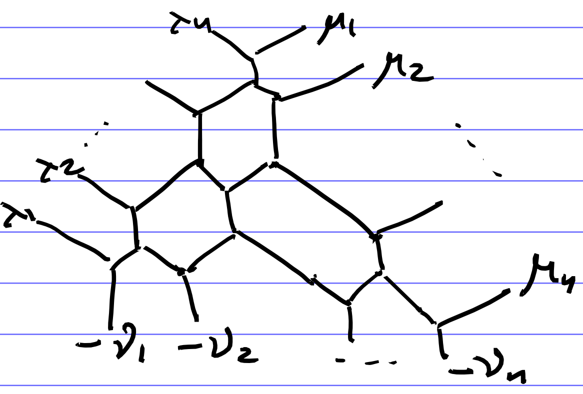
Fix n. Let $\lambda = (\lambda_1 .. \lambda_n)$, $\mu = (\mu_1 .. \mu_n)$, $\nu = (\nu_1 .. \nu_n)$ be 3 partitions such that $|\lambda| + |\mu| = |\nu|$.

Honeycombs are pictures like this drawn on the plane



<u>Saturation Theorem</u> (Knutson-Tao)

If $c^{k\nu}_{k\lambda, k\mu} \neq 0$ for some $k > 0$, then $c^{\nu}_{\lambda\mu} = 0$.

[KT] proved a stronger result:

<u>Theorem.</u> For any honeycomb H (not necessarily integer) there exists a honeycomb $\tilde{H}$ with the same positions of boundary rays such that all coordinates of all line segments in $\tilde{H}$ are integer linear combinations of the coordinates of the boundary rays.

Why does this theorem imply saturation?

<u>Proof of</u> (Theorem $\Rightarrow$ Saturation)

If $c^{k\nu}_{k\lambda, k\mu} \neq 0$, then there exists a honeycomb with boundary rays $k\lambda_1, .., k\lambda_n, k\mu_1, .., k\mu_n, -k\nu_1, .., -k\nu_n$.

$\Rightarrow$ There exists (not nec. integer) honeycomb with boundary rays $\lambda_1, \lambda_2, .., \lambda_n, \mu_1 .. \mu_n, -\nu_1, .., -\nu_n$ (multiply the first honeycomb by $\frac{1}{k}$)

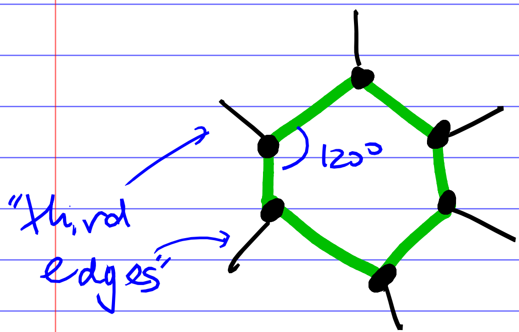
Thm.
$\Rightarrow$ There exist an <u>integer</u> honeycomb with boundary rays $\lambda_1, .., \lambda_n, \mu_1, .., \mu_n, -\nu_1, .., -\nu_n$

$\Rightarrow c^{\nu}_{\lambda\mu} \neq 0$. $\square$

Idea of the proof of the theorem.

Take any honeycomb H.
If H has a "cycle" C, then "deform" the cycle C until one of the edges contracts.

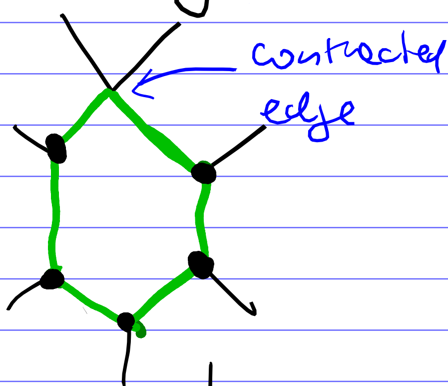
Examples of cycles:

