE 9

9/28/22

__Demazure operators__

(A.k.a. isobaric divided difference operators on $\mathbb{C}[x_1, \dots, x_n]$)

$$D_i \cdot f(x_1, \dots, x_n) \longmapsto \frac{f(x_1, \dots, x_n) - \frac{x_{i+1}}{x_i} f(x_1, \dots, x_{i+1}, x_i, \dots, x_n)}{1 - x_{i+1}/x_i}$$

$i = 1, \dots, n-1$

$$= \frac{(1 - \frac{x_{i+1}}{x_i} s)(f)}{1 - x_{i+1}/x_i} = \partial_i (x_i f)$$

__D-Hecke Relations__

(1)" $D_i^2 = D_i$

(2) $D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$

(3) $D_i D_j = D_j D_i$, $|i-1| \geq 2$

$D_w = D_{i_1} \cdots D_{i_\ell}$

for $w = s_{i_1} \cdots s_{i_\ell}$, reduced word

__Thrm__: For $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, $S_\lambda(x_1, \dots, x_n) = D_{w_0}(x^\lambda)$

where w_0 is longest permutation.

ASSIDE: $D_1 : x_1 \longmapsto x_1 + x_2$

$x_1 + x_2 \longmapsto x_1 + x_2$

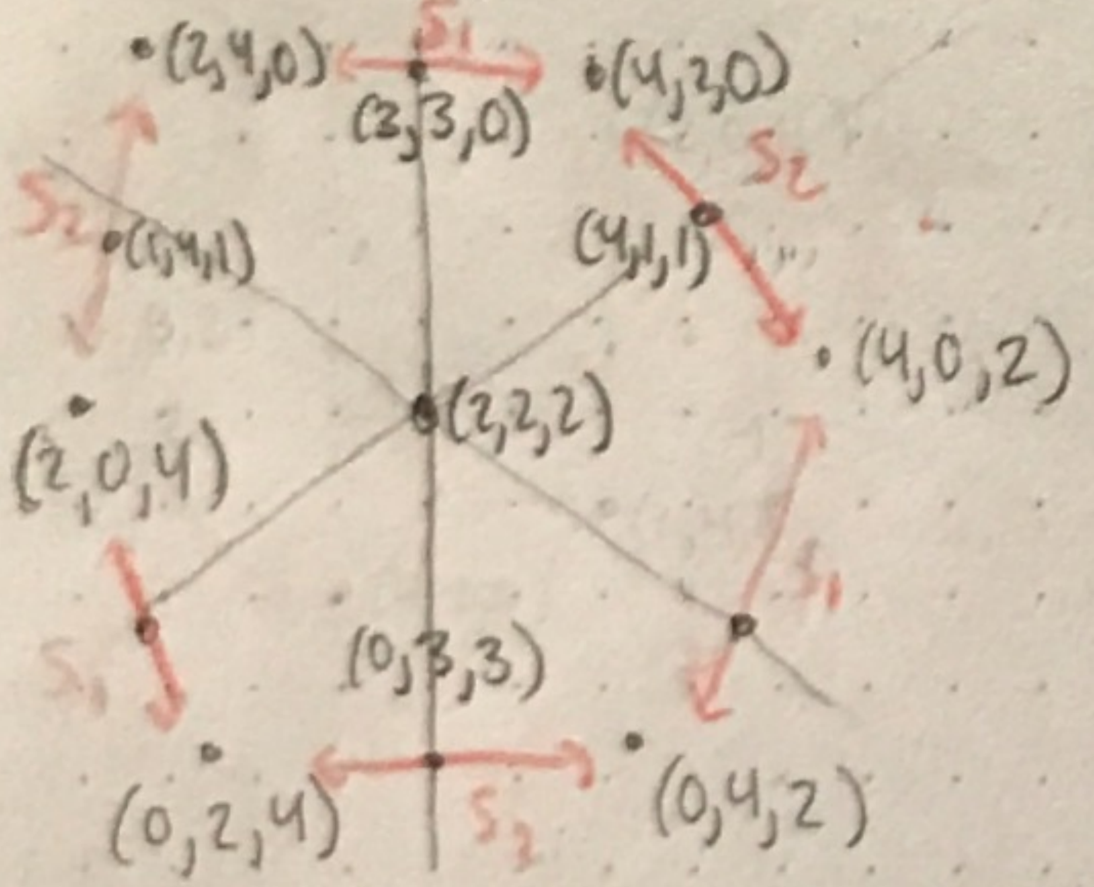
$x_2 \longmapsto 0$

__Problem__: $D_{w_0}(x_1^{\beta_1} \cdots x_n^{\beta_n}) = ?$ $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_{\geq 0}^n$

