18.217 **Lecture 3** 09/09/2020

last week: $m_\lambda, e_\lambda, h_\lambda, p_\lambda$

$\lambda = (\lambda_1, \ldots, \lambda_\ell)$ partitions

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today: Another basis of Λ given by Schur symmetric functions s_λ.

First, we define Schur polynomials

$s_\lambda(x_1, \ldots, x_n)$ in finitely many variables $x_1, \ldots, x_n$

Remark.

sym. polys vs sym. functions

polynomial ↘ inf. power ser. ↙

Ex. $e_\lambda(x_1 \dots x_n)$ vs $e_\lambda(x_1, x_2, \dots)$

$$e_\lambda(x_1, x_2, \dots) = \lim_{n\to\infty} e_\lambda(x_1, \dots, x_n)$$

Once a monomial $x_1^{i_1} \dots x_n^{i_n}$ appears in $e_\lambda(x_1, \dots x_n)$, it will appear in all $e_\lambda(x_1, \dots, x_N)$, for $N \geq n$, with the same coeff.

So, if $n \geq |\lambda|$, the sym. polynomial $e_\lambda(x_1, \dots, x_n)$ contains all info about sym. function $e_\lambda(x_1, x_2, \dots)$.

Classical definition of Schur polynomials $s_\lambda(x_1, \dots, x_n)$.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$

$$a_\alpha := \det \begin{bmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \vdots & \vdots & & \vdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & & x_n^{\alpha_n} \end{bmatrix}$$

a minor of Vandermonde matrix

$$= \det \left(x_i^{\alpha_j} \right)_{1 \leq i, j \leq n}$$

$$= \sum_{w \in S_n} \operatorname{sign}(w)\, x_{w_1}^{\alpha_1} \cdots x_{w_n}^{\alpha_n}$$

Clearly

- $a_\alpha = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$

- $a_\alpha = 0$ if $x_i = x_j$ for some $i \neq j$.

- a_α is anti-symmetric with respect to permutations of $x_1, \dots, x_n$ and w.r.t. perms of $\alpha_1 \dots \alpha_n$

$$a_\alpha(x_1, \dots x_i \dots x_j \dots x_n)$$

$$= - a_\alpha(x_1 \dots x_j \dots x_i \dots x_n)$$

So WLOG we may assume that $\alpha_1 > \dots > \alpha_n$ and write $\alpha = \lambda + \delta$ where λ is a partition and $\delta = (n-1, n-2, \dots, 1, 0)$.

a_α is divisible by

$$\prod_{1 \le i < j \le n} (x_i - x_j) = a_\delta$$

Vandermonde det.

So $\forall$ partition $\lambda = (\lambda_1, \dots \lambda_n)$

$$\boxed{s_\lambda(x_1, \dots, x_n) := \frac{a_{\lambda+\delta}}{a_\delta}}$$

"classical def. of s_λ

is a symmetric polyn. in $x_1, \dots x_n$

(in Lie theory)

Remark $\dfrac{a_{\lambda+\delta}}{a_\delta}$ is known as Weyl character formula for irreducible representations of GL_n (type A).

So Schur polynomials are the "characters of irreps. of GL_n"

In rep. theory, people use notation ρ instead of $\delta = (n-1, n-2, \dots, 1, 0)$

<u>Ex</u>. $n=2$, $\lambda=(1,0)=\square$

$\lambda+\delta=(2,0)$

$$S_{\square}(x_1,x_2) := \frac{\begin{vmatrix} x_1^2 & x_2^2 \\ x_1^0 & x_2^0 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ 1 & 1 \end{vmatrix}}$$

$a_{\lambda+\delta}$

a_δ

$$= \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2$$

<u>Observation</u>. This is a polynomial with <u>positive</u> integer coeff.

Remark. We allow $\lambda = (\lambda_1, \dots, \lambda_n)$ to have 0's in the end. One needs to check that

$$\frac{a_{\lambda+\delta}}{a_\delta} \text{ does not change}$$

if we append 0 to λ & subst. $x_{n+1} = 0$.

Lemma

$n \times n$ det $\qquad (n+1)\times(n+1)$ det

$$\frac{a_{(\lambda_1, \dots, \lambda_n)+\delta_n}}{a_{\delta_n}} = \left. \frac{a_{(\lambda_1 \dots \lambda_n, 0)+\delta_{n+1}}}{a_{\delta_{n+1}}} \right|_{x_{n+1} = 0}$$

where $\delta_n = (n-1, n-2, \dots, 0)$.

