18.217 Lecture 9 9/23/2020

<u>last time:</u> • $S_n \overset{\text{Schensted}}{\longleftrightarrow} \{(P,Q) \mid \text{SYT's of same shape}\}$

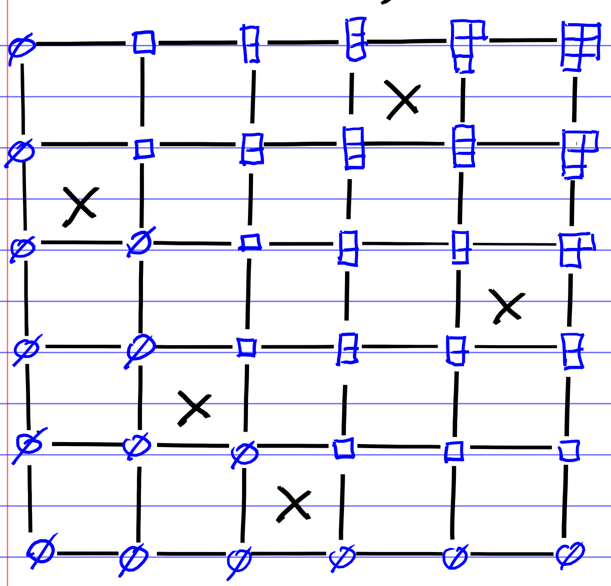
• $DU - UD = I$

this story is related to..

Fomin's growth diagrams

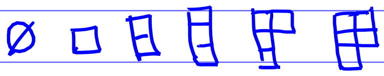
<u>Ex</u> $n=5$

Q

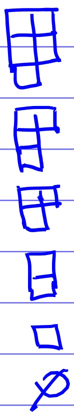


P

<u>a growth diagram</u>

top row: 

$$Q = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array}$$

last column: 

$$\rightsquigarrow P = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array}$$

• "X"'s form $n \times n$ permutation matrix (or rook placement)

• In each node, we have a Young diagram

• $\emptyset$ diagrams in 1^{st} column & bottom row

• For adjacent nodes, Young diagrams differ by at most 1 box

$\lambda - \mu$ $\quad$ $\begin{array}{c}\mu \\ | \\ \lambda\end{array}$

$\lambda = \mu$ or $\mu/\lambda = \square$

• "local rules" for each square:

$$\begin{array}{ccc} \lambda & - & \gamma \\ | & & | \\ \mu & - & \nu \end{array}$$

Local Rules:

$$\begin{tikzcd} \lambda \arrow[r, dash] \arrow[d, dash] & \gamma \arrow[d, dash] \\ \mu \arrow[r, dash] & \nu \end{tikzcd}$$

$\lambda = \mu$ or $\lambda \gtrdot \mu$

$\mu = \nu$ or $\mu \lessdot \nu$

$\lambda = \gamma$ or $\lambda \lessdot \gamma$

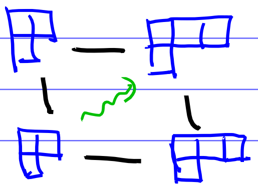
$\nu = \gamma$ or $\nu \lessdot \gamma$

$\lambda \gtrdot \mu$ means

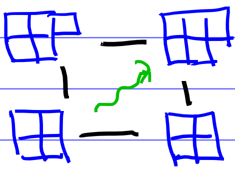
$\lambda \supset \mu$ & $\lambda/\mu = \square$

Young diagrams λ, μ, ν determine γ, as follows:

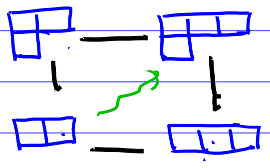
I. $\lambda = \mu \neq \nu \Rightarrow \gamma = \nu$



