18.217 Lecture 5 9/14/2020

last time: 4 formulas for s_λ,

$$s_\lambda^{\text{class.}} \overset{?}{=} s_\lambda^{\text{comb.}} \overset{?}{=} s_\lambda^{\text{Schub}} \overset{?}{=} s_\lambda^{\text{Dem.}}$$

$$s_\lambda^{\text{class.}} = \frac{\sum_{w \in S_n} (-1)^{\ell(w)} w(x^{\lambda+\delta})}{\prod_{i<j} (x_i - x_j)}$$

Weyl char. formula

$$s_\lambda^{\text{comb.}} = \sum_{SSYT\ T} x^{wt(T)}$$

$$s_\lambda^{\text{Schub}} = \partial_{w_0}(x^{\lambda+\delta})$$

div. diff. operator

$$s_\lambda^{\text{Dem.}} = D_{w_0}(x^\lambda)$$

Demazure operator

How to prove?

- Divided differences operators:

$$\partial_i : f \longmapsto \frac{(1 - s_i)(f)}{x_i - x_{i+1}}$$

- Demazure operators

$$D_i : f \longmapsto \frac{\left(1 - \frac{x_{i+1}}{x_i} s_i\right)(f)}{1 - \frac{x_{i+1}}{x_i}} = \partial_i(x_i f).$$

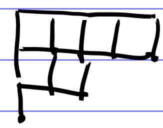
$$D_i(x_i^a x_{i+1}^b) = x_i^a x_{i+1}^b + x_i^{a-1} x_{i+1}^{b+1} + \dots$$

$a \geq b$

$$\dots + x_i^b x_{i+1}^a$$

D_i commutes with x_j, $j \neq i, i+1$

$$D_i(x_j f) = x_j D_i(f).$$

<u>Example</u> $n=3$, $\lambda = (4,2,0) =$ 

Let's calculate $S_{(4,2,0)}(x_1, x_2, x_3)$

using Demazure operators:

$$\underline{S_\lambda(x_1, x_2 x_3) =}\ D_1 D_2 D_1 (x^\lambda)$$

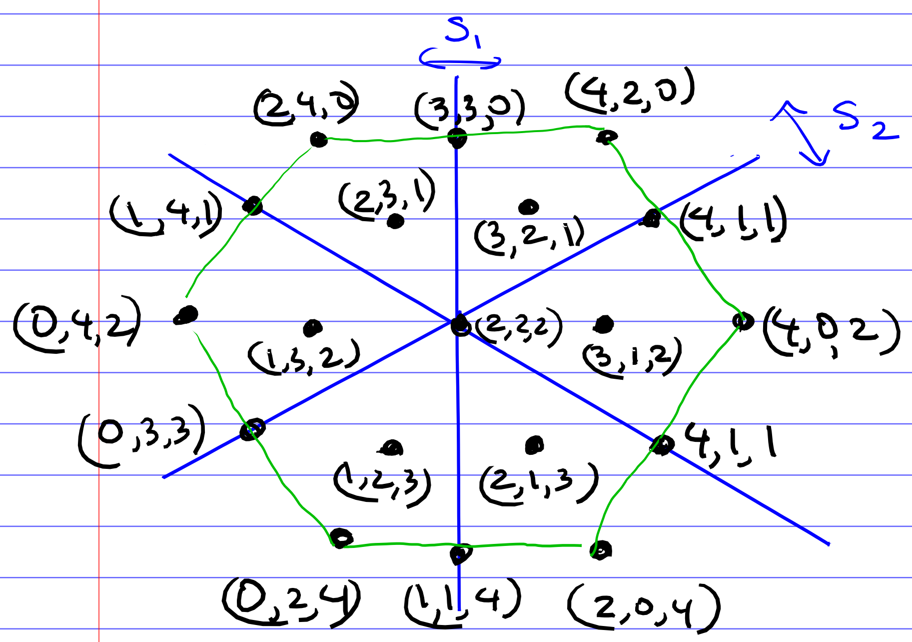
We'll represent a monomial

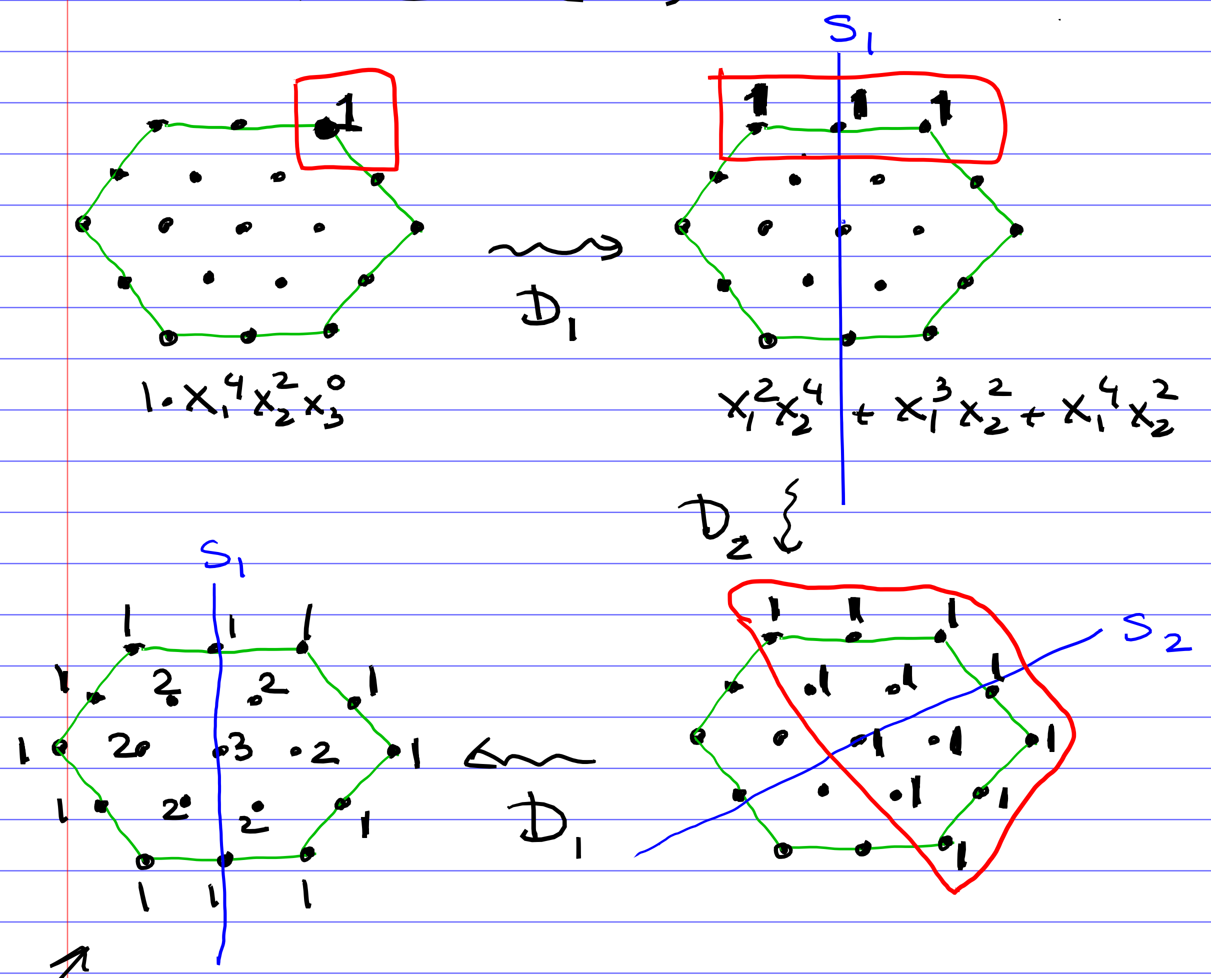
$x_1^a x_2^b x_3^c$ by a point (a, b, c)

in the affine plane

$$\{(x,y,z) \mid x+y+z = 6\} \subset \mathbb{R}^3$$

All monomials will have same degree $= 6$, so we are "living" in an affine plane $x+y+z = 6$





$$S_{(4,2,0)} = x_1^2x_2^4 + x_1^3x_2^3 + x_1^4x_2^2 +$$
$$+ x_1x_2^4x_3 + 2x_1^2x_2^3x_3 + 2x_1^3x_2^2x_3 + x_1^4x_2x_3$$
$$+ x_2^4x_3^2 + 2x_1x_2^3x_3^2 + 3x_1^2x_2^2x_3^3 + 2x_1^3x_2x_1^2 + x_1^4x_3$$
$$+ x_2^3x_3^3 + 2x_1x_2^2x_3^3 + 2x_1^2x_2x_3^3 + x_1^4x_2x_3$$
$$+ x_2^2x_3^4 + x_1x_2x_3^4 + x_1^2x_3^4$$