Combinatorial def. of s_λ

Def A semi-standard Young tableau (SSYT) of shape λ is a filling of boxes of the Young diagram λ by $1, 2, \ldots, n$ s.t. the numbers strictly increase in columns & weakly increase in rows of λ

$\leq$

Ex

$$\wedge \quad \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 6 \\ \hline 2 & 2 & 3 & 6 \\ \cline{1-4} 3 & 4 & 4 \\ \cline{1-3} 5 & 5 & 6 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}$$

shape $\lambda = (7, 4, 3, 3, 1)$

weight $\beta = (3, 4, 3, 2, 2, 3)$

weight $\beta = (\beta_1, \ldots, \beta_n)$

$\beta_i = \#\, i$'s in the tableau.

$$S_\lambda(x_1 \dots x_n) = \sum_{\substack{T:\ \text{SSYT} \\ \text{of shape } \lambda \\ \text{filled w/ } 1,\dots,n}} x^{\text{weight}(T)}$$

$$\text{weight}(T) = (\beta_1, \dots, \beta_n)$$

$$x^{\text{weight}(T)} = x_1^{\beta_1} \cdots x_n^{\beta_n}.$$

<u>Theorem</u>. Classical def. of S_λ $\Longleftrightarrow$ comb. def of S_λ

Since this is a course on combinatorics, we'll use the comb. def of S_λ & prove that $a_{\lambda+\delta}/a_\delta$ is the same thing

Speaking of Schur sym. functions...

$$s_\lambda(x_1, x_2, \ldots) = \sum_{\substack{T:\,\text{SSYT} \\ \text{of shape } \lambda \\ \text{filled with} \\ \text{any positive numbers}}} x^{\text{weight}(T)}$$

<u>Clearly,</u> $s_\lambda(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n, 0, 0, \ldots)$

But it is not immediatly clear (from the comb. def.) that s_λ is symmetric.

<u>Lemma</u> s_λ is a symmetric function.

Proof. Enough to show that

$$s_\lambda(\ldots x_i, x_{i+1}, \ldots) = s_\lambda(\ldots x_{i+1}, x_i \ldots)$$

$\forall\ i = 1, 2, \ldots$

(This implies s_λ is invariant w.r.t. *any* permutation of x_i's.

Ex. $s_\lambda(x, y, z) \overset{?}{=} s_\lambda(y, x, z)$

$= s_\lambda(y, z, x) = s_\lambda(z, y, x)$

Basically, adjacent transpositions generate all permutations.)

So we need to show that # SSYT's of shape λ and weight $\beta = (\ldots \beta_i\ \beta_{i+1} \ldots)$

$=$ # SSYT's of shape λ and weights $\tilde{\beta} = (\ldots \beta_{i+1}\ \beta_i \ldots)$

Let's construct a bijection

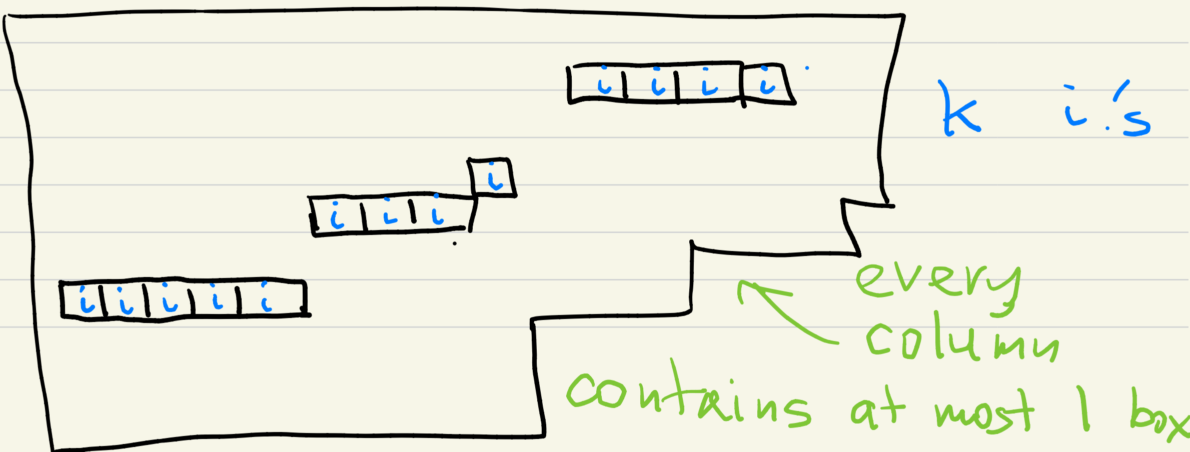
$$SSYT(\lambda, \beta) \xrightarrow{\sim} SSYT(\lambda, \tilde{\beta})$$

the set of SSYT's of shape λ & weight β.