II. $\lambda \neq \mu = \nu \Rightarrow \gamma = \lambda$

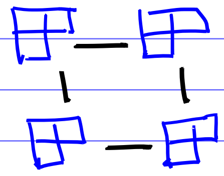


III. $\lambda \neq \mu,\ \mu \neq \nu,\ \lambda \neq \nu \Rightarrow \gamma = \lambda \cup \nu$



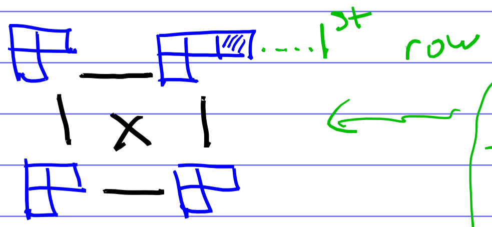
IV $\lambda = \mu = \nu$ & $\nexists$ "X" in the square

$\Rightarrow \quad \gamma = \lambda$



V $\lambda = \mu = \nu$ & $\exists$ "X" in the square

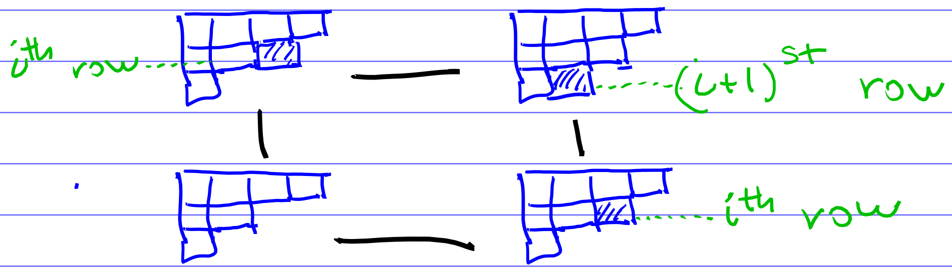
$\Rightarrow \quad \gamma = \lambda \cup$ box in 1^{st} row



This corresponds to "exceptional case" from the last lecture

VI $\lambda = \nu \gtrdot \mu$, λ/μ has one box in i^{th} row

$\Rightarrow \quad \gamma = \lambda \cup$ box in $(i+1)^{st}$ row

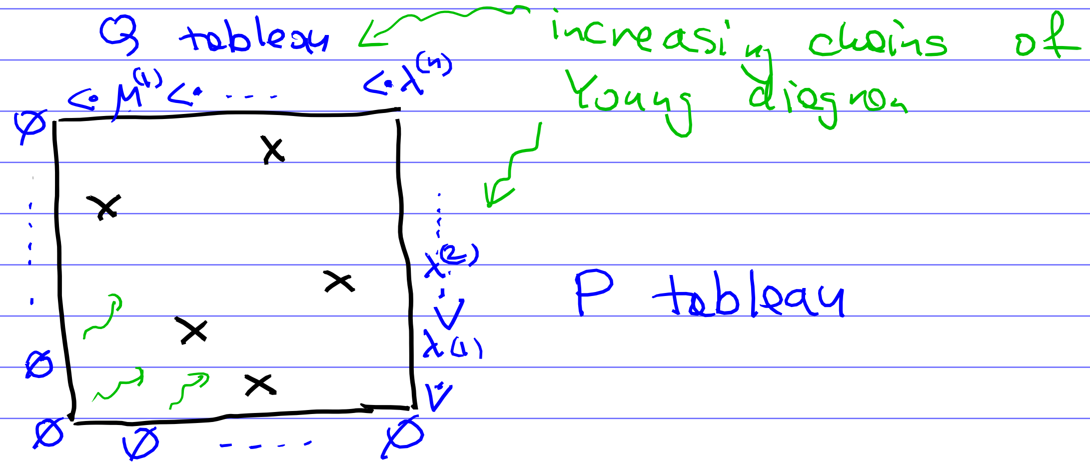


How does this work?

From a permutation to SYT's

- Start from a permutation $w \in S_n \rightsquigarrow$ rook placement & empty shapes $\emptyset$ in 1^{st} column & bottom row

- Use the local rules to fill all nodes with Young diagrams.
- Then in the top row & last column we get 2 sequences of increasing Young diagrams that give 2 SYT's : P & Q tableaux.



