18.217 Problem Set 1 (due Friday, October 15, 2021)

Solve 4 (or more) of the following problems.

Problem 1. The *Narayana number* $N(n,k)$ is the number of Dyck paths with $2n$ steps and k peaks. Prove bijectively that the Narayana number $N(n,k)$ equals the number of binary trees on n vertices with $k-1$ left edges.

Problem 2. An *increasing plane tree* is a plane tree with vertices labelled by $1, 2, \dots$ so that each child has a greater label than its parent. Find an explicit formula for the number of increasing plane trees on $n+1$ vertices.

Problem 3. A *left-increasing binary tree* is a binary tree with vertices labelled by $1, 2, \dots, n$ so that left childeren have greater labels than their parents. Show that the number of left-increasing binary trees on n vertices equals $(n+1)^{n-1}$.

Problem 4. Define the *q-Catalan number* as the sum $C_n(q) := \sum q^{\mathrm{area}(P)}$ over Dyck paths P with $2n$ steps, where $\mathrm{area}(P)$ is the area below P. Find a recurrence relation for the q-Catalan numbers and show that

$$\sum_{n\geq 0} C_n(q)\, x^n = \cfrac{1}{1-\cfrac{x}{1-\cfrac{qx}{1-\cfrac{q^2x}{1-\cfrac{q^3x}{1-\cdots}}}}}$$

Problem 5. Find a formula for the face number f_i, i.e., the numbers of i-dimensional faces, of the $(n-1)$-dimensional associahedron.

Problem 6. A permutation $w = w_1, \dots, w_n$ is *alternating* if it is satisfies $w_1 < w_2 > w_3 < w_4 > \cdots$. Show that the alternating sum

$$(-1)^{n(n+1)/2} \sum_{(a_1,\dots,a_n)} (-1)^{a_1+\cdots+a_n}$$

over all parking functions $(a_1, \dots, a_n)$ of size n equals the number of alternating permutations of size n.

Problem 7. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ with n nonzero parts, a sequence $(a_1, \ldots, a_n)$ of positive integers $a_i > 0$ is λ-*parking function* if the weakly decreasing rearrangement $b_1 \geq b_2 \geq \cdots \geq b_n$ of the numbers $a_1, \ldots, a_n$ satisfies $b_i \leq \lambda_i$ for $i = 1, \ldots, n$.

Show that the alternating sum

$$\sum_{(a_1,\ldots,a_n)} (-1)^{a_1+\cdots+a_n}$$

over all λ-parking functions $(a_1, \ldots, a_n)$ equals zero if and only if λ_n is even.

Problem 8. A *symmetric tree* T is a tree on $2n+1$ vertices labelled by $-n, -n+1, \ldots, -1, 0, 1, \ldots, n$ which is invariant under reversing the signs of all labels $i \mapsto -i$ (except 0). Construct a bijection between symmetric trees on $2n+1$ vertices and λ-parking functions for $\lambda = (2n+1, 2n-1, 2n-3, \ldots, 3, 1)$.

Problem 9. The *Shi arrangement* is the arrangement of the hyperplanes $H_{i,j,k}$ in $\mathbb{R}^n$ given by

$$H_{i,j,k} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = k\}, \text{ for } 1 \leq i < j \leq n,\ k \in \{0, 1\}$$

Find a bijection between regions of the Shi arrangement in $\mathbb{R}^n$ and parking functions of size n.

Problem 10. Recall that the *Stirling number* $S(n,k)$ *of the second kind* is the number of set partitions of $[n]$ with k non-empty blocks. The *Eulerian number* $A(n,k)$ is the number of permutations $w \in S_n$ with k descents. Give a combinatorial proof of the identity:

$$\sum_{i=0}^{n-1} (n-i)!\, S(n, n-i)\, t^i = \sum_{k=0}^{n-1} A(n,k)\, (t+1)^k$$

Problem 11. The Eulerian polynomials $A_n(x)$ are defined by

$$\sum_{i \geq 0} i^n\, x^i = \frac{x A_n(x)}{(1-x)^{n+1}}, \qquad \text{for } n \geq 1.$$

On the other hand,

$$\sum_{i \geq 0} (i)_n\, x^i = \frac{n!\, x^n}{(1-x)^{n+1}},$$

where $(i)_n := i(i-1)(i-2)\dots(i-n+1)$ is the n-th falling power of i. Use the identity

$$i^n = \sum_{k=0}^{n} S(n,k)\,(i)_k$$

to deduce an expression for the Eulerian polynomials $A_n(x)$ in terms of the Stirling numbers $S(n,k)$ of the second kind.

Problem 12. Find a combinatorial interpretation of the γ-vector of the $(n-1)$-dimensional permutohedron.