For example, Kostka number

$$K_{(4,2,0),(2,2,2)} = 3.$$

Let's check by counting SSYT's:

1	1	2	2
3	3		

1	1	2	3
2	3		

1	1	3	3
2	2		

Observation: All non-zero monomials "live" inside a certain polytope (hexagon in this example)

Def. The permutohedron

$\Pi(\lambda) := \mathrm{conv}((\lambda_{w(1)}, \dots, \lambda_{w(n)}) \mid w \in S_n)$

convex polytope in $\mathbb{R}^n$.

Fix $\lambda = (\lambda_1, \dots, \lambda_n)$.

$$s_\lambda = \sum_{\beta \in \mathbb{Z}^n} K_{\lambda\beta}\, x^\beta = \sum_{\substack{\mu = (\mu_1, \dots, \mu_n) \\ \text{partition}}} K_{\lambda\mu}\, m_\mu.$$

Kostka numbers

Theorem.

We have $K_{\lambda\beta} \neq 0$ iff

$$\beta \in \underbrace{\Pi(\lambda) \cap \mathbb{Z}^n}_{\text{integer lattice points of } \Pi(\lambda)}$$

We already mentioned a related result:

Theorem $K_{\lambda\mu} \neq 0$ (μ partition) iff $\lambda \geq \mu$ in the dominance order:

$$\lambda_1 \geq \mu_1,$$
$$\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2,$$
$$\lambda_1 + \lambda_2 + \lambda_3 \geq \mu_1 + \mu_2 + \mu_3,$$
$$\dots$$

and $|\lambda| = |\mu|$.

The equivalence of these two results follows from:

<u>Theorem</u> (Rado) Permutohedron $\Pi(\lambda) := \mathrm{conv}(w(\lambda) \mid w \in S_n) \subset \mathbb{R}^n$ is given by the following inequalities:

$$\Pi(\lambda) = \Big\{ (y_1, \dots, y_n) \in \mathbb{R}^n \text{ s.t.}$$

- $y_{i_1} + \dots + y_{i_k} \leq \lambda_1 + \dots + \lambda_k$ for any distinct $i_1, \dots, i_k$

- $y_1 + \dots + y_n = \lambda_1 + \dots + \lambda_n \Big\}$

In above example, $\Pi(4,2,0) = \{(x,y,z) \mid x, y, z \leq 4;\ x+y,\ x+z,\ y+z \leq 6;\ x+y+z = 6\}$

Indeed, for $(y_1, \dots, y_n) \in \mathbb{Z}^n$, Rado's inequalities $\Longleftrightarrow$

the weakly decreasing rearrangement $\mu = (\mu_1, \dots, \mu_n)$ of $(y_1, \dots, y_n)$ satisfies $\mu \leq \lambda$ in the dominance order.

BTW, there is another lesser known linear basis of Λ.

$$b_\lambda = \sum_{\mu:\, K_{\lambda\mu}\neq 0} m_\mu = \sum_{(\beta_1,\dots,\beta_n)\in \Pi(\lambda)\cap \mathbb{Z}^n} x^\beta$$

for a partition $(\lambda_1,\dots,\lambda_n)$.

i.e. b_λ is obtained from s_λ by replacing all non-zero coeffts. with 1.

<u>Lemma</u> $\{b_\lambda \mid \lambda$ any partition$\}$ is a linear basis of Λ.

<u>Proof</u>. Basically, the same argument as for s_λ: $\{b_\lambda\}$ is related to $\{m_\lambda\}$ by an upper-triangular matrix with 1's on the diagonal. $\square$

