18.217 Lecture 6 9/16/2020

last time: • To show that $s_\lambda^{class} = s_\lambda^{comb}$, it is "enough to check" Pieri rule (product w/ e_k).

• $\omega(s_\lambda) = s_{\lambda'}$ (Q: Is there a more direct proof?)

Lemma. Let $\{a_\lambda\}_{\lambda \text{ partition}}$ be a linear basis of Λ, $\deg(a_\lambda) = |\lambda|$. Let $\{\tilde{a}_\lambda\}$ be some sym. functions. Suppose that

(1) $a_\varnothing = \tilde{a}_\varnothing = 1$.

(2) $\forall\ \lambda, k$

$$e_k \cdot a_\lambda = \sum_\mu c_{\lambda\mu k}\, a_\mu$$

some coeffs.

$$e_k \cdot \tilde{a}_\lambda = \sum_\mu c_{\lambda\mu k}\, \tilde{a}_\mu$$

Then $a_\lambda = \tilde{a}_\lambda$ $\forall \lambda$.

Proof $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell} \cdot 1 =$

$$= e_{\lambda_1} \cdot (e_{\lambda_2} \cdot (\dots \cdot (e_{\lambda_{\ell-1}} \cdot (e_{\lambda_\ell} \cdot a_\varnothing)) \dots))$$

$$= \sum_\mu d_{\lambda\mu}\, a_\mu$$

some coeffs. that can be expressed in terms of $c_{\lambda\mu k}$.

Also $e_{\lambda_1} \cdot e_{\lambda_2} \cdots e_{\lambda_\ell} \cdot \tilde{a}_\varnothing = \sum_\mu d_{\lambda\mu}\, \tilde{a}_\mu$.

same coeffs.

So $\{e_\lambda\} = (d_{\lambda\mu}) \{a_\mu\}$

invertible matrix

and $\{e_\lambda\} = (d_{\lambda\mu}) \{\tilde{a}_\mu\}$.

$\Rightarrow a_\lambda = \tilde{a}_\lambda$ $\forall \lambda$. $\square$

How to prove that $\omega(s_\lambda) = s_{\lambda'}$?

1st approach: Compare Pieri rules for $e_k \cdot s_\lambda$ and $h_k \cdot s_\lambda$ and use the above lemma...

2nd approach: Clearly there is a linear map $\Lambda \to \Lambda$ s.t. $s_\lambda \mapsto s_{\lambda'}$ $\forall \lambda$.

Need to check that it is a homomorphism on the ring Λ. ($\Rightarrow$ this map $= \omega$, because $e_k = s_{(1,\dots,1)}$ and $h_k = s_{(k)}$.)

$$\boxed{s_\lambda \cdot s_\mu = \sum c_{\lambda\mu}^{\nu} \, s_\nu}$$

these are the Littelwood-Richardson coefficients given by the LR rule.

$$s_{\lambda'} \cdot s_{\mu'} = \sum c_{\lambda\mu}^{\nu} \, s_{\nu'}$$

Need to show $\boxed{c_{\lambda\mu}^{\nu} = c_{\lambda'\mu'}^{\nu'}}$

Problems
- Prove the Littlewood-Richardson rule for $c_{\lambda\mu}^{\nu}$.
- It is not obvious from the classical L-R rule, that $c_{\lambda\mu}^{\nu} = c_{\lambda'\mu'}^{\nu'}$.
- Need to reformulate it, so that this symmetry becomes clear, e.g. using Knutson-Tao puzzles.

we'll do this later in this course. But all this takes more efforts...

3rd approach: Use Cauchy identities...

Cauchy Identities

Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two infinite sets of variables.