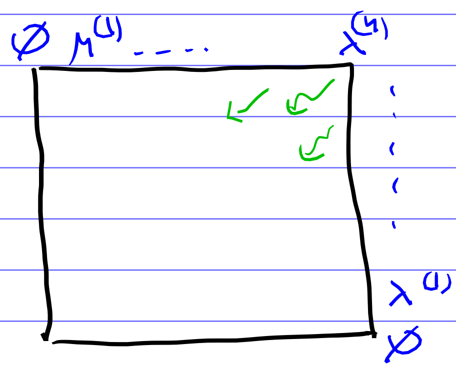
$S_n \ni w \longmapsto (P, Q)$ 2 SYT's of same shape with n boxes.

Claim. This map is exactly Schensted correspondence.

We can use the local rules "backward".

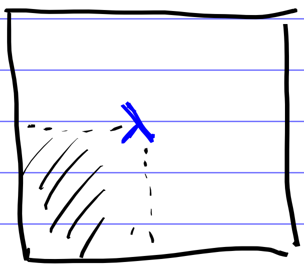
From (P, Q) to perm. w

- Start with (P, Q) & fill the top row & last column
- Then use the local rules to figure out all other Young diagrams & "X"'s.



In order to see that this works, we need to check a few things....

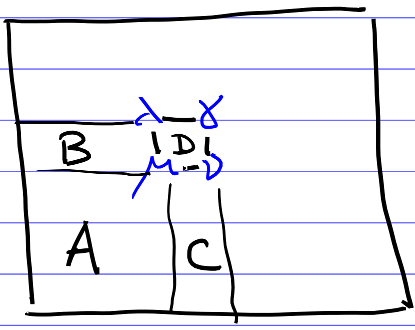
Lemma 1. In a growth diagram, for any node,



$|\lambda| = \#$ X's in the shaded rectangle

Proof Easy to check by induction using the local rules... $\square$

Corollary For each square in a growth diagram,



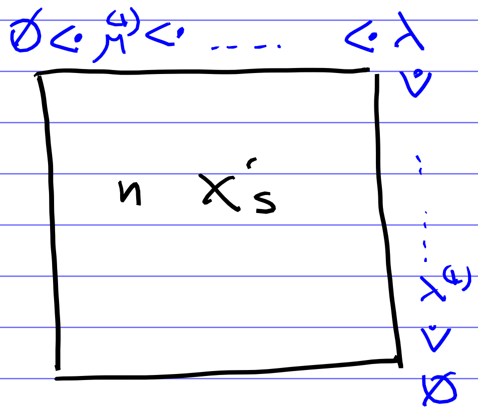
If there are no X's in B, then $\lambda = \mu$.

If there is one "X" in B, then $\lambda = \mu \cup$ one box.

Similarly, C is empty $\Rightarrow \mu = \nu$

C has one X $\Rightarrow \nu = \mu \cup$ one box

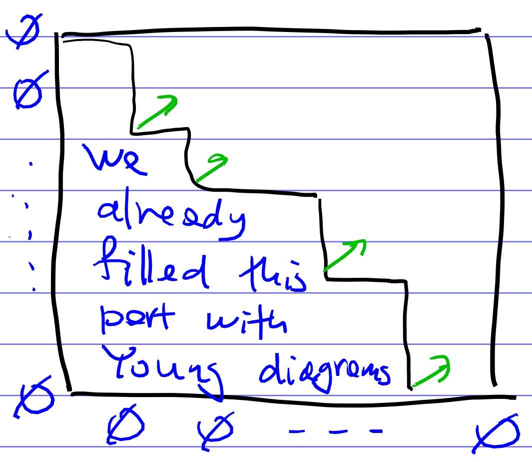
Corollary. In top row & last column, we get strictly increasing sequences of Young diagrams. In top right corner we have a Young diagram w n boxes.



$|\lambda| = n.$

So we actually get a pair (P, Q) of SYT's with n boxes.

There are many ways to "grow" a growth diagram (starting from a permutation)



a partially filled growth diagram

If we are somewhere in the middle of filling a growth diagram, there are several possible next steps.

