18.217 (Last) Lecture 38 12/09/2020

Positivity of symmetric functions

Def. A symmetric function $f \in \Lambda$ is

- monomial positive (m-positive) if
  $f = \sum_\lambda a_\lambda m_\lambda$, where $a_\lambda \geq 0 \ \forall \lambda$.
- Schur positive (s-positive) if
  $f = \sum_\lambda b_\lambda s_\lambda$, where $b_\lambda \geq 0 \ \forall \lambda$
- e-positive if
  $f = \sum_\lambda c_\lambda e_\lambda$, where $c_\lambda \geq 0 \ \forall \lambda$
- h-positive if
  $f = \sum_\lambda d_\lambda h_\lambda$, where $d_\lambda \geq 0 \ \forall \lambda$

There are a lot of results (and conjectures) on various kinds of positivity of some symmetric functions.

Often, a positivity (of some kind) holds if the coefficients in the exponsion of f (in the basis m_λ, s_λ, e_λ, or h_λ) have some representation-theoretic or geometric meaning
(e.g., the dim of some space, or the intersection numbers of some varieties)

Clearly,

$$\begin{matrix} \text{e-positivity} & \searrow & & & \\ & & \text{Schur positivity} & \Rightarrow & \text{monomial positivity} \\ \text{h-positivity} & \nearrow & & & \end{matrix}$$

Example. $s_{\lambda/\mu}$ is Schur positive (the coefficients = the LR-coefficients)

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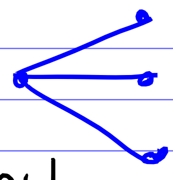
last time: Stanley's chromatic symmetric function X_G.

$G = (V, E)$ a simple graph

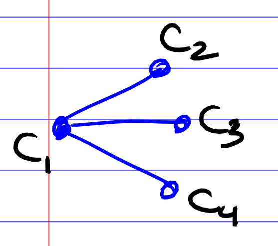
Def. $X_G = X_G(x_1, x_2, \dots) :=$

$$:= \sum_{\substack{c: V \to \mathbb{Z}_{>0} \\ \text{proper coloring of } G}} \prod_{v \in V} x_{c(v)}$$

Example. $G = K_n$. $X_{K_n} = n!\, e_n$ is e-positive $\Rightarrow$ Schur positive.

Example $G = K_{1,3} =$  the claw graph

$$X_{K_{1,3}} = \sum_{\substack{c_1, c_2, c_3, c_4 \geq 1 \\ c_1 \neq c_2,\ c_1 \neq c_3,\ c_1 \neq c_4}} x_{c_1} x_{c_2} x_{c_3} x_{c_4} =$$



$$= 4e_4 + 5e_{31} - 2e_{22} + e_{211}$$

$$= 4s_{1111} + 5s_{2111/1} - 2s_{2211/11} + s_{3211/21}$$

A helpful identity:

$$s_{\lambda/\mu} \cdot s_{\nu/\delta} = s_{\nu/\delta \,\cdot\, \lambda/\mu} = s_{a \leq b} + s_{a > b}$$

$a \leq b$ $\qquad$ $a > b$

$$X_{K_{1,3}} = 8s_{1111} + 6s_{211} + s_{31} - s_{221/1}$$

Another identity:

$$s_{221/1} = s_{211} + s_{22}$$

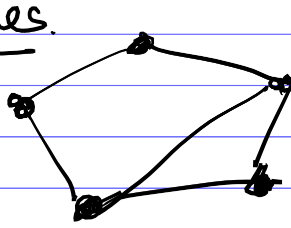
$$= 8s_{1111} + 5s_{211} + s_{31} - s_{22}$$

X_{claw} is not e-positive

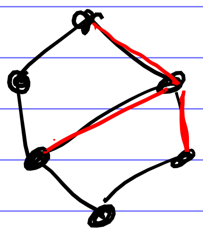
is not Schur positive

Definition A graph is called a claw-free graph if it does not have an induced subgraph isomorphic to $K_{1,3}$.

Examples.



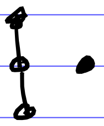
a claw-free graph



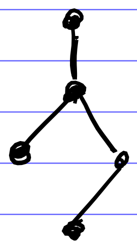
not a claw-free graph

Claw-free graphs have been studied for a while. E.g. Sumner & Las Vergnas ~1975 proved that any claw-free graph with an even # vertices has a perfect matching.