Consider two infinite products:

$$\prod_{\substack{i=1,2,\ldots\\ j=1,2,\ldots}} \frac{1}{1-x_i y_j} \quad \text{and} \quad \prod_{\substack{i=1,2,\ldots\\ j=1,2,\ldots}} (1+x_i y_j)$$

<u>Lemma.</u> • $\displaystyle\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda \text{ partition}}' h_\lambda(x) \cdot m_\lambda(y)$

• $\displaystyle\prod_{i,j} (1+x_i y_j) = \sum_{\lambda \text{ partition}}' e_\lambda(x) \cdot m_\lambda(y)$

(Here $h_\lambda(x) := h_\lambda(x_1, x_2, \ldots)$,
$m_\lambda(y) := m_\lambda(y_1, y_2, \ldots)$ etc.)

<u>Proof.</u> $\displaystyle\prod_{i,j} \frac{1}{1-x_i y_j} = \prod_{i,j} (1 + x_i y_j + (x_i y_j)^2 + \ldots)$

$$= \sum_{\beta_1, \beta_2, \ldots \geq 0}' \begin{pmatrix}\text{some}\\ \text{expression}\\ \text{in the } x_i\text{'s}\end{pmatrix} y_1^{\beta_1} y_2^{\beta_2} \cdots$$

inf. sequence w/ finitely many nonzero terms

coeff. of $y_1^{\beta_1}$ in $\prod_i (1 + x_i y_1 + (x_i y_1)^2 + \ldots)$

• "Same thing" for $y_2^{\beta_2}$. ....

$h_{\beta_1}(x)\, h_{\beta_2}(x) \ldots$

So $\displaystyle\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\beta_1, \beta_2, \ldots \geq 0} h_{\beta_1}(x)\, h_{\beta_2}(x) \cdots y_1^{\beta_1} y_2^{\beta_2} \cdots$

$$= \sum_{\lambda \text{ partition}} h_\lambda(x)\, m_\lambda(y).$$

<u>Note.</u> If you don't like infinite products... We can use finite products

$$\prod_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}} \frac{1}{1-x_i y_j} = \sum_{\substack{\lambda=(\lambda_1 \ldots \lambda_n)\\ \text{partition with}\\ \text{at most } n \text{ parts}}}' h_\lambda(x_1 \ldots x_m)\, m_\lambda(y_1 \ldots y_n)$$

2^{nd} product is similar

$$\prod_{i,j}(1+x_iy_j) = \sum_{\beta_1,\beta_2,\ldots\geq 0} (\ldots)\, y_1^{\beta_1} y_2^{\beta_2}\cdots$$

$$e_{\beta_1}(x)\cdot e_{\beta_2}(x)\cdots$$

$$= \sum_{\beta_1,\beta_2,\ldots\geq 0} e_{\beta_1}(x)\, e_{\beta_2}(x)\cdots\, y_1^{\beta_1} y_2^{\beta_2}\cdots$$

$$= \sum_{\lambda \text{ partition}} e_\lambda(x)\, m_\lambda(y). \qquad \square$$

Cauchy Identity:

$$\boxed{\prod_{i,j\geq 1} \frac{1}{1-x_iy_j} = \sum_{\lambda \text{ partition}}' s_\lambda(x)\, s_\lambda(y)}$$

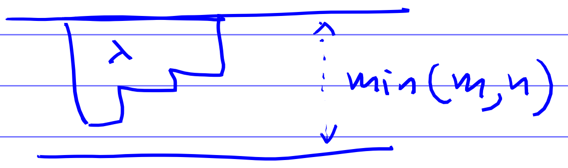
Dual Cauchy Identity

$$\boxed{\prod_{i,j\geq 1} (1+x_iy_j) = \sum_{\lambda \text{ partition}}' s_\lambda(x)\, s_{\lambda'}(y)}$$

the conjugate partition to λ

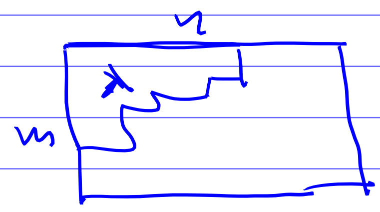
We can use finite #'s of variables:

- $$\prod_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}} \frac{1}{1-x_iy_j} = \sum_{\substack{\lambda \text{ partition}\\ \text{with at most}\\ \min(m,n) \text{ parts}}}' s_\lambda(x_1..x_m)\, s_\lambda(y_1..y_n)$$



$$\prod_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}} (1+x_iy_j) = \sum' s_\lambda(x_1..x_m)\, s_{\lambda'}(y_1..y_n)$$

λ partition with Young diagram with at most m rows and at most n columns



λ fits inside the $m\times n$ rectangle

Both forms (for inf. & finite #s of var.) are equivalent to each other.

We'll prove Cauchy & dual Cauchy for combinatorially defined s_λ's...

But before, let's show that they imply:

<u>Corollary</u>. $\omega(s_\lambda) = s_{\lambda'}$

<u>Proof</u>. Compare two formulas

$$(C) \quad \sum_\lambda h_\lambda(x)\, m_\lambda(y) = \sum_\lambda s_\lambda(x)\, s_\lambda(y)$$

$$(C^*) \quad \sum_\lambda e_\lambda(x)\, m_\lambda(y) = \sum_\lambda s_{\lambda'}(x)\, s_\lambda(y)$$

Apply the involution ω acting on symmetric functions in $x_1, x_2, \ldots$

$\omega: h_\lambda(x) \mapsto e_\lambda(x)$

ω "does nothing" with $m_\lambda(y)$

LHS of (C) $\overset{\omega}{\rightsquigarrow}$ LHS of (C^*)

So RHS of (C) $\overset{\omega}{\rightsquigarrow}$ RHS of (C^*)

Taking the coeff. in front of $s_\lambda(y)$, we get $s_\lambda(x) \overset{\omega}{\longmapsto} s_{\lambda'}(x)$. $\square$

How to prove Cauchy Ident.?

$$\prod_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \left(\sum_{a_{ij}\geq 0} (x_i y_j)^{a_{ij}} \right) \overset{?}{=} \sum_{\lambda} s_\lambda(x_1 \dots x_m) s_\lambda(y_1 \dots y_n)$$

Expand LHS : All terms correspond to $m \times n$ matrices $A = (a_{ij})$ with nonnegative integer entries a_{ij}.

$$LHS = \sum_{\substack{A=(a_{ij}) \\ a_{ij} \in \mathbb{Z}_{\geq 0}}} x_1^{r_1(A)} \dots x_m^{r_m(A)} y_1^{c_1(A)} \dots y_n^{c_n(A)},$$

where $r_i(A)$ are row sums of A
and $c_j(A)$ are column sums of A

$$RHS = \sum_{(P, Q)} x^{\text{weight}(P)} y^{\text{weight}(Q)}$$

pair of SSYT's
of the <u>same</u> shape λ
with at most $\min(m,n)$ parts

Example. $m = n = 2$

LHS : $$\sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ a,b,c,d \geq 0}} x_1^{a+b} x_2^{c+d} y_1^{a+c} y_2^{b+d}$$

RHS : $$\sum_{\substack{\lambda = (\lambda_1, \lambda_2) \\ \lambda_1 \geq \lambda_2 \geq 0}} x^{\text{weight}(P)} y^{\text{weight}(Q)}$$

P, Q - SSYTs of shape λ filled with 1's & 2's

$$P = \begin{array}{l} \overbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}}^{e}\boxed{2}\boxed{2}\boxed{2} \quad \lambda_1 \\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{2} \quad \lambda_2 \end{array}$$

$$Q = \begin{array}{l} \overbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}}^{f}\boxed{2}\boxed{2} \quad \lambda_1 \\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{2} \quad \lambda_2 \end{array}$$

$$(P, Q) \longleftrightarrow \begin{array}{ccc} \lambda_2 & \leq & e \\ \wedge\!\!| & & \wedge\!\!| \\ f & \leq & \lambda_1 \end{array}$$