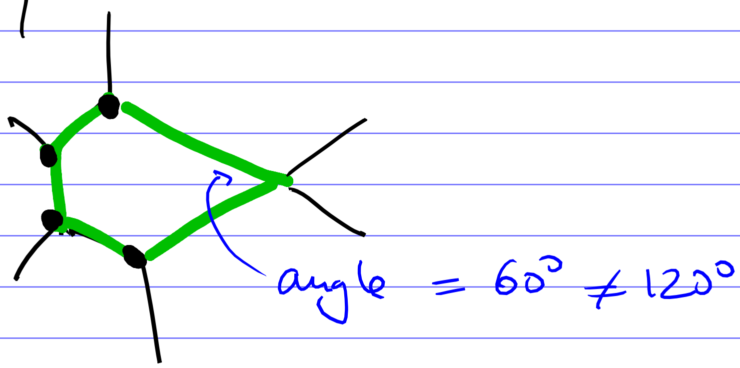


Conditions

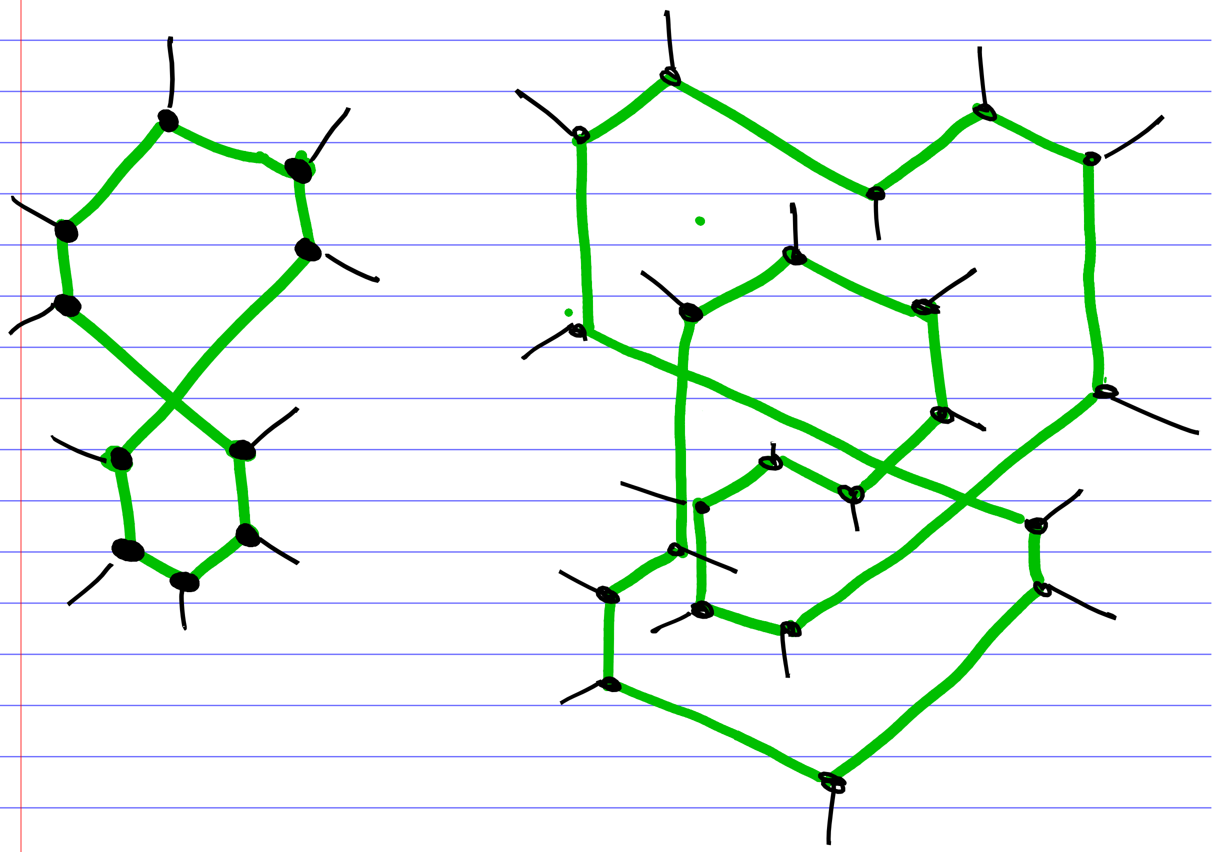
- all angles in C are $120°$
- all edges in C and also the "third edges" at each vertex of C have lengths $\gneq 0$.