<u>Lemma 2.</u> The final result (e.g. the completely filled growth diagram) <u>does not depend</u> on a way how we were filling the diagram.

<u>Equivalently</u>, for any permutation $w \in S_n$ (rook placement) there exists <u>exactly one</u> valid growth diagram.

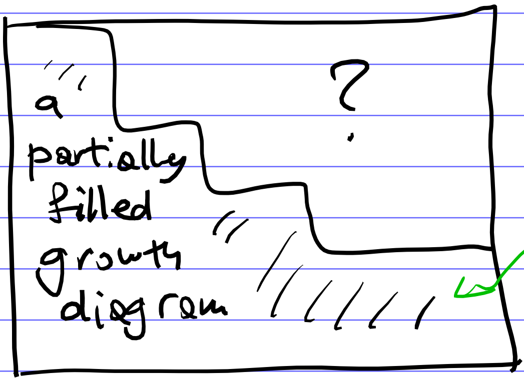
<u>In particular</u>, the pair (P, Q) of SYT's is defined <u>only</u> by <u>permutation</u> $w \in S_n$, and does not depend on our particular choice of growing the growth diagram.

A property of this kind is called "<u>confluence</u>"

when $\exists$ several ways to calculate something, but the result is the same

<u>Remark.</u> Another well-known example of a confluent process is the <u>chip firing game</u> (aka <u>Abelian Sandpile Model</u>)

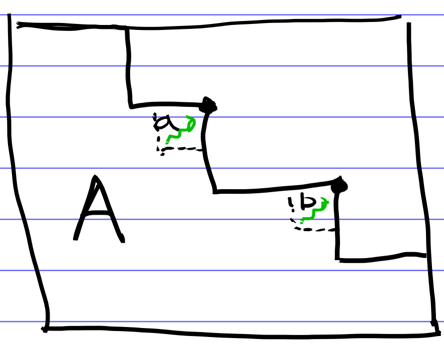
Proof "Diamond Lemma"-type argument.



Assume by induction that the filling of this area does not depend on a choice of growing

(Induction on the size of the area)

By induction, we know that the "independence" holds for all areas with $\leq m-1$ squares. We want to show the independence for an area with m boxes.



an area A with m squares

Assume that there 2 or more possible "last steps" that can lead to a filling of A

Let $A' = A \setminus$ square a ← $n-1$ square

$A'' = A \setminus$ square b

$A''' = A \setminus$ squares a & b ← $n-2$ square

$$\begin{tikzcd}
 & A & \\
A' \arrow[ur] & & A'' \arrow[ul] \\
 & A''' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

this is why it is called the diamond lemma

We know by induction that any way to fill each of the areas A' or A'' gives the same result.

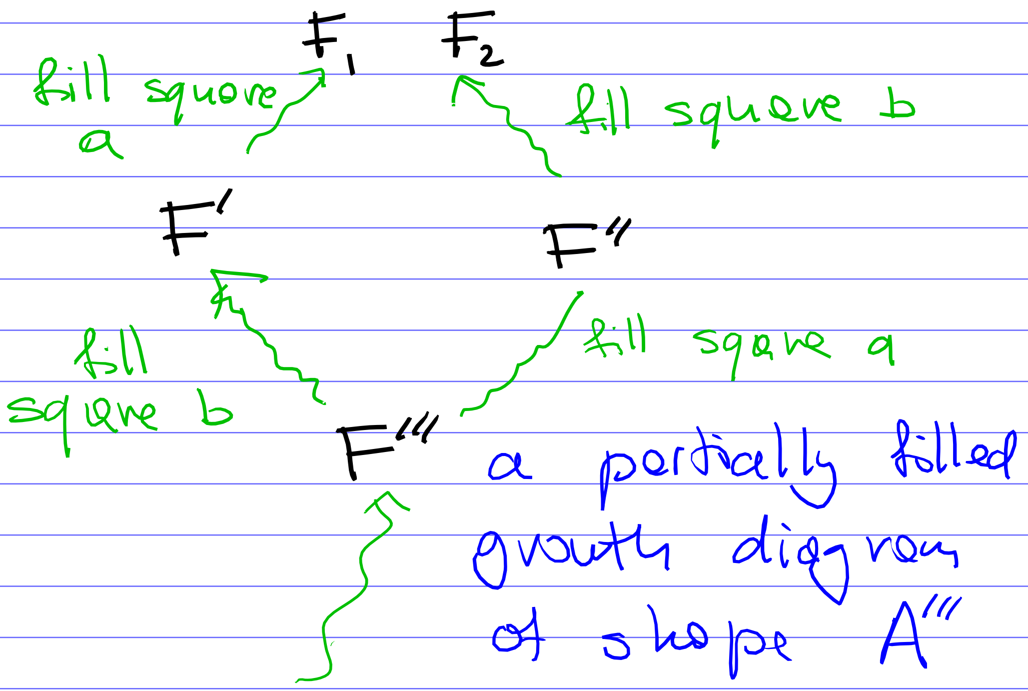
We want to show that the same holds for area A.

