Young lattice $\mathcal{Y}$ = the poset of Young diagrams ordered by inclusion, $\mathcal{Z}[\mathcal{Y}]$ = the space of formal linear combinations of Young diagrams.

Some elements of $\mathcal{Z}[\mathcal{Y}]$: $\square$, $27\cdot \varnothing + 3 \Box - \emptyset$

Up & Down operators acting on $\mathcal{Z}[\mathcal{Y}]$

$U : \lambda \mapsto \sum_{\mu : \mu \triangleright \lambda} M_{\mu \lambda}$

$D : \lambda \mapsto \sum_{\lambda : \mu \triangleright \lambda} M_{\lambda \mu}$

Ex $U : \square \mapsto \square + \Box$

$D : \Box \mapsto \square + \emptyset$

$(D \cdot U) : \square \mapsto 2 \square + \emptyset$

$(U \cdot D) : \square \mapsto \square + \emptyset$

$(DU-UD) : \square \mapsto \square$

Lemma $[(DU-UD) = I]$

Commutator $[D, U] = DU-UD$
Proof.

The coeff. of \( \lambda \) in \([D \cup U] (\lambda)\) is equal to the number of such \( \mu \)'s minus the number of such \( \hat{\mu} \)'s.

2 cases:

(I) \( \lambda \neq \emptyset \)

We first add a box to \( \lambda \) and then remove a different box.

We can do these operations in a different order (first remove & then add) to get the same result.

So the coeff. of \( \lambda \) in \([D \cup U] (\lambda)\) is 0 (if \( \lambda \neq \emptyset \)).

(II) \( \lambda = \emptyset \).

- \( \frac{k}{2} \) ways to add a box to \( \lambda \) & then remove the same box

- \( \frac{k+1}{2} \) ways to remove a box from \( \lambda \) & then add the same box

In \( Y \):
Why?

So the coeff. of $\lambda$ in

$$\text{LD}_k \cup \text{U}_k (\lambda) = (k+1) - k = 1$$

we obtain

$$\text{LD}_k \cup \text{U}_k (\lambda) = \lambda.$$

The properties of $\mathcal{Y}$ “responsible” for the identity $\text{LD}_k \cup \text{U}_k = I$:

- $\lambda \neq \emptyset$

- $\lambda = \emptyset$

Def. A differential poset is a ranked poset with a unique minimal elt. $\emptyset$ s.t.

$$\text{LD}_k \cup \text{U}_k = I$$

holds for the up & down oper. on $\mathbb{Z}[P]$.

- $\forall \lambda \neq \emptyset$ on the same level

So $\mathcal{Y}$ is a “prototypical” differential poset.
Why “differential”? 

Because the operators

\[ x : f(x) \rightarrow x f(x) \quad \& \quad \frac{d}{dx} : f(x) \rightarrow f'(x) \text{ acting on } \mathbb{Z}[x] \]

satisfy the same relation \[ \left[ \frac{d}{dx}, \; x \right] = 1. \]

Another differential poset

**Fibonacci lattice:**

\[ \begin{align*}
F_0 &= 0 \\
F_1 &= 1 \\
F_2 &= 1 \\
F_3 &= 2 \\
F_4 &= 3 \\
F_5 &= 5 \\
F_6 &= 8 \\
F_7 &= 13 \\
F_8 &= 21 \\
F_9 &= 34 \\
F_{10} &= 55
\end{align*} \]

- Start with \( F_0 = 0 \) to level \( F_5 \).

Then recursively construct level \( k \) by “reflecting” the covering relations between levels \( k-1 \) & \( k-2 \) and then adding elements covering \( \hat{0} \)‘s on level \( k-1 \).

Remark: \( F \neq \mathbb{F} \) because the Fibonacci numbers \( F_{n+1} \neq \) the partition numbers \( p(n) \), for \( n > 5 \):

\[
\begin{array}{cccccccc}
\text{F(n)} & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 \\
\text{p(n)} & 1 & 1 & 2 & 3 & 5 & 8 & 11 & 15 & 22
\end{array}
\]
Theorem. For any differential poset (e.g., for $\mathcal{Y}$ or $\mathcal{F}$), we have

$$D^n U^n \hat{\sigma} = n! \hat{\sigma}$$

This is the min. elt. off, not 0.