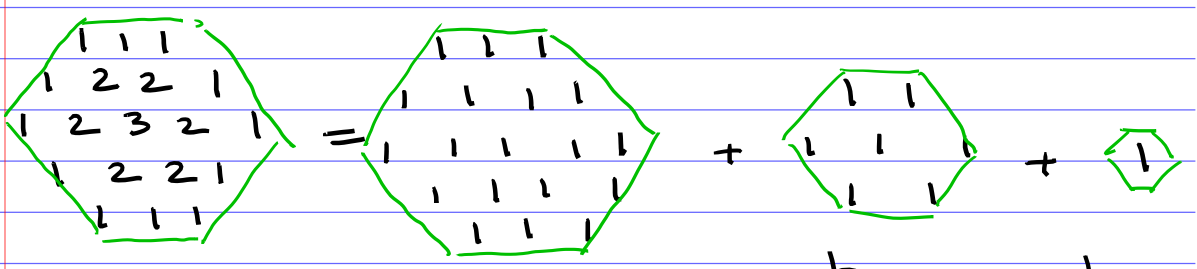
We have

$$s_\lambda = \sum_\mu A_{\lambda\mu}\, b_\mu$$

<u>Problem:</u> A combinatorial formula for $A_{\lambda\mu}$? Is it true that $A_{\lambda\mu}\geq 0$.

<u>Example</u>: $\lambda = (4,2,0)$

$$s_{(4,2,0)} = b_{(4,2,0)} + b_{(3,2,1)} + b_{(2,2,2)}$$



s_{420} b_{420} b_{321} b_{222}

Here we are using sym. polynomials $f(x_1,x_2,x_3)$, i.e. we only keeping partitions w/ at most 3 parts.

<u>Theorem</u>. $b_\lambda(x_1,\dots,x_n) = \sum_{w\in S_n} w\left(\dfrac{x^\lambda}{\prod_{i=1}^{n-1}\left(1-\frac{x_{i+1}}{x_i}\right)}\right).$

Can be deduced from Brion's formula, which gives Σ over lattice points of a polytope

Compare w/

$$s_\lambda(x_1,\dots,x_n) = \sum_{w\in S_n} w\left(\frac{x^\lambda}{\prod_{i<j}\left(1-\frac{x_j}{x_i}\right)}\right).$$

Back to 4 formulas for s_λ...

Fix n.

$$S_\lambda^{\text{Schub}} := \partial_{w_0}(x^{\lambda+\delta}) \stackrel{?}{=} S_\lambda^{\text{Dem.}} := D_{w_0}(x^\lambda)$$

Let $X_i : f \mapsto x_i f$ (operator of mult. by x_i)

$$X^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

Then $D_i = \partial_i X_i$.

Theorem. $\boxed{D_{w_0} = \partial_{w_0} X^\delta}$

Example $n = 3$.

$$D_{w_0} := D_1 D_2 D_1 = \partial_1 X_1 \partial_2 X_2 \partial_1 X_1$$

$$\stackrel{?}{=} \underbrace{\partial_1 \partial_2 \partial_1}_{\partial_{w_0}} X_1^2 X_2.$$

this is not completly trivial, becau ∂_i does not commute w/ X_j for $j = i, i+1$.

$$D_{w_0} = D_2 D_1 D_2 =$$

$$= \partial_2 X_2 \partial_1 X_1 \partial_2 X_2$$

$$= \partial_2 \partial_1 \partial_2 \; X_1^2 X_2$$

But we can still "move" X_i's through ∂_j's if we do it smartly:

x_2 and ∂_1 don't commute

$$\partial_1 X_1 \partial_2 X_2 \partial_1 X_1 =$$

$$= \partial_1 \partial_2 X_1 X_2 \partial_1 X_1$$

$$= \partial_1 \partial_2 \partial_1 X_1 X_2 X_1 .$$

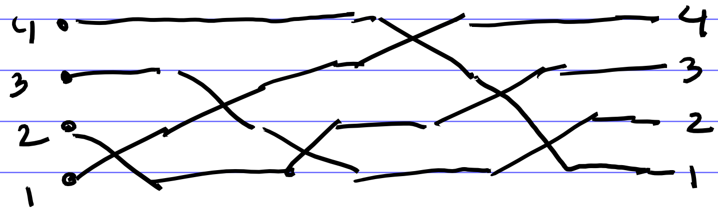
Lemma
$f(x_1 \ldots x_n)$ commutes w/ ∂_i if $f = s_i(f)$.

We can do this for any n, if we pick a reduced decomposition for w_0 smartly.