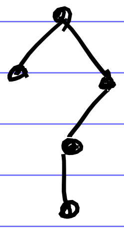
Definition A poset P is called (3+1)-free, aka (3+1)-avoiding if it does not contain an induced subposet isomorphic to



Examples



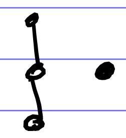
a (3+1)-free poset

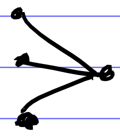


not a (3+1)-free poset

Definition For a poset P on the set of elements V, the <u>incomparability graph</u> of P is the graph $G = (V, E)$ s.t. $(u, v) \in E$ whenever the elements u & v are incomparable in P.

Example

$P =$ 

the incomparability graph is $G =$ 

(the claw graph)

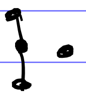
Lemma. For a poset P,

$$X_{\text{incomp. graph of } P} = \sum_{C_1, \dots, C_\ell}{}' \, m_{|C_1|, \dots, |C_\ell|}$$

the sum over decomositions of P into a disjoint union of chains $C_1, C_2, \dots, C_\ell$

monomial sym. funct. for the partition given by the sizes of the chains

Example

$P =$  , $X_G = m_{31} + 3m_{211} + m_{1111}$

Lemma. If G is the incomparability graph of a poset P, then G is claw-free iff P is $(3+1)$-free.

Remark This is <u>not</u> a bijection between claw-free graphs and $(3+1)$-posets. There are claw-free graphs which are not incomparability graphs of any poset.

<u>Conjecture</u> (Stanley 1995 ~ Stanley–Stembridge 1993)

If G is the incomparability graph of a $(3+1)$-free poset, then X_G is e-positive.

<u>Theorem</u> (Gasharov '1996)

If G is the incomparability graph of a $(3+1)$-free poset, then X_G is Schur positive.

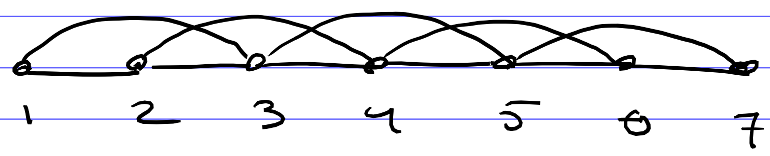
<u>Conjecture</u> (Gasharov) If G is claw-free, then X_G is Schur positive.

<u>Remark</u> There are known claw-free graphs, for which X_G is not e-positive.

For $n \geq k \geq 1$, let G be the graph on $V = \{1, 2, \ldots, n\}$ with the edge set

$$E = \{ (i,j) \mid |i-j| \leq k-1 \}$$

<u>Example</u> $n = 7$, $k = 3$

$G =$ 

The chromatic symmetric function X_G of G is

$$P_{n,k} = \sum_{\substack{i_1, \ldots, i_n \in \mathbb{Z}_{>0} \text{ s.t.} \\ \text{any } k \text{ consecutive} \\ \text{terms are distinct}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

<u>Conjecture</u> (A special case of above Stanley's conjecture)

$P_{n,k}$ is e-positive.

The case $k = 2$ follows from the identity due to Carlitz:

$$\sum_{n \geq 0} P_{n,2} t^n = \frac{\sum_{i \geq 0} e_i t^i}{1 - \sum_{i \geq 1} (i-1) e_i t^i}$$

$\Rightarrow P_{n,2}$ is e-positive.

The case $k = 3$ is also known.

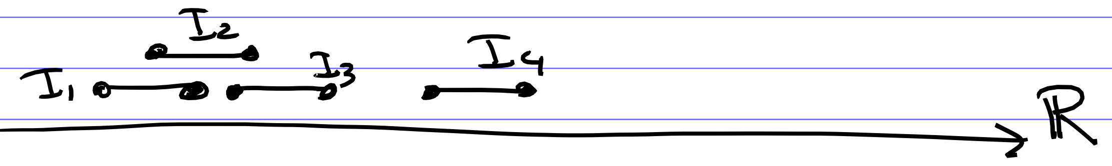
For more details, see R. Stanley's slides:

A chromatic symmetric function conjecture

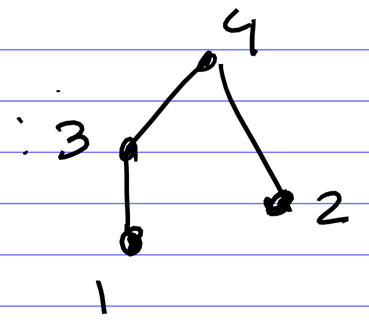
Def. A poset P is called a <u>unit interval order</u> (aka <u>semiorder</u>) if we can find a collection of unit intervals $I_1, \dots, I_n$ on the line $\mathbb{R}$ (associated with elements of P) s.t.

$i <_P j$ iff the interval I_i lies to the left of I_j

<u>Example</u>

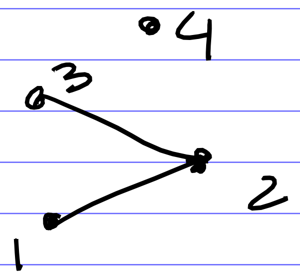


P :



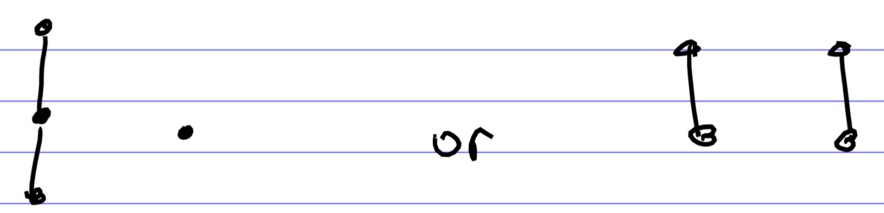
the incomparability graph of P

$G =$



i & j are connected by an edge in G iff the intervals I_i & I_j overlap: $I_i \cap I_j \neq \emptyset$.

<u>Theorem</u> (Scott, Suppes 1958)
A poset P is a unit interval order iff it is (3+1)-avoiding & (2+2)-avoiding i.e. it does not contain induced subposets of the form:

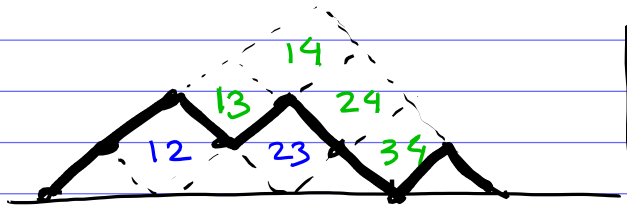


3+1 or 2+2.

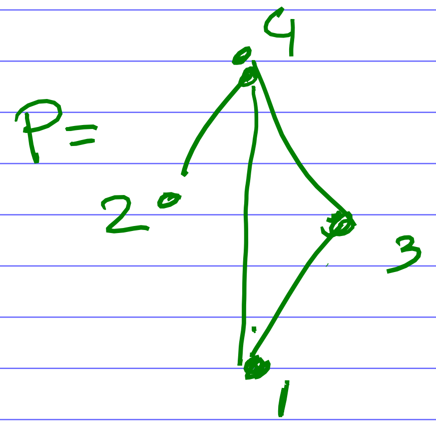
The number of (unlabelled) unit interval orders on n vertices equals the <u>Catalan</u> number C_n.

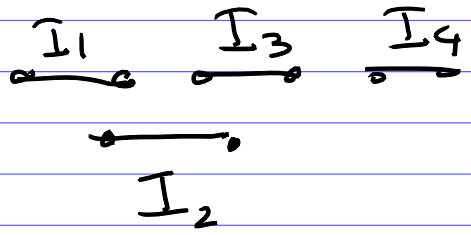
A bijection with Dyck paths:

<u>Example</u>: $n = 4$

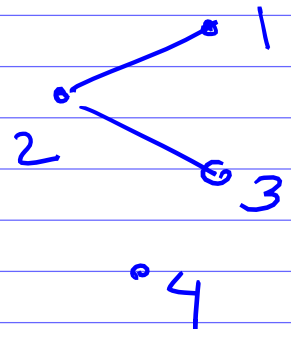


a Dyck path





$i <_P j$ if box (i,j) is above the Dyck path



edges (i,j) of the incomp. graph G correspond to boxes (i,j) <u>below</u> the Dyck path

Any Dyck path $\rightsquigarrow$ graph G

$\rightsquigarrow$ the chromatic symm. function X_G.