Equiv. $\sum \text{# saturated chains}^2 = n!$

For $\mathcal{Y}$: $\sum \lambda^2 = n!$

Proof. This follows from

1. $\mathcal{D} U \mathcal{Y} = \mathcal{I}$
2. $\mathcal{D} \hat{\sigma} = 0$

Ex. $n=2$

$$\mathcal{D} \mathcal{D} U U(\hat{0}) = \mathcal{D} U \mathcal{D} U(\hat{0}) + DU(\hat{0})$$

$D$ either "jump" over $U$ to the right, or "annihilates" with $U$.

$$= (DUU D + DU + UD + I)(\hat{0})$$

$$= (UD + I + I)(\hat{0})$$

$$= 2 \hat{0}.$$
Each D moves to the left by "jumping over" U's until it "annihilates" with some U. 
(If it cannot jump over all U's, because we would get 
$D U \ldots U \overset{\hat{0}}{\longrightarrow} U \ldots U D \overset{\hat{0}}{\longrightarrow} 0$
There are $n!$ ways to match $n$ D's with $n$ U's into $n$ pairs of D & U which annihilate each other.
Ex. D D D D U U U U \overset{\hat{0}}{\longrightarrow}

Another argument: The operators 
$X: f \rightarrow xf \quad \& \quad \frac{d}{dx}$
satisfy the same relations

1. $\left[\frac{d}{dx}, X\right] = 1$
2. $\frac{d}{dx}(1) = 0$

So the calculation $D^n U^n(\hat{0}) = ?$ is equivalent to
$(\frac{d}{dx})^n x^n(1) = (\frac{d}{dx})^n (x^n) = n! \cdot 1$
$\Rightarrow ? = n! \square$
How about more general paths in $Y$ (or in any other $\text{alt. poset}$)?

Consider any word $w$ with $n$ U's & $n$ D's.

$$Ex. \text{ } D D D U U \text{ } D U U$$

$DDD UU D U U (\emptyset) = ? \emptyset$

$? = \# \text{oscillating tableaux}$

**Def.** An oscillating tableau is any path going along the edges of Hesse diagram of $Y$ with fixed initial & final Young diagram & prescribed sequence of Up & Down steps given by word $w$ in USDD (read backward).

In our case, the initial & final Young diagrams are $\emptyset$ and $w$ consists of $n$ U's and $n$ D's.

**Example.** An oscillating tableau

$$\emptyset \xrightarrow{U} \square \xrightarrow{U} \square \xrightarrow{D} \square \xrightarrow{D} \square \xrightarrow{U} \square \xrightarrow{D} \square \xrightarrow{D} \square$$

$w = D D D U U D U U.$

The same argument as before shows that

$? = \# \text{ ways to match all D's with all U's s.t. each D is matched with a U to the right of D}$

**Ex.** $w = D D D, U, U, D, U, U$

$? = \# \text{such matchings}$
Such matchings correspond to rook placements.

Example:

Ex. 

If $D_i$ is matched with $U_j$, then we place a rook in column $i$ & row $j$ (labelled from the bottom).

The rooks should not attack each other.

Theorem: Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, $n > \omega_1 > \omega_2 > \ldots > \omega_n > 0$.

Young diagram that fits inside an $n \times n$ square.

Then the number of ways to place $n$ non-attacking rooks into the shape $\omega$ equals

$$\text{# rook placements} = \omega_n (\omega_{n-1} - 1)(\omega_{n-2} - 2) \cdots (\omega_1 - n + 1)$$

Or 0 if some term is ≤ 0.