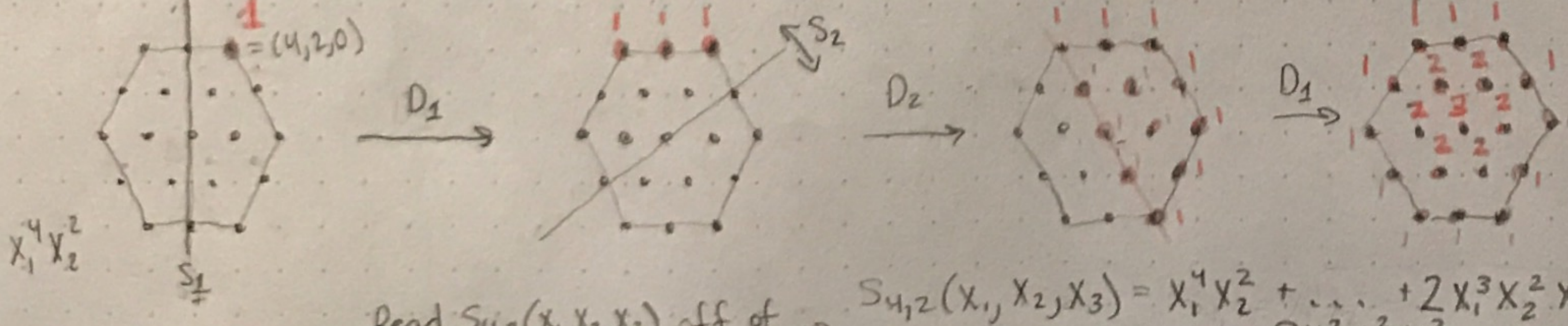
(β not necessarily a partition here)

__Ex.__ $n = 3$ $\lambda = (4,2,0) =$ $\boxed{}\boxed{}\boxed{}\boxed{}$ / $\boxed{}\boxed{}$

$S_\lambda(x_1, x_2, x_3) = \sum x_1^a x_2^b x_3^c$ ⇝ a, b, c



$D_1 D_2 D_1 (x_1^4 x_2^2 x_3^0)$



Read $S_{4,2}(x_1, x_2, x_3)$ off of last diagram → $S_{4,2}(x_1, x_2, x_3) = x_1^4 x_2^2 + \dots + 2 x_1^3 x_2^2 x_3 + \dots + 3 x_1^2 x_2^2 x_3^2 + \dots$

Now let's check:

$K_{(4,2,0),(2,2,2)} = 3$

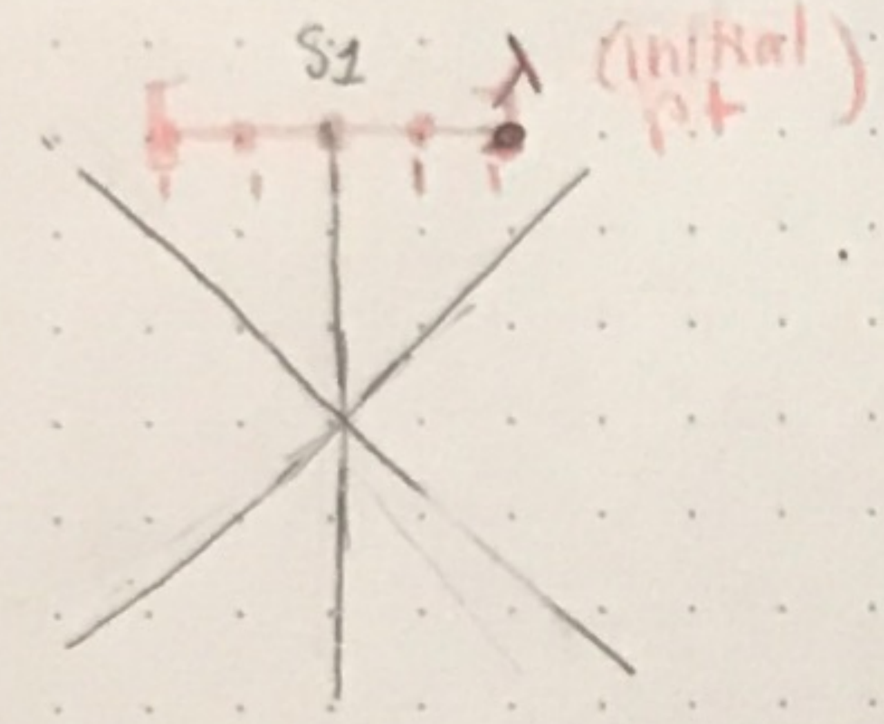
1	1	2	2
3	3		

1	1	2	3
2	3		

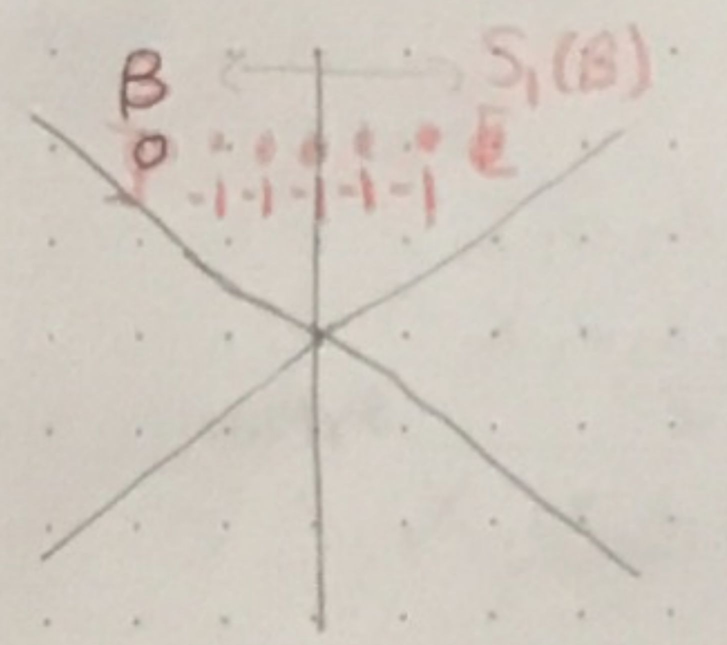
1	1	3	3
2	2		

which matches the coefficient we got for $x_1^2 x_2^2 x_3^2$ ✓

Ex.



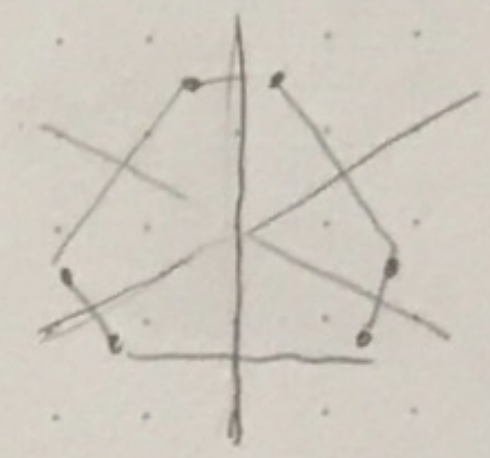
if we instead start with β not a partition



sometimes an a_j sometimes an 0

Def: The permutohedron is the polytope

$$\Pi(\lambda) := \operatorname{conv}\left((\lambda_{w(1)}, \lambda_{w(2)}, \dots, \lambda_{w(n)}) \mid w \in S_n \right)$$
$\lambda \in \mathbb{R}_n$



Observation: Fix $\lambda = (\lambda_1, \dots, \lambda_n)$ partition.

$$S_\lambda(x_1, \dots, x_n) = \sum_{\beta \in \mathbb{Z}^n} K_{\lambda\beta} x^\beta$$

Thrm: $K_{\lambda\beta} \neq 0$ iff $\beta \in \Pi(\lambda) \cap \mathbb{Z}^n$ (i.e. β is a lattice pt. of the permutohedron for λ).

$K_{\lambda\beta} = K_{\lambda\tilde{\beta}}$ $\forall$ permutations $\tilde{\beta}$ of β.

So assume $\mu = (\mu_1 \geq \dots \geq \mu_n)$ is a partition.

Thrm: $K_{\lambda\mu} \neq 0$ iff

$$\left[\begin{array}{l} \lambda_1 \geq \mu_1 \\ \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2 \\ \vdots \\ \lambda_1 + \dots + \lambda_k \geq \mu_1 + \dots + \mu_k \end{array} \quad \text{and } |\lambda| = |\mu| \right]$$

The dominence order on partition

so $K_{\lambda\mu} \neq 0$ if $\lambda > \mu$ in the dominence order

Rodo's Thrm: $\Pi(\lambda)$ is given by the following inequalities:

$$\Pi(\lambda) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x_{i_1} + \dots + x_{i_k} \leq \lambda_1 + \dots + \lambda_k \ \forall k \text{ and distinct } i_1, \dots, i_k \\ x_1 + \dots + x_n = \lambda_1 + \dots + \lambda_n \end{array} \right\}$$

NOTE:
(This Thrm immediately implies thrm above)

$$\partial_{w_0}(x^{\lambda+\delta}) \underset{\text{WHY}}{\overset{?}{=}} D_{w_0}(x^\lambda) \quad \text{where } \delta = (n-1, n-2, \dots, 1, 0)$$

Notation: $X_i : f \longmapsto x_i f$

$$X^\delta = X_1^{\delta_1} X_2^{\delta_2} \cdots X_n^{\delta_n}$$

<u>Thrm</u>: $D_{w_0} = \partial_{w_0} X^{\delta}$ (Comp. of operators on $\mathbb{C}[x_1, \ldots, x_n]$)

<u>Ex.</u> $n=3$

$\boxed{\text{we know } D_i = \partial_i X_i}$

$$D_{w_0} = D_1 D_2 D_1 = \partial_1 X_1 \partial_2 X_2 \partial_1 X_1$$

$$\stackrel{?}{=} \partial_1 \partial_2 \partial_1 X_1^2 X_2$$

$\boxed{\text{we also know } \partial_i X_j = X_j \partial_i \text{ if } j \neq i, i+1}$

$$\hookrightarrow \partial_1 X_1 \partial_2 X_2 \partial_1 X_1$$

$$= \partial_1 \partial_2 \underbrace{X_1 X_2}_{} \partial_1 X_1$$

symmetric w.r.t. perm. of X_1, X_2, so commutes w/ ∂_1

$$= \partial_1 \partial_2 \partial_1 \underbrace{X_1 X_2} X_1$$