$$T \longmapsto \tilde{T}$$

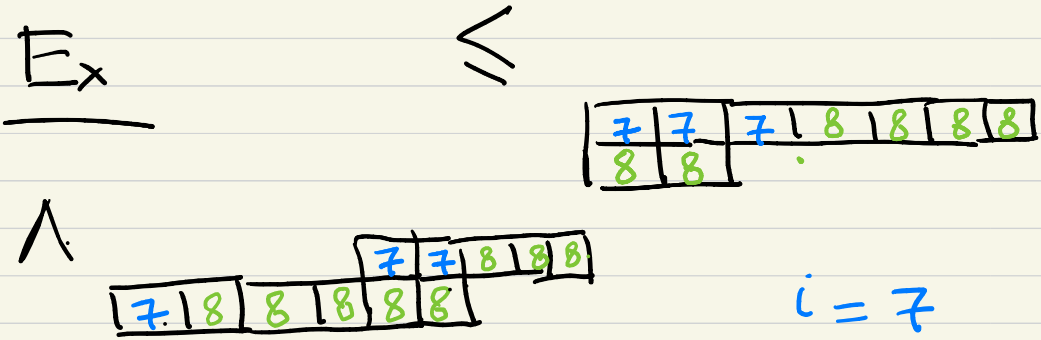
a SSYT with k i's & ℓ $(i+1)$'s , SSYT with ℓ i's and k $(i+1)$'s

All boxes of T filled w/ i's form a <u>horizontal k-strip</u>



We will only modify boxes filled with i's & $(i+1)$'s

Ex



$i = 7$

$i+1 = 8$

6 7's

14 8's

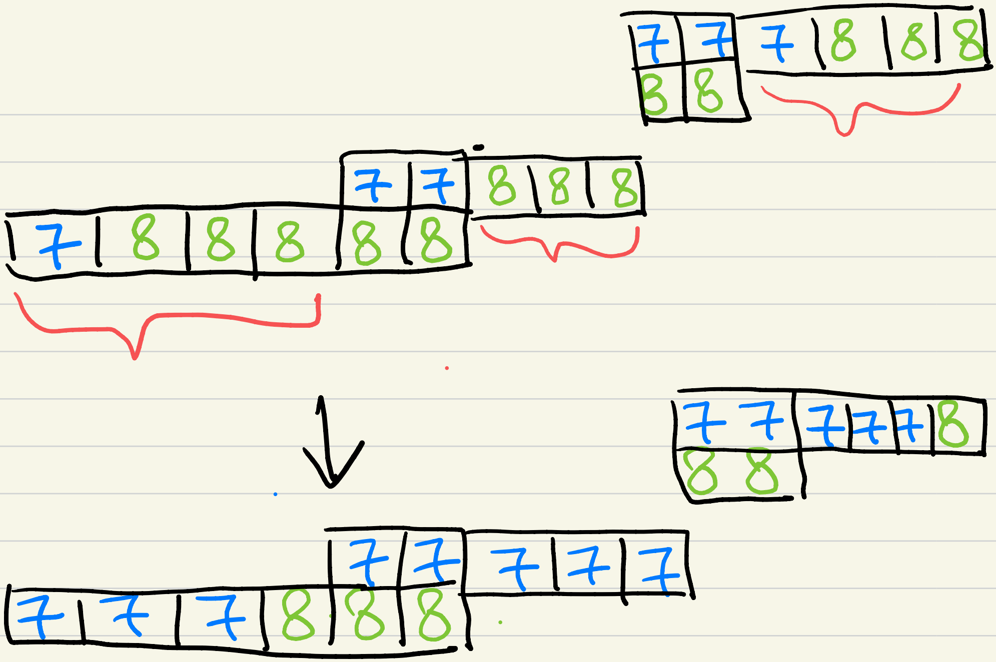
We want to modify (only) this part of the tableau and replace it by 14 7's and 6 8's.

We might have some vertical dominos $\boxed{\begin{smallmatrix}7\\ \hline 8\end{smallmatrix}}$ $\wedge$ which we cannot modify. All other boxes come in several rows of the form

$$\boxed{\underbrace{7 \cdots 7}_{a}\ \underbrace{8 \cdots\cdots 8}_{b}}\ .$$

Replace them by

$$\boxed{\underbrace{7 \cdots\cdots\cdots 7}_{b}\ \underbrace{8 \cdots 8}_{a}}$$



If we repeat the operation, we go back to the orig. tableau T. So this is a bijection

$$SSYT(\lambda, \beta) \longleftrightarrow SSYT(\lambda, \tilde{\beta}).$$

This proves that s_λ is symmetric. $\square$

Another construction for S_λ.