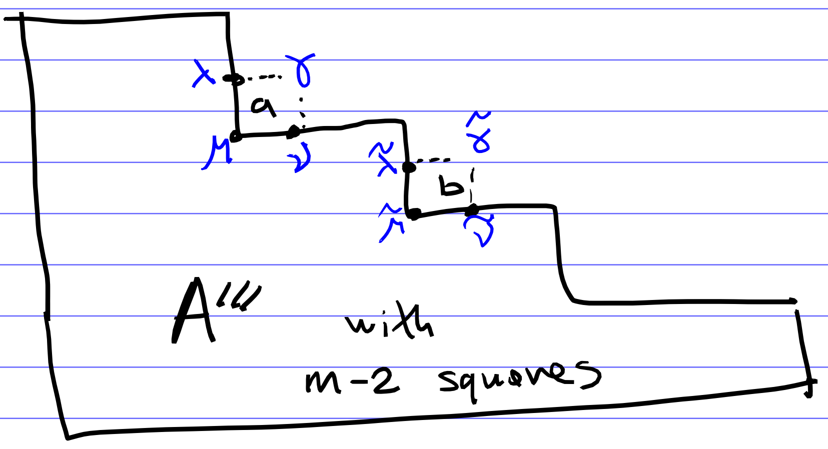
Let F_1 be a filling of A that "arrived" to A through of filling of F' of A'

Let F_2 be a filling of A that "arrived" to A through a filling F'' of A''.

We want to show that $F_1 = F_2$.

Since we already prove the independence for A' & A'', we may assume that both fillings F' & F'' arrived to A' & A'', resp. through a filling F''' of A'''.





Since squares a & b are <u>not adjacent</u> (they can share a corner, but they cannot share a side), filling the square a (i.e. figuring out γ from λ, μ, ν) does not affect filling the square b, and vice versa.

So we will arrive to the same filling of A if we first fill box a and then box, or if we first fill box b and then a.

So $F_1 = F_2$,

as needed. $\square$

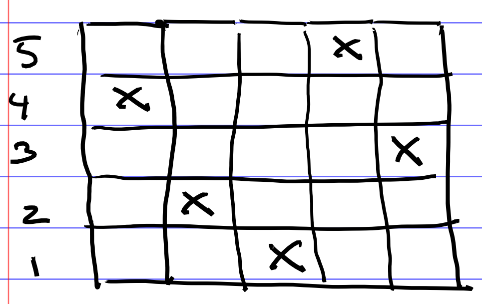
This implies that growth diagrams give a well defined bijection between S_n and $\{(P, Q) \mid$ SYT's of some shape $\lambda \vdash n\}$

How does it related to Schensted correspondence (constructed via Insertion Algorithm)?

<u>Claim</u> Schensted insertion process is one particular way to grow a growth diagram.