RHS : $$\sum_{\substack{\begin{pmatrix} \lambda_2 \leq e \\ \wedge\!| \quad \wedge\!| \\ f \leq \lambda_1 \end{pmatrix} \\ \lambda_1, \lambda_2, e, f \in \mathbb{Z}_{\geq 0}}} x_1^{e} x_2^{\lambda_1 - e + \lambda_2} y_1^{f} y_2^{\lambda_1 - f + \lambda_2}$$

such arrays are called reverse plane partitions (RPP)

Robinson-Schensted-Knuth corresp.

is a correspondence between such matrices A & pairs (P,Q) of SSYT's

In this case (for 2×2):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{RSK} \begin{pmatrix} \lambda_2 \le e \\ \wedge\!\!| \quad \wedge\!\!| \\ f \le \lambda_1 \end{pmatrix}$$

$$\begin{aligned} a+b &= e \\ c+d &= \lambda_1 - e + \lambda_2 \\ a+c &= f \\ b+d &= \lambda_1 - f + \lambda_2 \end{aligned}$$

$$RHS = \begin{pmatrix} \lambda_1 & e \\ f & \lambda_2 \end{pmatrix} = \begin{pmatrix} ? \le a+b \\ \wedge\!\!| \quad \wedge\!\!| \\ a+c \le ? \end{pmatrix}$$

$$= \begin{pmatrix} \min(b,c) & a+b \\ a+c & a+d+\max(b,c) \end{pmatrix}.$$

let's check:
- $e = a+b$ ✓
- $\lambda_1 - e + \lambda_2 = a+d+\max(b,c) - (a+b) + \min(b,c) = c+d$ ✓
- $f = a+c$ ✓
- $\lambda_1 - f + \lambda_2 = a+d+\max(b,c) - (a+c) + \min(b,c) = b+d$ ✓

How to construct such correspondence in general?

Special case:

A - $n \times n$ permutation matrix

P, Q - standard Young tableaux of the same shape λ with $|\lambda| = n$ boxes

Standard Young Tableau (SYT) is a SSYT whose weight is $(1, \dots, 1)$

Ex.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

an SYT

Schensted corresp.

$$S_n \xleftrightarrow{\text{Schested}} \left\{ (P, Q) \ \text{SYTs of same shape with } n \text{ boxes} \right\}$$

Terminological Remark:

Many people call this RSK (even Wikipedia).

But this is Schensted corresp. not RSK.

RSK is the corr. for semistandard tableaux. You can not trust Wikipedia!

RSK : classical construction

(due to Knuth, following the case of permutations due to Schensted)

Example $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}$

$\rightsquigarrow$

biword w with a_{ij} columns $\binom{i}{j}$ ordered lexicographically

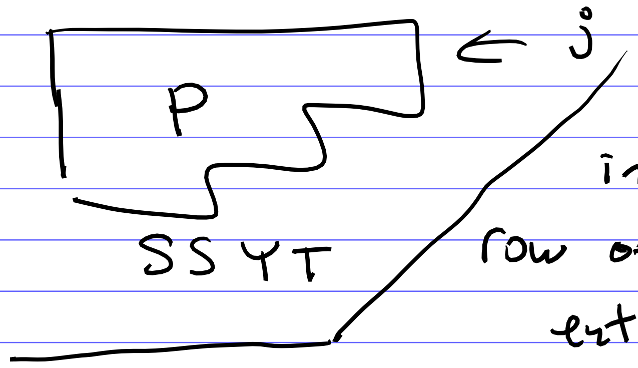
$$w = \underbrace{\binom{1}{1}\binom{1}{1}}_{a_{11}=1}\underbrace{\binom{1}{3}}_{a_{13}=1}\underbrace{\binom{2}{1}}_{a_{21}=1}\underbrace{\binom{2}{2}\binom{2}{2}\binom{2}{2}}_{a_{22}=3}$$