According to <u>Stanley's conj.</u> such X_G should be e-positive...

More Schur positivity results & conjectures.

Q: when is $\boxed{s_\nu \cdot s_\gamma - s_\lambda \cdot s_\mu}$ Schur positive (or monomial positive)?

Okounkov's Conjecture '1997

Let $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n)$ be two partitions such that all parts of $\lambda + \mu = (\lambda_1 + \mu_1, \dots, \lambda_n + \mu_n)$ are even.

Let $\frac{\lambda+\mu}{2} := \left(\frac{\lambda_1 + \mu_1}{2}, \dots, \frac{\lambda_n + \mu_n}{2} \right)$

Then $\left(s_{\frac{\lambda+\mu}{2}} \right)^2 - s_\lambda \cdot s_\mu$ is Schur positive.

(Okounkov proved monomial positivity of this expression.)

We, with Lam and Pylyavskyy, proved Okounkov's conjecture, and more general Fomin–Fulton–Li–Poon's conjecture.

We'll write $f \geq_s g$ if $f - g$ is Schur positive.

<u>Theorem</u> (Lam–P.–Pylyavskyy, 2007)
For any partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ & $\mu = (\mu_1, \dots, \mu_n)$, let

$$\left\lfloor \frac{\lambda+\mu}{2} \right\rfloor := \left(\left\lfloor \frac{\lambda_1+\mu_1}{2} \right\rfloor, \dots, \left\lfloor \frac{\lambda_n+\mu_n}{2} \right\rfloor \right)$$

$$\left\lceil \frac{\lambda+\mu}{2} \right\rceil := \left(\left\lceil \frac{\lambda_1+\mu_1}{2} \right\rceil, \dots, \left\lceil \frac{\lambda_n+\mu_n}{2} \right\rceil \right)$$

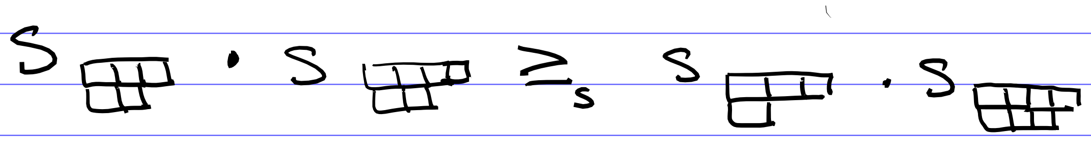
Then we have

$$s_\lambda \cdot s_\mu \leq_s s_{\lfloor \frac{\lambda+\mu}{2} \rfloor} \cdot s_{\lceil \frac{\lambda+\mu}{2} \rceil}$$

<u>Example</u> $\lambda = (3, 1)$, $\mu = (4,3)$

$$\frac{\lambda+\mu}{2} = (3.5, 2).$$

$$s_{32} \cdot s_{42} \geq_s s_{31} \cdot s_{43} \quad \text{or}$$



Log-concave functions $f(x)$:

$$f\left(\frac{x+y}{2}\right)^2 \geq f(x)\, f(y)$$

Schur log-concavity

We can view the previous result as a "Schur log-concavity" property of Schur functions s_λ (viewed as functions of λ)

$$\left(s_{\frac{\lambda+\mu}{2}}\right)^2 \geq_s s_\lambda \cdot s_\mu$$

$$\overset{\lambda}{\bullet} \text{———} \tfrac{\lambda+\mu}{2} \text{———} \overset{\mu}{\bullet}$$

Rougly speaking, $s_\lambda \cdot s_\mu$ "becomes larger" (in Schur positivity sense) if λ, μ become "closer to each other"

$$\overset{\lambda}{\circ} \rightarrow \nu \text{————} \gamma \leftarrow \overset{\mu}{\circ}$$