Example. $w = DDDUUUDUU$

\[
\begin{array}{c|cccc|c}
\hline
 & & & & \uparrow \\
\hline
 & & & & 4 \\
\hline
 & & & & 3 \\
\hline
 & & & & 3 \\
\hline
\end{array}
\]

$\omega = (4, 4, 3, 3)$

$\# \text{ rook placements} = 3 \cdot 2 \cdot 2 \cdot 1 = 12$

$\Rightarrow \# \text{ oscillatory tableaux s.t.}$

$$\rho_1 \cdots \rho_k$$

$= 12$
If \( W = D U \cdots U \),

\[ x = n \times n \text{ square} \]

\# placements of 

\[ \text{in } n \times n \text{ square} \]

\[ \geq n! \]

Now we know that \( n! \text{ polylines in } \{0,1,2,3,4\} \) with particular sequence of up & down steps

\[ \text{from } 0 \text{ to } n \]

\( \Theta \). Can we construct a bijection between oscillating tableaux and rook placements?

For any \( \lambda \in \mathcal{P} \), we need to fix the following correspondence between all \( k \) \( \nu \)'s covering \( \lambda \) (except 1) and \( k \) \( \nu \)'s covered by \( \lambda \).

\( \begin{array}{c}
\text{\usepackage[utf8]{inputenc}}
\begin{array}{c}
\text{One special unwatched}
\end{array}
\end{array} \)

- Label the inner/outer corners of \( \lambda \) by \( 1, 2, \ldots, k \)
- Match \( i \)'th inner corner with \( i \)'th inner corner
- Leave the rightmost outer corner unwatched.
Given a word $w$ in $S_n$ and a tableau $T$, we have a bijection $\phi_w$ from $\mathfrak{S}_n$ to $\mathfrak{S}_n$ of shape $\lambda$ given by the word $w$.

Want to construct a rook placement $R = \psi_w(T)$ in $\lambda$.

We will construct $\psi_w$ by induction.

**Base case:** If $w$ is an empty word, then $\psi_w$ is empty.

**Inductive step:**

If any element of $\mathfrak{S}_n$ is $\mathfrak{S}_n$, let $w = w_1 \cdots w_k$.

For corresponding shapes, we have

In the oscillating tableau $T$, we have

In all cases, we have a correspondence between possible $\lambda$ and $\tilde{\lambda}$'s except for the box when $\lambda = 2$.

If $\lambda$ is obtained from $\lambda$ by adding a box in the first row, (special outer corner box)
Then \( y_w(T) = y_{\tilde{w}}(T) \)

in non-exceptional cases

In the exceptional case

* Place the rook in box 4 of \( \alpha \)

* \( \tilde{w} \) obtained from \( w \) by removing his fragment "DU"

and \( \tilde{T} \) obtained from \( T \) by removing his up \& down steps.

* Place all other rooks as in \( y_{\tilde{w}}(\tilde{T}) \)
Example:

\[ T = (\varnothing - \square - \varnothing - \square - \varnothing - \square - \varnothing - \varnothing) \]

\[ w = \text{DDDUDUDUU} \]

\[ \tilde{T} = (\varnothing - \square - \varnothing - \square - \varnothing - \square - \varnothing - \varnothing) \]

\[ \tilde{T} = (\varnothing - \square - \varnothing - \square - \varnothing - \square - \varnothing - \varnothing) \]

\[ \text{We get } \varphi_w(T) = \begin{array}{|c|c|c|}
\hline
x & x & x \\
\hline
x & x & x \\
\hline
\end{array} \]
Theorem. We've got a bijection \( \text{osc. tableaux} \rightarrow \text{placements} \).

It is clear from the construction that it is symmetric.

\[ T \mapsto \text{osc. tableau} \]
\[ \text{rev}(T) \text{ is reversal} \]
\[ (\text{reverse the path}) \]
\[ \text{if } T \mapsto R \]
\[ \text{rev}(T) \mapsto \text{transpose}(R) \]

Special case \( w = D_\infty U_\infty U_\infty \)

\( \mathcal{R} = n \times n \text{ square} \)

In this case, this constr. gives Schensted corresp.

\[ (P, Q) \xrightarrow{\text{Schensted}} \text{permutation } w \]

pair of SYT's viewed as an oscillating tableau

Corollary. Schensted corresp. is symmetric:

if \( (P, Q) \mapsto w \)

then \( (Q, P) \mapsto w^t \)