Lemma

$w_0 = (s_1 s_2 \ldots s_{n-1})(s_1 s_2 \ldots s_{n-2}) \ldots (s_1 s_2)(s_1)$, is a reduced decomposition

For $n=4$:



a wiring diagram for w_0

$(s_1)(s_2 s_1)(s_3 s_2 s_1)$

Proof of $D_{w_0} = \partial_{w_0} x^\delta$:

$$D_{w_0} = (\partial_1 x_1 \, \partial_2 x_2 \cdots \partial_{n-1} x_{n-1})(\partial_1 x_1 \, \partial_2 x_2 \cdots \partial_{n-2} x_{n-2}) \cdots (\partial_1 x_1 \, \partial_2 x_2)(\partial_1 x_1) =$$

$$= \partial_1 \partial_2 \cdots \partial_{n-1} (x_1 \cdots x_{n-1}) \; \partial_1 \partial_2 \cdots \partial_{n-2} (x_1 \cdots x_{n-2}) \cdots \partial_1 \partial_2 (x_1 x_2) \; \partial_1 x_1$$

$$= \underbrace{(\partial_1 \cdots \partial_{n-1})(\partial_1 \cdots \partial_{n-2}) \cdots (\partial_1 \partial_2)(\partial_1)}_{\partial_{w_0}} \cdot \underbrace{(x_1 \cdots x_{n-1})(x_1 \cdots x_{n-2}) \cdots (x_1 x_2)\, x_1}_{x^\delta}$$

$$= \partial_{w_0} x^\delta . \qquad \square$$

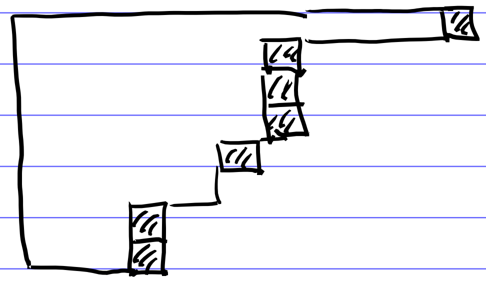
So we proved $S_\lambda^{\text{Schub}} = S_\lambda^{\text{Dem.}}$

Theorem. $s_\lambda^{\text{class.}} = s_\lambda^{\text{comb.}}$

By the fund. thm. of symm. functions, elem. functions e_k generate Λ. So in order to prove that two linear bases of Λ, or of $\Lambda_n = \mathbb{Z}[x_1 \dots x_n]^{S_n}$ coincide it is enough to show that they satisfy the same product rule with e_k.

Def: A skew Young diagram λ/μ is a <u>vertical k-strip</u> if any row of λ/μ contains at most 1 box, and $|\lambda/\mu| = k$.

Ex.



a vertical 7-strip.

Pieri Rule (e version)

This is identil for symm functions

$$e_k \cdot s_\lambda = \sum_{\substack{\mu \text{ any partition s.t.} \\ \mu/\lambda \text{ is a} \\ \text{vertical } k\text{-strip}}} s_\mu .$$

For symm polynomials in n variables, we have

$$e_k(x_1 .. x_n) \cdot s_\lambda(x_1 ... x_n) =$$

$$= \sum_{\substack{\mu \text{ w/ at most } n \text{ parts} \\ \mu/\lambda \text{ is a vert. } k \text{ strip}}} s_\mu(x_1 ... x_n)$$

Both versions are equivalent to each other:

$\Lambda \Rightarrow \Lambda_n$: specialize $x_{n+1} = x_{n+2} = .. = 0$

$\Lambda_n \Rightarrow \Lambda$: take n sufficiently large

Ex: $e_1 \cdot s_{\ydiagram{2,1}} = s_{\ydiagram{3,1}} + s_{\ydiagram{2,2}} + s_{\ydiagram{2,1,1}}$

$e_2 \cdot s_{\ydiagram{2,1}} = s_{\ydiagram{3,2}} + s_{\ydiagram{3,1,1}} + s_{\ydiagram{2,2,1}} + s_{\ydiagram{2,1,1,1}}$

In order to show that $s_\lambda^{\text{class}} = s_\lambda^{\text{comb.}}$ is it enough to prove that both s_λ^{class} & $s_\lambda^{\text{comb.}}$ satisfy Pieri rule.

For $s_\lambda^{\text{comb.}}$, Pieri rule will follow from RSK (Robinson - Schensted - Knuth correspondence), which we'll discuss later.