Not cycles





But these are cycles

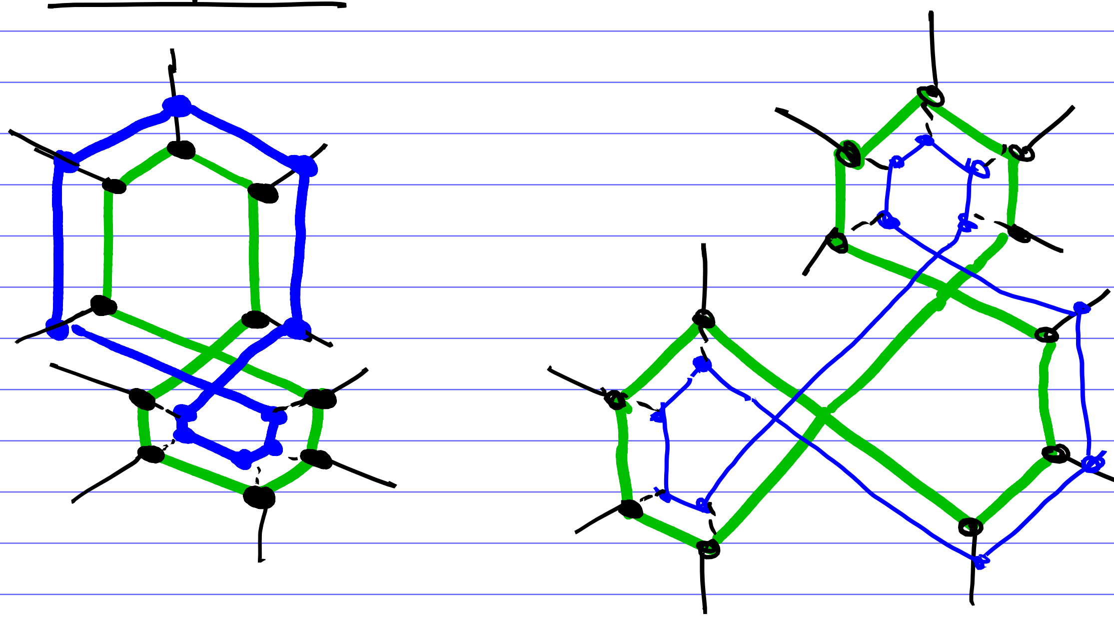


a "cycle" might have self-intersections, segments of edges lying on top of each other.

Basically, a cycle is any picture "like this" that you draw in the honeycomb.

<u>Lemma</u>. these "cycles" can "breathe". More precisely, for a honeycomb H with a cycles C, there is 1-parameter family of deformations of H that change only the coordinates of the lines containing the edges of C.