Gelfand-Tsetlin patterns

- triangular arrays with nonnegative integers
- Top row is $\lambda_1, \lambda_2, \ldots \lambda_n$
- Adjacent rows are weakly interlaced.

$$
\begin{array}{ccccccc}
\lambda_1 & & \lambda_2 & \cdots\cdots & & & \lambda_n \\
& \mu_1 & & \mu_2 \cdots & & \mu_{n-1} & \\
& & \nu_1 & & \nu_2 \cdots \nu_{n-2} & & \\
& & & \cdots\cdots & & & \\
& & & \delta & & &
\end{array}
$$

Example. $n=6$ $\lambda=(7,5,5,4,1,0)$

$$
\begin{array}{ccccccccccc}
7 & & 5 & & 5 & & 4 & & 1 & & 0 \\
& 6 & & 5 & & 5 & & 2 & & 1 & \\
& & 6 & & 5 & & 3 & & 1 & & \\
& & & 5 & & 4 & & 2 & & & \\
& & & & 4 & & 3 & & & & \\
& & & & & 3 & & & & &
\end{array}
$$

Gelfond-Tsetlin patterns are in bijection with SSYT's filled with $1,\dots,n$

Ex The above GT-patt. corresponds to the tableau

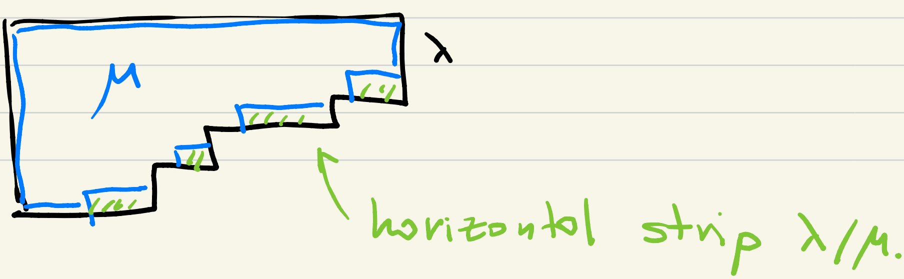
1	1	1	2	3	4	6
2	2	2	3	4		
3	3	4	5	5		
4	5	6	6			

.

Lemma The interlacing condition

$$\begin{array}{ccccccccc} \lambda_1 & & \lambda_2 & & \cdots & & & & \lambda_n \\ & \begin{smallmatrix}\geq & \geq\end{smallmatrix} & & \begin{smallmatrix}\geq & \geq\end{smallmatrix} & & & \begin{smallmatrix}\geq & \geq\end{smallmatrix} & & \\ & \mu_1 & & \mu_2 & & \cdots & \mu_{n-1} & & \end{array}$$

is equivalent to the condition that boxes between λ & μ form a horizontal strip.



Corollary

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$

$$S_\lambda(x_1, \dots, x_n) = \sum_{\substack{P:\ \text{Gelfand-} \\ \text{Tsetlin patterns} \\ \text{with top row } \lambda}} x^{\text{weight}(P)}$$

$$\text{weight}(P) = (\beta_1, \dots, \beta_n)$$

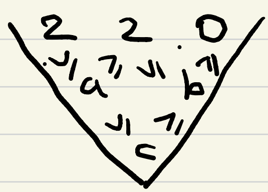
$\beta_i = (n+1-i)^{th}$ row sum of P

$\quad - (n-i)^{th}$ row sum of P

Example

$n = 3, \quad \lambda = (2, 2, 0)$

$$S_{\boxplus}(x, y, z) = \sum_{a,b,c \in \mathbb{Z}}{}' \; x^c \, y^{a+b-c} \, z^{4-a-b}$$



Remark. We can view such expression for S_λ as a sum over lattice points of a certain <u>polytope</u> $P(\lambda)$.

Theorem. Schur symmetric functions s_λ form a $\mathbb{Z}$-linear basis of Λ.

Proof We'll show that

Schur funct s_λ's & monomial funct. m_μ's are

<u>related by a triangular matrix</u>

By def.
$$\boxed{s_\lambda = \sum_{\mu} K_{\lambda\mu}\, m_\mu}$$

sum over partitions μ s.t. $|\mu| = |\lambda|$.

where $K_{\lambda\mu} :=$ # SSYTs of shape λ and weight μ.