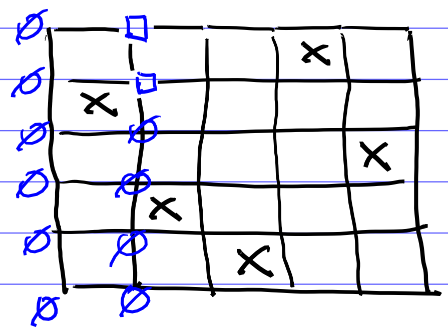
Namely, it is exactly the way to grow a growth diagram by first filling the first column, then filling the second column, etc.

Example (Same as in the beginning of lecture)



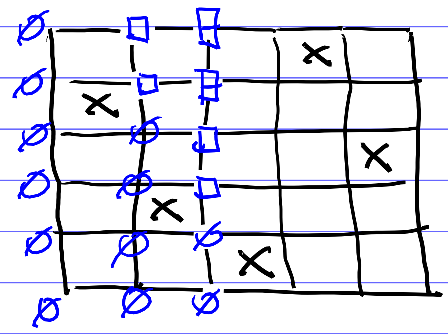
$w = 4\ 2\ 1\ 5\ 3$

1st insertion step



$\emptyset \leftarrow 4 \rightsquigarrow \boxed{4}$

2nd insertion



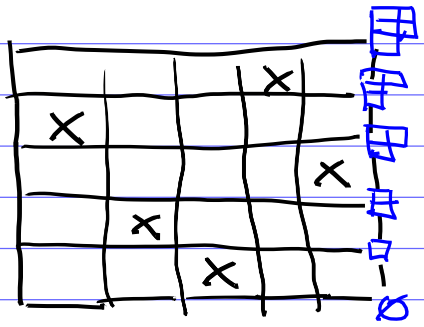
this column corresponds to $\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \end{array}$

This is exactly Schensted insertion

$\boxed{4} \leftarrow 2 \rightsquigarrow \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \end{array}$

etc.

The last column



gives $P = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$

$= ((((\emptyset \leftarrow 4) \leftarrow 2) \leftarrow 1) \leftarrow 5) \leftarrow 3$

Exercise Check that Schensted Insertion Algorithm is equivalent to filling 1 column of a growth diagr.

Benefits of Growth diagrams

- Insertion Steps are broken into more elementary & simple "local rules"
  ($\sim$ up & down operators and the relation $DU - UD = I$)

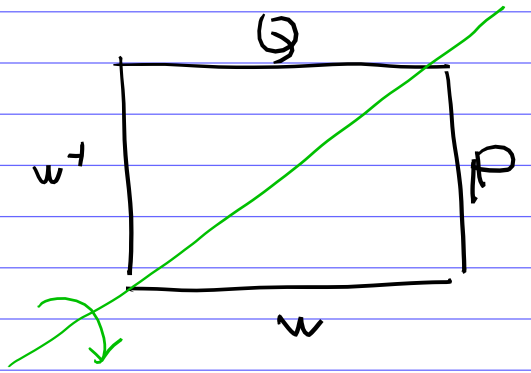
- The symmetry

$$w \longmapsto (P, Q) \iff w^{-1} \longmapsto (Q, P)$$

is a mystery from the point of view of Schensted insertion
(P & Q play different roles in the Insertion Alg.)

But the symmetry is clear in growth diagrams.

The local rules are symmetric w.r.t. the reflection



$w \Longleftrightarrow w^{-1}$

$P \Longleftrightarrow Q$

By the diamond lemma, it does not matter if we fill a growth diagram by columns or by rows.