Examples



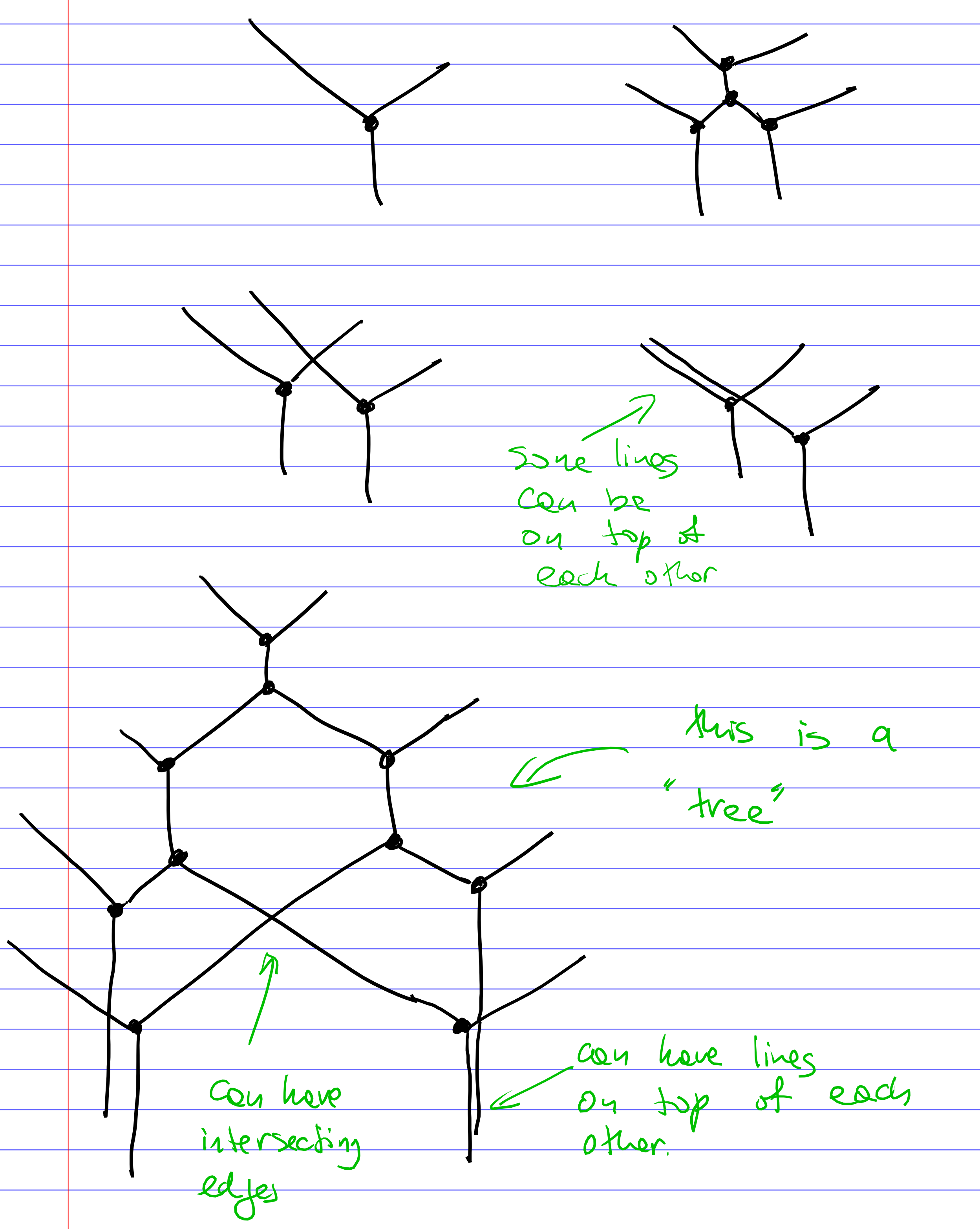
We can deform a cycle like this untill one of its edges or one of "3rd edges" at some vertex of C contracts.

These deformations don't change the positions of the boundary rays of the honeycomb.

We can keep contracting edges like this untill we obtain a honeycomb without cycles.

Lemma Any honeycomb without cycles is a "forest", i.e. one or more "trees" drawn on top of each other.

Examples of "trees" and "forests".



Lemma. In a "tree" honeycomb, we can express coord. of all lines as integer linear comb of the coord. of boundary edges.

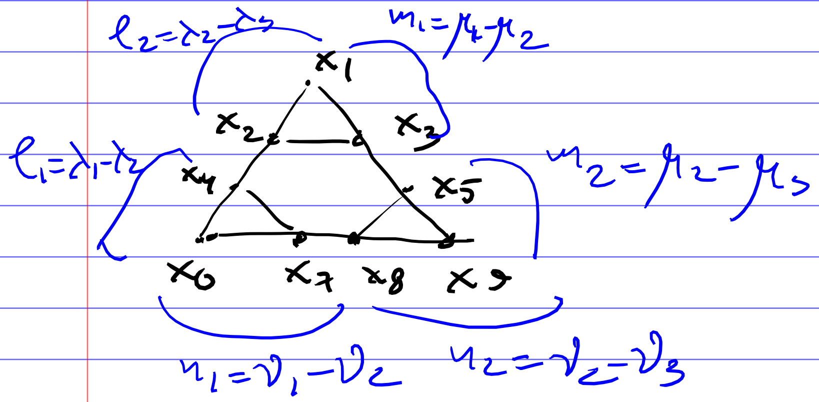
For more details and careful definitions, see

[Knutson, Tao] The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: proof of the saturation conjecture, J. AMS, (1999), 1055–1090.

Berenstein-Zelevinsky polytope

$BZ(\lambda,\mu,\nu) :=$ the polytope in $\mathbb{R}^N$ (where $N \sim \frac{3}{2}n^2$) whose points are $\mathbb{R}$-valued BZ-triangles with boundary conditions given by λ, μ, ν.