Kostka numbers

Def. The <u>dominance partial order</u> (aka <u>majorization order</u>) on partitions of n.

$\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_\ell)$

$\lambda \geq \mu$ if

- $|\lambda| = |\mu|$

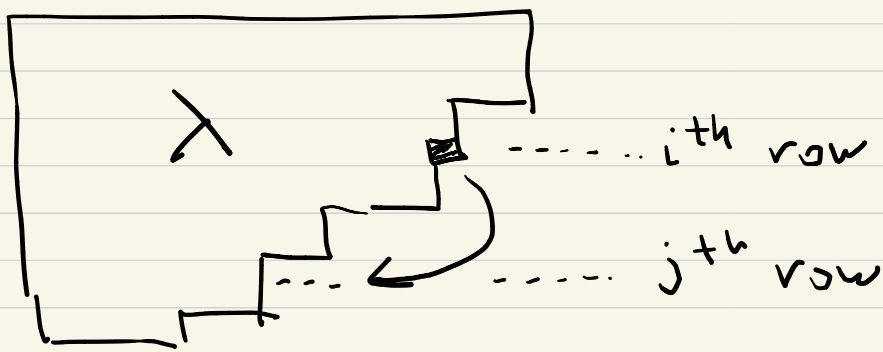
- $\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i \quad \forall i$

(assuming $\lambda_i = \mu_j = 0$ for $i > k$, $j > \ell$).

Lemma. The dominance order is generated by the operations R_{ij} on Young diagrams

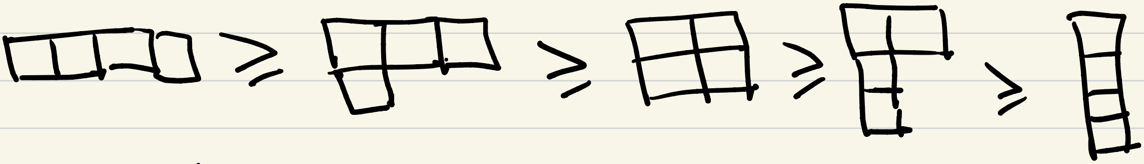
R_{ij} :

$i<j$



move a box in λ from i^{th} row to j^{th} row (if the result is a valid Young diagr.)

Example. $n = 4$



(In this case, it is a total order. But in general it is a partial order.)

Theorem. $K_{\lambda\lambda} = 1$

$K_{\lambda\mu} \neq 0$ iff $\lambda \geq \mu$.

Proof of $K_{\lambda\mu} \neq 0 \Rightarrow \lambda \geq \mu$.

For any SSYT of shape λ & weight μ, boxes containing $1, 2, \ldots, i$ appear only in first i rows of λ.

So $\mu_1 + \dots + \mu_i \le \lambda_1 + \dots + \lambda_i \quad \forall i$

$\Updownarrow$

$\lambda \ge \mu$ in the dominance order.

Thus $\{s_\lambda\} = (K_{\lambda\mu}) \{m_\mu\}$

an upper triangular matrix with 1's on the diagonal for any linear extension of the dominance order on partitions of n.

$\Longrightarrow$ s_λ is a basis of Λ.

For $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$
the set S_λ

$$= \left\{ (i_1, \ldots, i_n) \in \mathbb{Z}^n \;\middle|\; \begin{array}{l} \text{monomial} \\ x_1^{i_1} \cdots x_n^{i_n} \text{ occurs} \\ \text{in } s_\lambda(x_1, \ldots, x_n) \\ \text{w/ non-zero coeff.} \end{array} \right\}$$

is the set of all integer lattice points in a certain convex polytope $\Pi(\lambda) \subset \mathbb{R}^n$, called the

<u>permutohedron</u>

$$\Pi(\lambda) = \mathrm{conv}\big((\lambda_{w_1}, \ldots, \lambda_{w_n}) \,\big|\, w \in S_n \big)$$

<u>Theorem</u>. $S_\lambda = \Pi(\lambda) \cap \mathbb{Z}^n$

The fact $K_{\lambda\mu} \neq 0 \Leftrightarrow \mu \leq \lambda$

is related to

<u>Theorem</u> (Rado)

$$\Pi(\lambda) = \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \bullet\ y_1 + \ldots + y_n = |\lambda| \\ \bullet\ y_{w_1} + \ldots + y_{w_i} \leq \\ \quad \lambda_1 + \ldots + \lambda_n \\ \quad \forall\, i = 1, \ldots, n \\ \quad \forall\, w \in S_n \end{array} \right\}$$