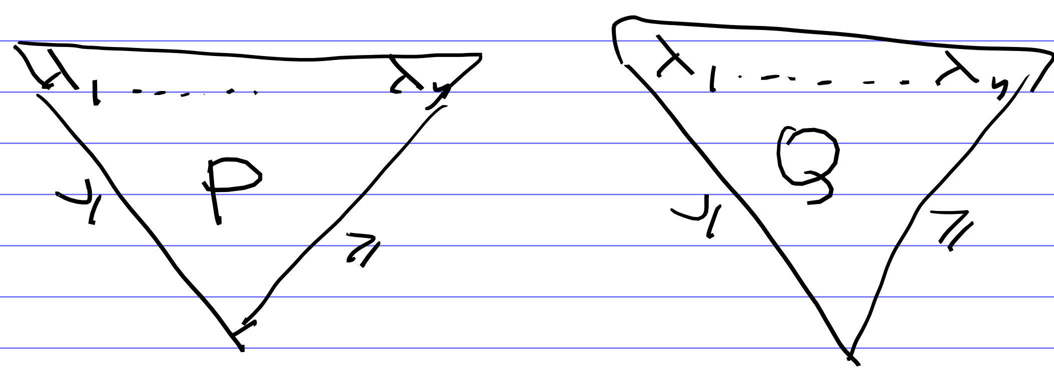
How to generalize this to RSK
& <u>semi</u>standard Young tabl.?

<u>RSK:</u>

A $n \times n$ matrix filled with nonnegative integers

P & Q tableaux are SSYT's with entries $\in [n]$

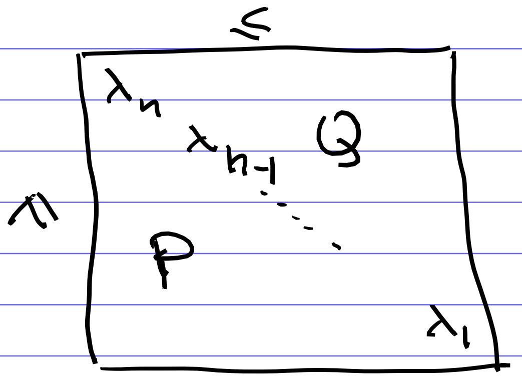
$\rightsquigarrow$ Gelfand-Tsetlin patterns



with the same top row

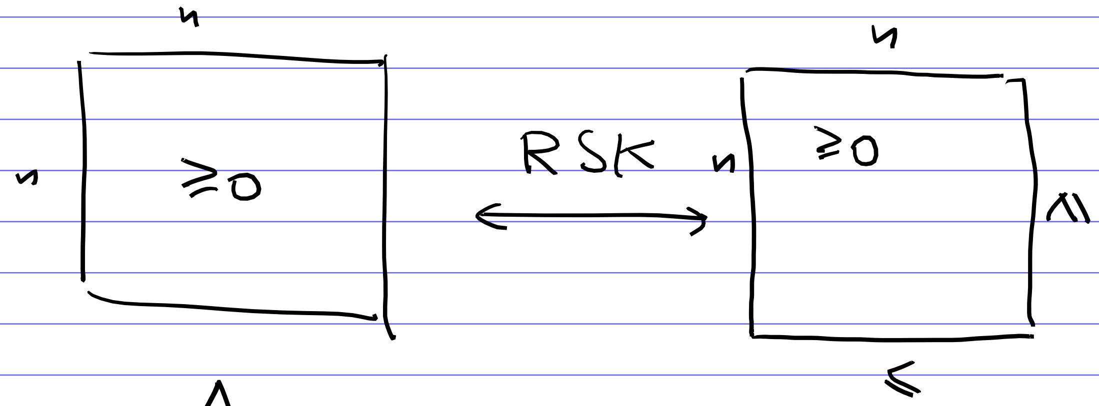
$(\lambda_1, \dots, \lambda_n)$ = shape of P-tableau
= shape of Q-tableau

Let's glue these two triangles into 1 square along its λ-edge



we get a <u>reverse plane partition</u> (or RPP) of shape $n \times n$

So RSK can be viewed as a bijection.



A

nonnegative integer $n \times n$ matrices

RPP's

nonnegative integer $n \times n$ matrices with <u>weakly increasing</u> entries in all rows & columns

<u>Example</u> $n=2$ (we did this in lecture 6)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{\text{RSK}} \begin{pmatrix} \min(b,c) & a+b \\ a+c & a+d+\max(b,c) \end{pmatrix}$$

matrix RPP

We'll explain how to construct this map for any n using a generaliz. of growth diagrams...