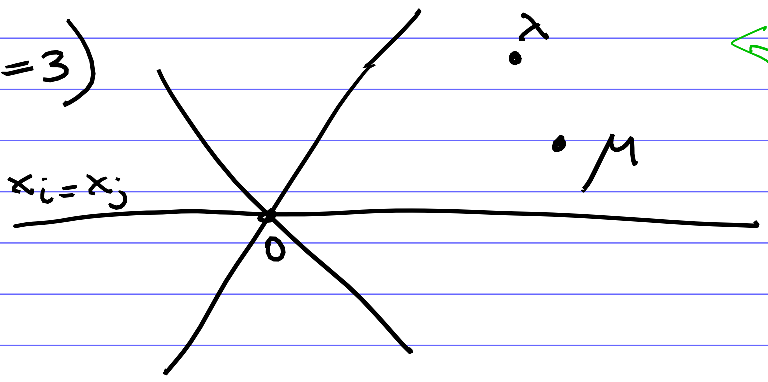
$$s_\nu \cdot s_\gamma \geq_s s_\lambda \cdot s_\mu$$

Let's formulate this more precisely...

Take any partitions

$\lambda = (\lambda_1, \dots, \lambda_n)$ $\qquad \mu = (\mu_1, \dots, \mu_n)$

Ex. $n=3$

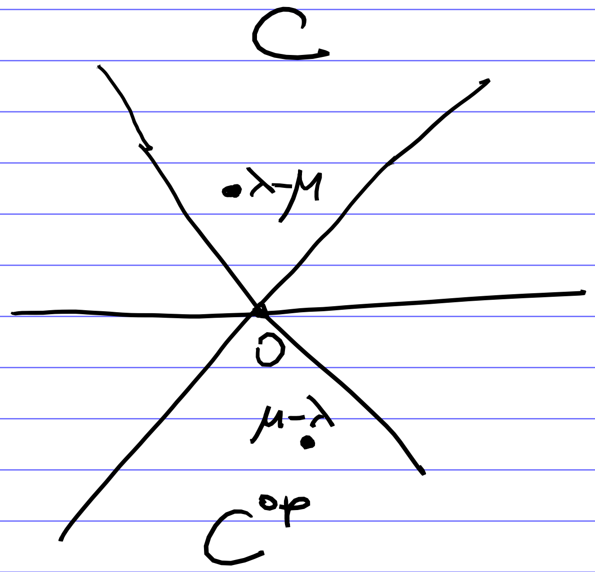


the fundamental Weyl chamber $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n\}$

the braid hyperplane arrangement in $\mathbb{R}^n$

$\binom{n}{2}$ hyperplanes : $x_i = x_j$ cut $\mathbb{R}^n$ into $n!$ regions, called the Weyl chambers

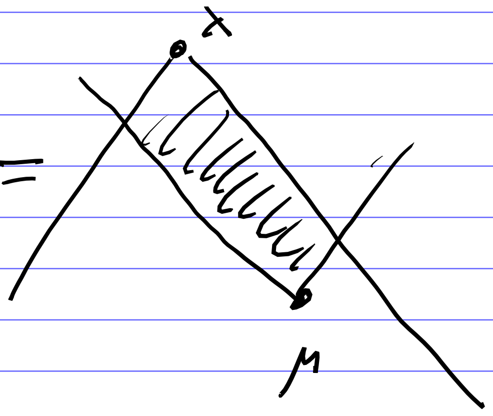
Let us take the Weyl chamber C containing the vector $\lambda - \mu$. (Then the opposite chamber $C^{op} = -C$ contains $\mu - \lambda$).



Define the parallelotope

$$\Pi(\lambda, \mu) := (\lambda + C^{op}) \cap (\mu + C).$$

$\Pi(\lambda, \mu) =$

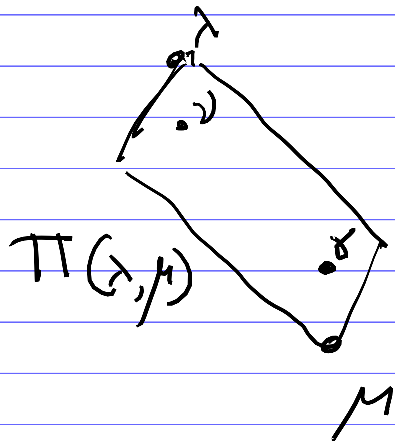


Conjecture (LPP)

Let $\lambda, \mu, \nu, \gamma$ be partitions s.t.