Example $n = 3$



$$BZ(\lambda,\mu,\nu) = \{ (x_1,\dots,x_9) \in \mathbb{R}^9_{\geq 0} \mid$$

boundary conditions
$$\begin{cases} x_6 + x_4 = \ell_1, & x_2 + x_1 = \ell_2 \\ x_1 + x_3 = m_1, & x_5 + x_9 = m_2 \\ x_6 + x_7 = n_1, & x_8 + x_9 = n_2 \end{cases}$$

hexagon conditions
$$\begin{cases} x_2 + x_3 = x_8 + x_7 \\ x_3 + x_5 = x_7 + x_4 \end{cases} \}$$

Equivalently $BZ(\lambda,\mu,\nu)$ is the polytope whose points corresp. to honeycombs with boundary rays given by λ, μ, ν.

Even if λ, μ, ν are integer,

the polytope $BZ(\lambda, \mu, \nu)$ might have non-integer vertices.

<u>Exercise</u>. Find 3 partitions λ, μ, ν such that $BZ(\lambda, \mu, \nu)$ has a non-integer vertex.

Present this vertex by a honeycomb or a BZ-triangle.

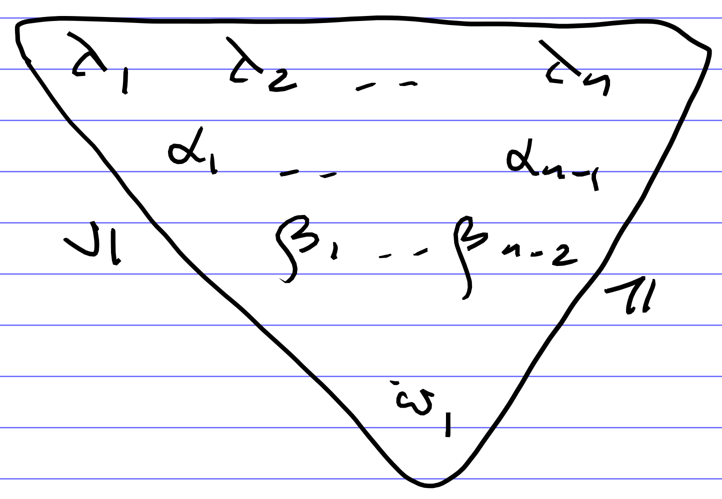
<u>Theorem</u> (Knutson-Tao) For integer λ, μ, ν, the polytope $BZ(\lambda, \mu, \nu)$ has at least 1 integer vertex.

The Berenstein-Zelevinsky polytopes include, as a special case, the Gelfand-Tsetlin polytopes

Gelfand-Tsetlin polytope

$GT(\lambda,\mu) \subset \mathbb{R}^M$ $\quad (M = \binom{n}{2})$

:= the polytope of $\mathbb{R}$-valued Gelfand-Tsetlin patterns for λ & μ.

