18.217 Problem Set 1

due Friday, October 9, 2020

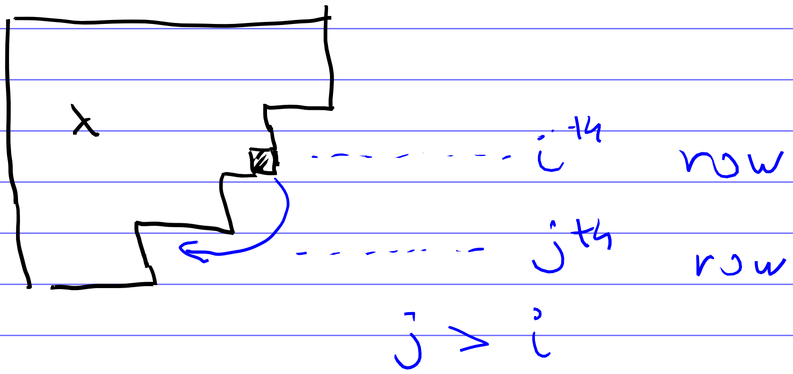
① The Kostka number $K_{\lambda\mu}$ is the # of SSYT of shape λ and weight μ (λ, μ partitions)

Prove that the following are equivalent:

(A) $K_{\lambda\mu} \neq 0$

(B) $\lambda \geq \mu$ in the dominance order

(C) The Young diagram μ can be obtaind from λ by a sequence of operations that move 1 box from i^{th} to j^{th} row, for $j > i$ (and produce a valid Young diagram)



(2) Prove that any two reduced decompositions of permutation $w \in S_n$ can be obtained from each other by a sequence of moves

$$A\, s_i s_{i+1} s_i\, B \longleftrightarrow A\, s_{i+1} s_i s_{i+1}\, B$$

$$A\, s_i s_j\, B \longleftrightarrow A\, s_j s_i\, A$$

$$|i-j| \geq 2$$

(without using the relation $s_i^2 = id$.)

(3) Let $b_\lambda = \sum_{\mu \leq \lambda} m_\mu$
(in dominance order)

The basis $\{b_\lambda\}$ of Λ is related to the basis $\{s_\lambda\}$ by some matrix $(A_{\lambda\mu})$

$$s_\lambda = \sum_\mu A_{\lambda\mu}\, b_\mu.$$

Is it always true that $A_{\lambda\mu} \geq 0$?

④ Prove that the h-version of Pieri rule

$$h_k \cdot S_\lambda = \sum_{\substack{\mu \text{ s.t.} \\ \mu/\lambda \text{ is} \\ \text{a horizontal} \\ k\text{-strip}}} S_\mu$$

holds for "classically defined" Schur polynomials

$$S_\lambda := \frac{a_{\lambda+\delta}}{a_\delta}.$$

⑤ Let $\partial_i = \frac{1}{x_i - x_{i+1}}(1 - s_i)$ be the divided difference operators acting on the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$.

Also let D_i be the Demazure operators

$$D_i = \frac{1}{1 - \frac{x_{i+1}}{x_i}}\left(1 - \frac{x_{i+1}}{x_i} s_i\right)$$

Check the relations:

(1)′ $\partial_i \partial_i = 0$

(2)′ $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_i$

(3)′ $\partial_i \partial_j = \partial_j \partial_i$, $|i-j| \geq 2$

and

(1)″ $D_i D_i = D_i$

(2)″ $D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$

(3)″ $D_i D_j = D_j D_i$, $|i-j| \geq 2$

⑥ For a reduced decomp.

$$w_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell} \quad \left(\ell = \binom{n}{2}\right)$$

of the longest permutation $w_0 \in S_n$, let

$$\partial_{w_0} := \partial_{i_1} \partial_{i_2} \cdots \partial_{i_\ell}$$

Prove that

$$\partial_{w_0}(f) = \frac{\sum_{w \in S_n} (-1)^{\ell(w)} w(f)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}.$$

⑦ Prove the dual Cauchy identity for (combinatorially defined) Schur functions

$$\prod_{i,j}(1+x_i y_j) = \sum_{\lambda} s_\lambda(x)\, s_{\lambda'}(y)$$

You can construct and prove a dual version of RSK correspondence.

⑧ Prove that

$$\sum_{\lambda \vdash n} f_\lambda = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k-1)!!$$

#SYT's of shape λ

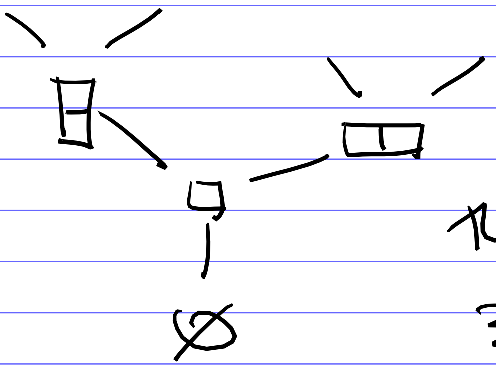
where $(2k-1)!! := 1 \cdot 3 \cdot 5 \cdots (2k-1)$

(9) Consider paths that go along the edges of the Hasse diagram of Young's lattice $\mathbb{Y}$ s.t.