$$\nu, \gamma \in \Pi(\lambda, \mu)$$

$$\lambda + \mu = \nu + \gamma$$

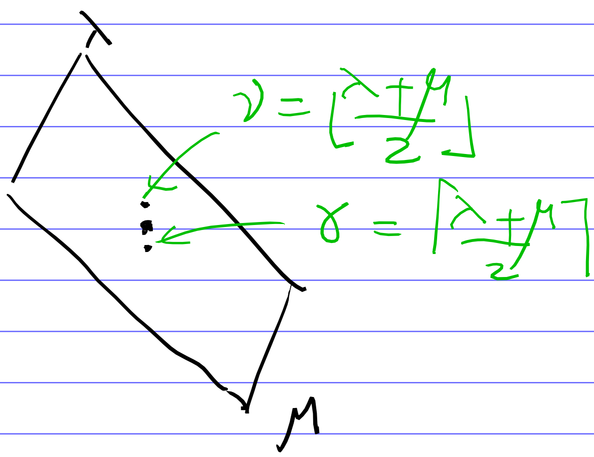


Then

$$s_\nu \cdot s_\gamma \geq_s s_\lambda \cdot s_\mu .$$

There are lot of confirmations of this conjecture, and proven special cases.

Previous Theorem



these are 2 lattice points closest to the midpoint $\frac{\lambda+\mu}{2}$.

