18.217 PROBLEM SET 2 (due Friday, December 3, 2021)

Solve 5 (or more) of the following problems.

Problem 1. A tree T on vertices labelled $1, 2, 3, \ldots$ is called *alternating* if it has no pair of edges (a, b) and (b, c) with $a < b < c$. A tree T is called *non-crossing* if it has no pair of edges (a, c) and (b, d) with $a < b < c < d$. A tree T is called *non-nesting* if it has no pair of edges (a, d) and (b, c) with $a < b < c < d$.

Prove bijectively that the number of non-crossing alternating trees on $n+1$ vertices equals the number of non-nesting alternating trees on $n+1$ vertices equals the Catalan number C_n.

Problem 2. Prove that the number of alternating trees on $n+1$ vertices equals the number of binary trees on n vertices labelled by $1, \ldots, n$ such that the left child of a vertex in always greater than its parent and the right child of a vertex is always less than its parent.

Problem 3. Find a formula for the number of alternating trees on n vertices. (Your formula might involve a single summation.)

Problem 4. Find a formula for the number of non-crossing trees on n vertices.

Problem 5. Calculate the value $\mu_{NC_n}(\hat{0}, \hat{1})$ of the Möbius function for the lattice NC_n of non-crossing partitions.

Problem 6. Prove transitivity of the Hurwitz action for the symmetric group S_n. In other words, show that any two factorizations $t_1 t_2 \cdots t_{n-1}$ and $t'_1\, t'_2 \cdots t'_{n-1}$ of the long cycle $c = (1, 2, \ldots, n) \in S_n$ into products of $n-1$ transpositions can be obtained from each other by a sequence of Hurwitz moves σ_i, $i = 1, \ldots, n-2$:

$$\sigma_i : \cdots t_{i-1}\, t_i\, t_{i+1}\, t_{i+2} \cdots \longrightarrow \cdots t_{i-1}\, t_{i+1}\, (t_{i+1}^{-1} t_i t_{i+1})\, t_{i+2} \cdots$$

Problem 7. For the Hurwitz moves σ_i acting on factorizations of the long cycle $c \in S_n$ into products of $n-1$ transpositions (as in the previous problem) show that $(\sigma_1 \sigma_2 \cdots \sigma_{n-2})^{n(n-1)}$ is the identity operator.

Problem 8. Prove that the number of m-tuples $(t_1, \ldots, t_m)$ of transpositions $t_i \in S_n$ such that
(a) $t_1\, t_2 \cdots t_m = 1 \in S_n$,
(b) $t_1, \ldots, t_m$ generate the symmetric group S_n, and
(c) $m = 2n - 2$

equals $n^{n-3}\,(2n-2)!$.

Problem 9. In class, we showed that the lattice of non-crossing partitions NC_n is isomorphic to the interval $[1, c]_{\mathrm{abs}}$ between the identity permutation 1 and the long cycle $c = (1, 2, \ldots, n)$ in the absolute reflection order on the symmetric group S_n. Thus the Kreweras complementation map $K : NC_n \to NC_n$ induces a map on saturated chains in the poset $[1, c]_{\mathrm{abs}}$. These saturated chains correspond to factorizations $t_1 t_2 \cdots t_{n-1}$ of c in products on $n-1$ reflections.

Show that the map $t_1 t_2 \cdots t_{n-1} \to t_1'\, t_2' \cdots t_{n-1}'$ acting on factorizations of c obtained from the Kreweras complementation can be described as follows:
$t_1' = t_{n-1}$,
$t_2' = t_{n-1}^{-1}\, t_{n-2}\, t_{n-1}$,
$t_3' = t_{n-1}^{-1}\, t_{n-2}^{-1}\, t_{n-3}\, t_{n-2}\, t_{n-1}$, etc.

Problem 10. Let $\mathcal{A}$ be the affine hyperplane arrangement in the space $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\} \simeq \mathbb{R}^{n-1}$ with $\binom{n}{2}$ affine hyperplanes H_{ij}, $1 \le i < j \le n$, given by the equations

$$x_i - x_j = a_{ij}\,,$$

where a_{ij} are some fixed generic real numbers. Show that
(a) The number of vertices of the arrangement $\mathcal{A}$ (i.e., the number of 0-dimensional intersections of some hyperplanes H_{ij}) equals the number n^{n-2} of trees on n labelled vertices.
(b) The number of regions of the arrangement $\mathcal{A}$ equals the number of forests on n labelled vertices.

Problem 11. Find an explicit formula for the number of regions of the hyperplane arrangement in $\mathbb{R}^n$ with $5\binom{n}{2}$ hyperplanes given by the equations

$$x_i - x_j = -2, -1, 0, 1, 2,$$

for $1 \le i < j \le n$.