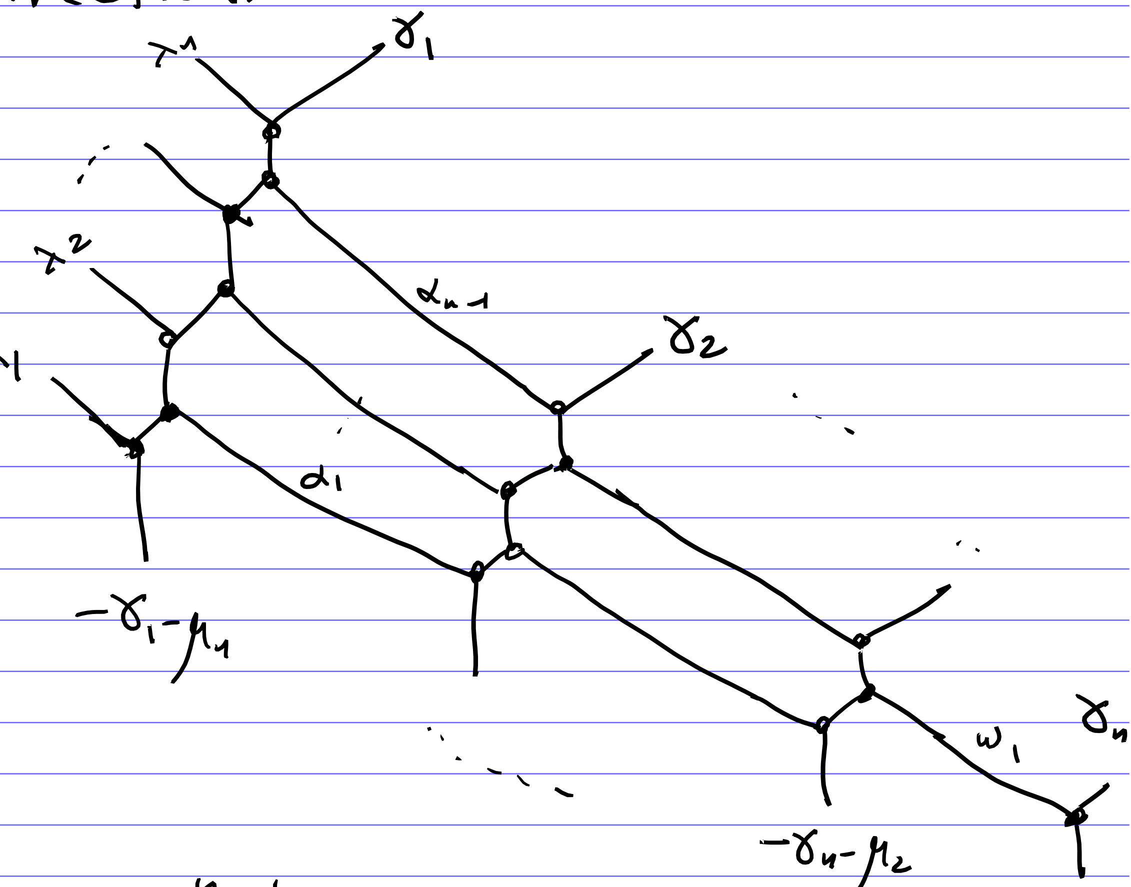


$\mu_i = (n-i)^{th}$ row sum $- (n-i+1)^{st}$ row sum

The Kostka number

$K_{\lambda\mu} := \# \, GT(\lambda,\mu) \cap \mathbb{Z}^M$

Consider honeycombs which are "very streched" in one direction.



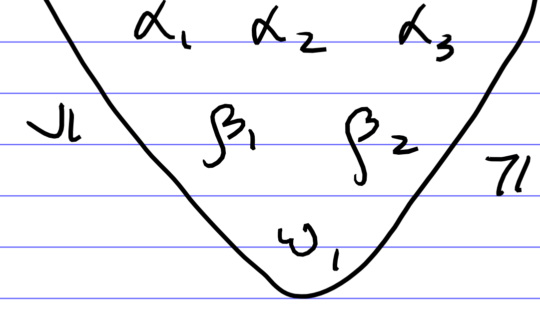
Assume that $-\gamma_n-\mu_1$

all $\gamma_i - \gamma_{i+1}$ are "very large" (compared to the $\lambda_i - \lambda_{i+1}$).

$BZ(\lambda, \gamma, \gamma + (\mu_n, \dots, \mu_1))$ is given by the inequalities "edge lengths" ≥ 0

If parts of γ are very "spread apart" then the inequalities for edges of 1^{st} direction ⟍ are automatically satisfied, so we only need to require the inequalities for edges of directions / and | which are exactly the interlacing conditions for GT patterns.

the above honeycomb $\rightsquigarrow$ GT pattern



<u>Corollary</u> $GT(\lambda, \mu) =$

$$= BZ(\lambda, \gamma, \gamma + (\mu_n, \dots, \mu_1))$$

and, in particular

$$K_{\lambda\mu} = c_{\lambda\gamma}^{\gamma + (\mu_n \dots \mu_1)},$$

the Kostka number

the Littlewood-Richardson coeff.

if $\gamma_1 \gg \gamma_2 \gg \dots \gg \gamma_n$

"much larger"

Actually the Gelfand-Tsetlin polytope $GT(\lambda, \mu)$ (for integer λ & μ) might have non-integer vertices.

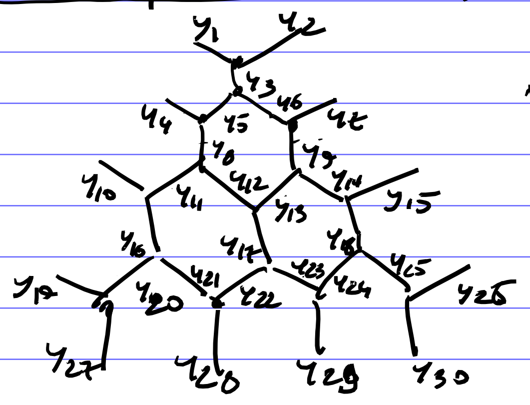
<u>Exercise</u>: Find such non-integer vertex of $GT(\lambda, \mu)$ using honeycombs.