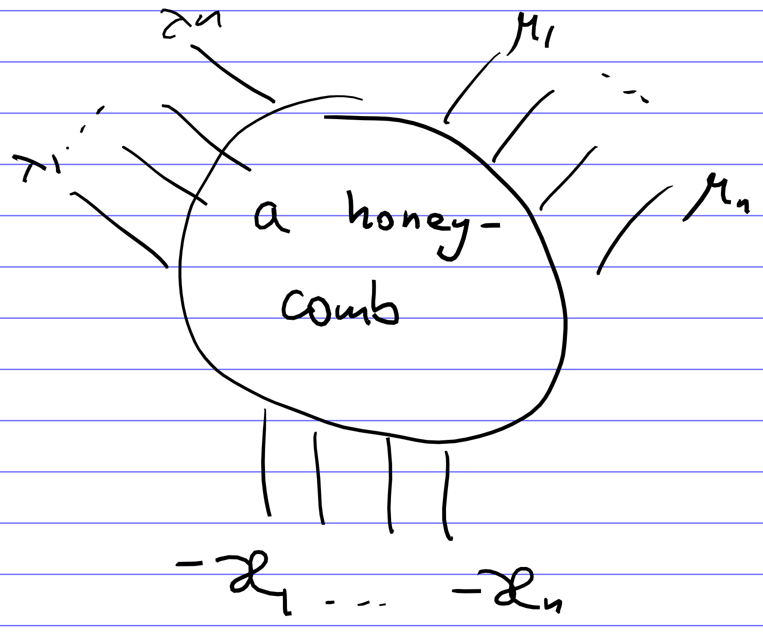
Schur positivity via honeycombs

$$s_\lambda \cdot s_\mu \le_s s_\nu \cdot s_\gamma$$

iff $\quad c^{\varkappa}_{\lambda\mu} \le c^{\varkappa}_{\nu\gamma}$

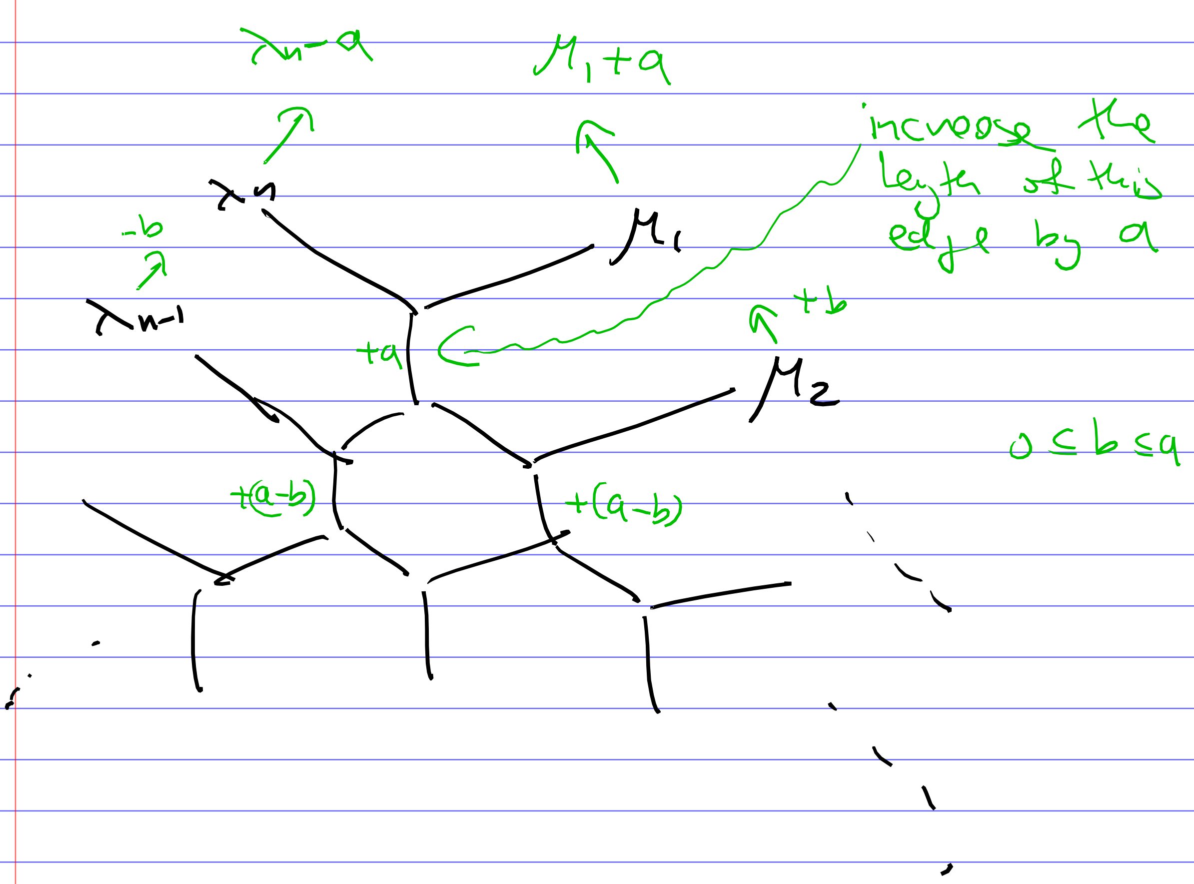
$\forall$ partition $\varkappa$

The Honeycomb LR-rule for $c^{\varkappa}_{\lambda\mu}$:



Q: How can we move the boundary rays for $\lambda_1, \ldots, \lambda_n$ & $\mu_1 \ldots \mu_n$ (without moving the rays for $\varkappa$) so that the space of honeycombs "becomes larger"?

Here is one way to do this:



Theorem. Let $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ be 3 partitions such that $\lambda_n \ge \alpha_1$.

Let $\nu := \lambda - w_0(\alpha)$
$= (\lambda_1 - \alpha_n, \lambda_2 - \alpha_{n-1}, \dots, \lambda_n - \alpha_1)$

$\gamma = \mu + \alpha = (\mu_1 + \alpha_1, \dots, \mu_n + \alpha_n)$.

Then $S_\nu \cdot S_\gamma \ge_s S_\lambda \cdot S_\mu$.

Proof Easy to see from the honeycomb LR-rule that $c_{\lambda\mu}^{\varkappa} \le c_{\nu\gamma}^{\varkappa}$ $\forall \varkappa$. $\square$

Remark In the above theorem, we have $w_0(\lambda) + \mu = w_0(\nu) + \gamma$. In the conjecture we have $\lambda + \mu = \nu + \gamma$.

Can the also prove this conjecture using honeycombs?

A stronger conjecture?

For $\lambda, \mu, \nu, \gamma$ satisfying the conditions $\nu, \gamma \in \Pi(\lambda, \mu)$ & $\lambda + \mu = \nu + \gamma$ and any $\varkappa$ the "polytope of honeycombs" (the BZ-polytope) for $\lambda, \mu, \varkappa$ is contained in a "some transformation" of the polytope for $\nu, \gamma, \varkappa$.

THE END