Let's prove Pieri rule for $s_\lambda^{\text{class.}}(x_1, \dots x_n)$

Proof. $S_\lambda^{\text{class}} := \dfrac{a_{\lambda+\delta}}{a_\delta}$

$$a_\alpha := \sum_{w\in S_n} (-1)^{\ell(w)}\, w(x^\alpha).$$

Since $e_k = e_k(x_1 \dots x_n)$ is a symmetric polynomial, we have

$$e_k \cdot a_\alpha = \sum_w (-1)^{\ell(w)} e_k w(x^\alpha)$$

$$= \sum_w (-1)^{\ell(w)} w(e_k \cdot x^\alpha)$$

$$= \sum_w (-1)^{\ell(w)} w\Big(\sum_{i_1<\dots<i_k} x_{i_1}\dots x_{i_k} x^\alpha\Big)$$

$$= \sum_{i_1<\dots<i_k} a_{\underbrace{\alpha + \vec{e}_{i_1} + \dots + \vec{e}_{i_k}}}$$

these are the coord. vectors in $\mathbb{R}^n$

only the terms where this is a strictly decreasing vector are non-zero.

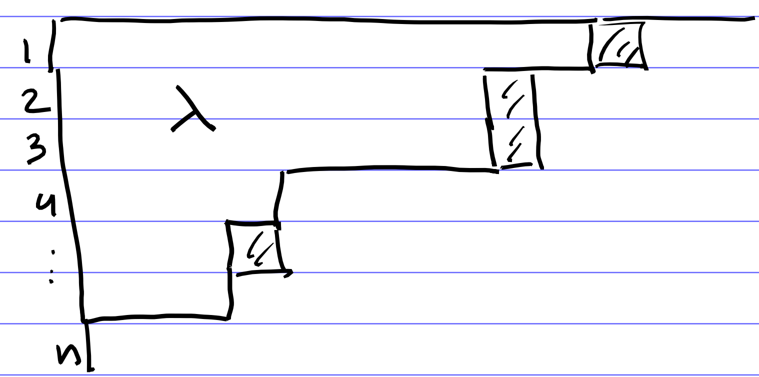
Thus

$$e_k s_\lambda(x_1 \ldots x_n) = \sum_{\mu = \lambda + \vec{e}_{i_1} + \ldots + \vec{e}_{i_k}} s_\mu(x_1 \ldots x_n)$$

where the sum is over $i_1 < \ldots < i_k$ such that μ is a <u>weakly</u> decreasing vector, i.e., μ is a valid partition.

This exactly means that the sum is over all μ's obtained from λ by adding a vertical k-strip.

Ex.



$\leftarrow i_1 = 1$

$\leftarrow i_2 = 2$

$\leftarrow i_3 = 3$

$\leftarrow i_4 = 5.$

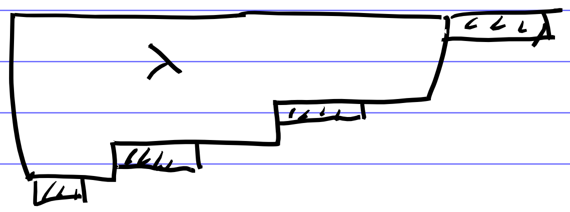
So we've got Pieri Rule $\square$

We also have a similar Pieri rule for h_k

Pieri rule (h-version)

We have in Λ:

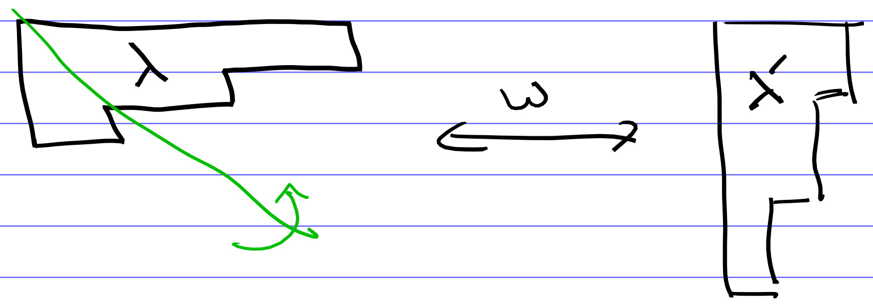
$$h_k s_\lambda = \sum_{\substack{\mu:\ \mu/\lambda \\ \text{is a horizontal} \\ k\text{-strip}}} s_\mu$$



The 2 versions of Pieri rule are related by the involution $\omega : \Lambda \to \Lambda$

$$\omega : e_k \longleftrightarrow h_k$$

Theorem. We have $\omega(s_\lambda) = s_{\lambda'}$.



λ' is the conjugate partition to λ.