- paths start and end at $\emptyset$
- paths have $2n$ steps
- the steps can be up & down steps in any order.

Prove that # such paths is

$(2n-1)!!$

Ex. 

For $n=2$ there are $3 \cdot 1 = 3$ such paths:

$\emptyset - \square - \square\square - \square - \emptyset$

$\emptyset - \square - \boxminus - \square - \emptyset$

$\emptyset - \square - \emptyset - \square - \emptyset$

(10) Prove that filling Fomin's growth diagrams by <u>columns</u> is equivalent to Schensted corresp. (as we explained in class)

In other words, filling 1 column of a growth diagram using local rules is exactly doing the same thing as 1 Schensted insertion step.

(11) Prove that

$$\Big(\sum_{\substack{\mu \text{ partitions} \\ \text{with all } \underline{\text{even}} \\ \text{parts}}} s_\mu\Big)(e_0+e_1+e_2+\dots) = \sum_{\lambda:\ \text{all partitions}} s_\lambda .$$

(12) Consider the following operators acting on the space $\mathbb{Z}[\mathbb{Y}]$ of formal linear combination of Young diagrams by adding / removing horizontal / vertical strips

For $k \geq 1$

$$U_k^{hor.} : \lambda \mapsto \sum_{\substack{\mu \supseteq \lambda \text{ s.t.} \\ \mu/\lambda \text{ horizontal} \\ k\text{-strip}}} \mu$$

$$D_k^{hor.} : \lambda \mapsto \sum_{\substack{\mu \subseteq \lambda \text{ s.t.} \\ \lambda/\mu \text{ horizontal} \\ k\text{-strip}}} \mu$$

$$U_k^{vert.} : \lambda \mapsto \sum_{\substack{\mu \supseteq \lambda \text{ s.t.} \\ \mu/\lambda \text{ vertical} \\ k\text{-strip}}} \mu$$

$$D_k^{vert.} : \lambda \mapsto \sum_{\substack{\mu \subseteq \lambda \text{ s.t.} \\ \lambda/\mu \text{ vertical} \\ k\text{-strip}}} \mu$$

Also $U_0^{hor.} = D_0^{hor.} = U_0^{vert} = D_0^{vert} =$
$= Id.$

and $U_k^{hor.} = D_k^{hor.} = U_k^{vert} = D_k^{vet} = 0$
for $k < 0$.

Prove the following identities:

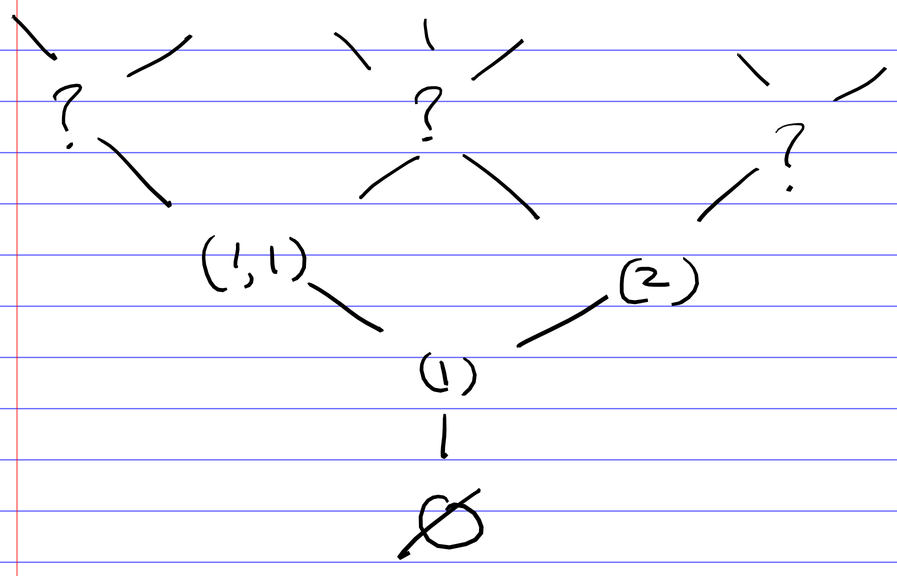
(1) The operators $U_k^{hor.}$, U_ℓ^{hor}, U_m^{vert}, U_n^{vert} commute with each other for all k, ℓ, m, n

(and the same holds for the down operators)