- Start with $(P, Q) = (\emptyset, \emptyset)$
- Insert pairs $\binom{i}{j}$ into (P, Q) with j's going into P and i's going into Q

P is called insertion tableau

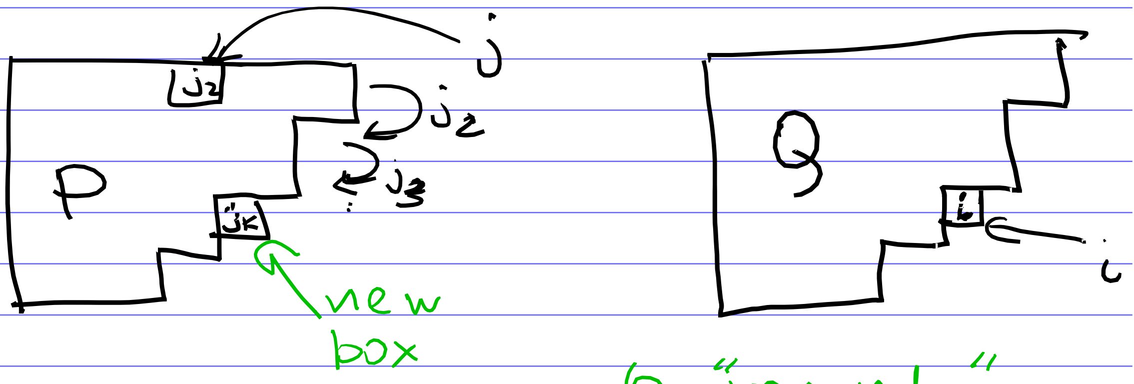
Q is called recording tableau

Schensted Bumping Algorithm



- j is appended in the end of 1^{st} row of P if $j \geq$ all entries of 1^{st} row.
- Otherwise, find the smallest entry j_2 of 1^{st} row s.t. $j_2 > j$. j "bumps" j_2, i.e. j goes in the position of j_2
- Repeat the same procedure with inserting j_2 into 2^{nd} row: j_2 is either appended in the end of 2^{nd} row, or bumps entry j_3.
- In the latter case, insert j_3 into 3^{rd} row, etc
- At some point a new box will be appended to some row of P.

Add a box filled with i in the same position in Q (recording tableau)



Q "records" the position of the new box

Ex: $A \rightsquigarrow \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}$

$$\begin{array}{cc} P & Q \\ \downarrow & \downarrow \end{array}$$

$$\varnothing, \quad \varnothing \qquad \leftarrow \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline\end{array} \qquad \leftarrow \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \qquad \leftarrow \begin{pmatrix}1\\3\end{pmatrix}$$

$$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \qquad \leftarrow \begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array} \qquad \leftarrow \begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2\\\cline{1-1}\end{array} \qquad \leftarrow \begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline 2\\\cline{1-1}\end{array} \qquad \leftarrow \begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&2&2\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&2&2\\\hline 2\\\cline{1-1}\end{array}$$

this is the resulting P tableau

and this is the Q tableau

A more interesting example of Schensted algorithm:

$j = 2$

$$P: \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 & 3 & 8 \\ \hline 2 & 2 & 2 & 4 & 5 \\ \cline{1-5} 3 & 4 & 6 & 6 & 6 \\ \cline{1-5} 8 & 8 \\ \cline{1-2} \end{array}$$

$$P \leftarrow j: \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 3 & 8 \\ \hline 2 & 2 & 2 & 3 & 5 \\ \cline{1-5} 3 & 4 & 4 & 6 & 6 \\ \cline{1-5} 6 & 8 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

<u>Theorem.</u> This procedure gives a valid bijection between matrices A & pairs (P, Q) of SSYT's of same shape satisfying all needed properties

<u>Proof.</u> We need to check that we get valid SSYT's of needed weights, and that this procedure is <u>invertible.</u>

Not really hard to check, but one needs to pay attention to all little details.... $\square$