Problem 12. Find an explicit formula for the number of regions of the hyperplane arrangement in $\mathbb{R}^n$ with $4\binom{n}{2}$ hyperplanes given by the equations

$$x_i - x_j = -1, 0, 1, 2,$$

for $1 \le i < j \le n$.

Problem 13. For a finite matroid M without loops, let $L(M)$ be the lattice of flats of M. Prove that

(a) $L(M)$ is a geometric lattice, and
(b) any geometric lattice L is isomorphic to $L(M)$ for some M.

Problem 14. Let $M = (M, \mathcal{B})$ be a matroid, where E is the ground set of M, and $\mathcal{B}$ is the set of bases of M. Let $M^* = (E, \mathcal{B}^*)$, where $\mathcal{B}^* := \{E \setminus I \mid I \in \mathcal{B}\} \subset 2^E$. Prove that M^* is a matroid. In other words, show that the Exchange Condition for the set of bases $\mathcal{B}$ is equivalent to the Exchange Condition for the set of bases $\mathcal{B}^*$.

Problem 15. Let $A_{m,n}$ be the number of acyclic orientations of the complete bipartite graph $K_{m,n}$. Prove that

(a) $A_{m,n}$ equals the number of $m \times n$ matrices filled with 0's and 1's such that no 2×2 submatrix equals $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
(b) $A_{m,n}$ equals the number of $m \times n$ matrices filled with 0's and 1's such that no 2×2 submatrix equals $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(A "2×2 submatrix" means a submatrix located in any 2 rows and any 2 columns of a matrix, not necessarily consecutive rows/columns.)

Problem 16. Find an explicit formula for the number $A_{m,n}$ of acyclic orientations of $K_{m,n}$. (Your answer that might involve a single summation.)

Problem 17. An *increasing forest* is a forest with vertices labelled by $1, 2, 3, \ldots$ that contains no pair of edges (a, c) and (b, c) with $a < b < c$.

Prove bijectively that the number of increasing forests on n labelled vertices with k edges equals the Stirling number of the first kind $s(n, n-k)$, i.e., the number of permutations in S_n with $n - k$ cycles.

Problem 18. Construct a linear basis of the Orlik-Solomon algebra for the Catalan hyperplane arrangement. Describe a bijection between elements of your basis and some set of combinatorial objects of cardinality $n!\, C_n$.

Problem 19. In class, we discussed the following map ϕ from the set of Young diagrams λ that fit inside the staircase shape $(n-1, n-2, \dots, 1)$ and certain posets P on n *labelled* vertices $1, \dots, n$. (Clearly, such Young diagrams λ correspond to Dyck paths with $2n$ steps.) The poset $P = \phi(\lambda)$ is given by $i <_P j$ if and only if the box $(i, n+1-j)$ belongs to λ.

Prove that the map ϕ induces a bijection between Dyck paths with $2n$ steps and all *unlabelled* semiorders on n vertices.

Problem 20. Let $v_1, \dots, v_N$ be a collection of vectors that span a vector space $V \simeq \mathbb{R}^d$, and let $\Lambda \subset V$, $\Lambda \simeq \mathbb{Z}^d$, be the $\mathbb{Z}$-span of these vectors v_i (i.e., the set of their integer linear combinations). We say that a collection of vectors $v_1, \dots, v_N$ is *unimodular* if, whenever a subset of these vectors forms a linear basis of V, the $\mathbb{Z}$-span of this subset equals Λ. For a example, the collection of vectors $(1,0), (0,1), (1,1) \in \mathbb{Z}^2 \subset \mathbb{R}^2$ is unimodular, but the collection of vectors $(1,0), (0,1), (1,2)$ it not unimodular because the $\mathbb{Z}$-span of $(1,0)$ and $(1,2)$ is not $\mathbb{Z}^2$.

In class, we discussed the graphical arrangement $\mathcal{A}_G$ and the co-graphical arrangement $\mathcal{A}_G^*$ associated with a graph G. Show that, for each of these arrangements, one can pick normal vectors to the hyperplanes so that they form a unimodular collection of vectors.

Problem 21. For $n \geq 4$, the *wheel graph* W_n is the simple graph on n vertices obtained from the $(n-1)$-cycle graph C_{n-1} by adding one extra vertex connected to all vertices of C_{n-1}. Show that the number of acyclic orientations of the wheel graph W_n equals the number of totally cyclic orientations of W_n.

Problem 22. Show that the evaluation $T_{K_{n+1}}(1,-1)$ of the Tutte polynomial for the complete graph K_{n+1} equals the number of alternating permutations in S_n. (Recall that a permutation $w \in S_n$ is *alternating* if $w_1 < w_2 > w_3 < w_4 > \cdots$.)