(2) $[D_k^{hor.}, U_\ell^{hor.}] = D_{k-1}^{hor.} U_{\ell-1}^{hor.}$

(3) $[D_k^{hor.}, U_\ell^{vert}] = U_{\ell-1}^{vert.} D_{k-1}^{hor.}$

(13) Give a <u>non-recursive</u> construction of the Fibonacci lattice $\mathbb{F}$ as a certain poset on <u>compositions</u> $(c_1, \ldots, c_\ell)$ with all parts $c_i \in \{1, 2\}$

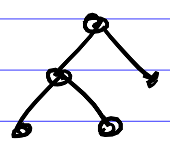


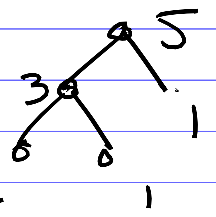
(14) Prove the "<u>hook length formula</u>" for trees

T - a rooted tree on n vertices

$h(v)$ = # descendents of vertex v (including v)

Then # linear extensions of T $= \dfrac{n!}{\prod_{v \text{ vertex of } T} h(v)}$

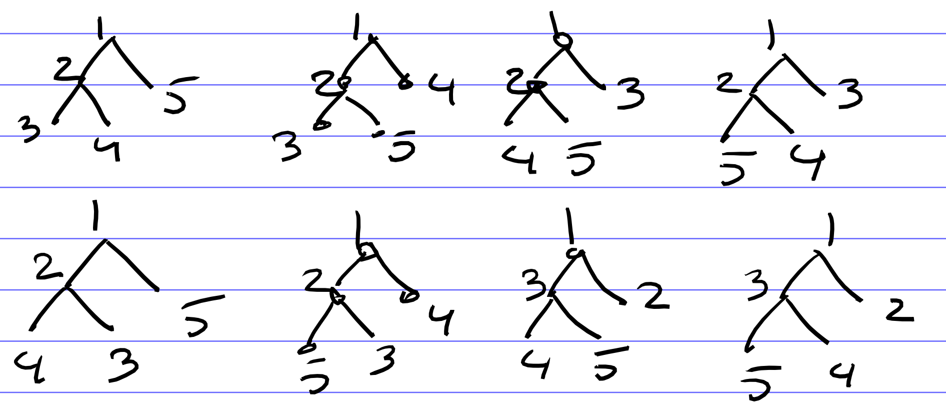
<u>Example</u> $T =$  $n=5$

"hook lengths" in T 

linear ext. of T

$$= \frac{5!}{5 \cdot 3 \cdot 1 \cdot 1 \cdot 1} = 4 \cdot 2 = 8$$

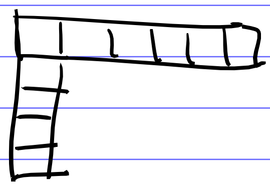
lin. extensions of T:



(15) In class, we constructed 2 bijections φ_λ^{RSK} & φ_λ^{HG} (the generalized RSK correspondence and the Hillman–Grassl corresp.) between "matrices" of shape λ and reverse plane partitions of shape λ.

(1) Give an explicit formula for $\varphi^{HG}_{\boxplus}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for any non-negative integer 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(2) Prove or disprove the following claims:

(A) $\varphi_\lambda^{RSK} = \varphi_\lambda^{HG}$ if λ is a hook $\lambda =$ 

(B) $\varphi_\lambda^{RSK} = \varphi_\lambda^{HG}$ for all Young diagrams λ.

16) Prove combinatorially that Stanley's hook-content formula for $s_\lambda(\underbrace{1,\dots,1}_{n})$ is equivalent to Weyl's dimension formula (of type A).

In other words, show that for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ we have

$$\prod_{a \in \lambda} \frac{n + c_a}{h_a} \overset{?}{=} \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

where h_a is the <u>hook lengths</u> and $c_a = j - i$ is the <u>content</u> of box (i,j) in λ.

17 (A) Prove combinatorially that the number of semi-standard Young tableaux of the staircase shape $\lambda = (n-1, n-2, \dots, 1, 0)$ with all entries $\in \{1, 2, \dots, n\}$ equals $2^{\binom{n}{2}}$.

(B) More generally, find a piecewise linear volume preserving Ehrhart polynomial preserving bijection between the Gelfand-Tsetlin polytope and the hypercube:

$$GT(n-1, n-2, \dots, 0) \xrightarrow[\sim]{?} [0,1]^{\binom{n}{2}}.$$

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18 For $\lambda = (\lambda_1 \geq \dots \geq \lambda_n) \in \mathbb{Z}^n$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, let $GT(\lambda, \mu)$ be the polytope of real-valued Gelfand-Tsetlin patterns with top row λ and row sums $\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \dots$ (from the bottom). Prove or disprove that $GT(\lambda, \mu)$ is always a lattice polytope.