Define the <u>Berenstein-Zelevinsky cone</u> $BZ(n) \subset \mathbb{R}^M$ $(M \sim Cn^2)$ as the cone of all honeycombs of size n (without fixing the positions of the boundary rays)

$\big(BZ(\lambda, \mu, \nu)$ = the section of the cone $BZ(n)$ by the affine subspace given by fixing the coord. of the boundary rays.$\big)$

(Basically, $BZ(n) \sim$ cone of BZ-triangles of size n without boundary conditions.)

<u>Example</u>. $n = 4$



$BZ(4) \subset \mathbb{R}^{30}$
$\ni (y_1, \dots, y_{30})$
y_i's coord. of all lines in the honeycomb.

$$BZ(4) = \{(y_1, \dots, y_{30}) \in \mathbb{R}^{30} \mid y_1 + y_2 + y_3 = 0, \dots, \; y_4 \geq y_6 \geq y_1, \dots \}$$

Let $p: \mathbb{R}^M \to \mathbb{R}^{3 \cdot n}$ be the projection $(y_1, \dots, y_M) \to$ coord. of boundary rays.

<u>Ex</u> For $n = 4$

$$p: (y_1, \dots, y_{30}) \longmapsto (y_1, y_4, y_{10}, \dots)$$

only coord. of the boundary ray

The Klyachko cone is the projection of the Berenstein-Zelevinsky cone

$$\mathrm{Klyachko}(n) = p\left(BZ(n)\right).$$

Equivalently, $(\lambda, \mu, \nu) \in \mathbb{R}^{3n}$ belongs to Klyachko (n) iff $\exists$ ($\mathbb{R}$-valued) honeycomb with boundary rays given by λ, μ, ν.

<u>Saturation theorem</u> $\Longleftrightarrow$

$$\mathrm{Klyachko}(n) \cap \mathbb{Z}^{3 \cdot n}$$

$$= p\left(BZ(n) \cap \mathbb{Z}^M\right).$$

Last time we mentioned some inequalities for the Klyachko cone.

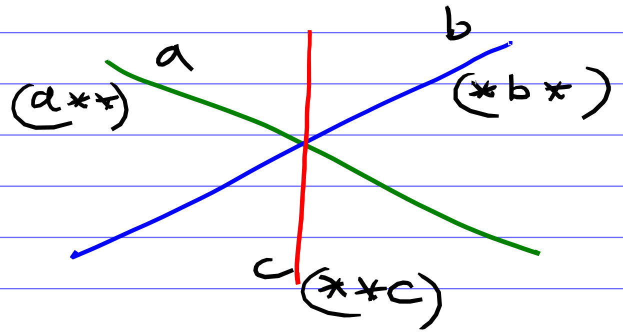
For example, for $(\lambda, \mu, \nu) \in \text{Klyachko}(n)$

$\lambda_1 + \mu_1 \geqslant \nu_1$

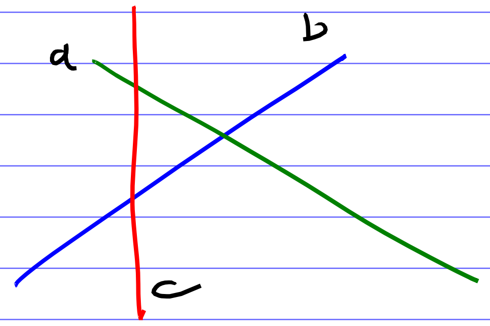
In terms of Hermitian matrices $C = A + B$, this mean the largest eigenvalue ν_1 of C is less than or equal the sum of the largest eigenvalues λ_1 and μ_1 of A and B.

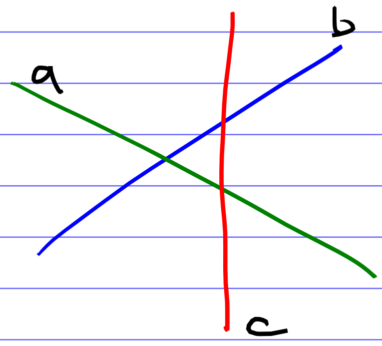
Lets explain this inequality in terms of honeycombs.

We have

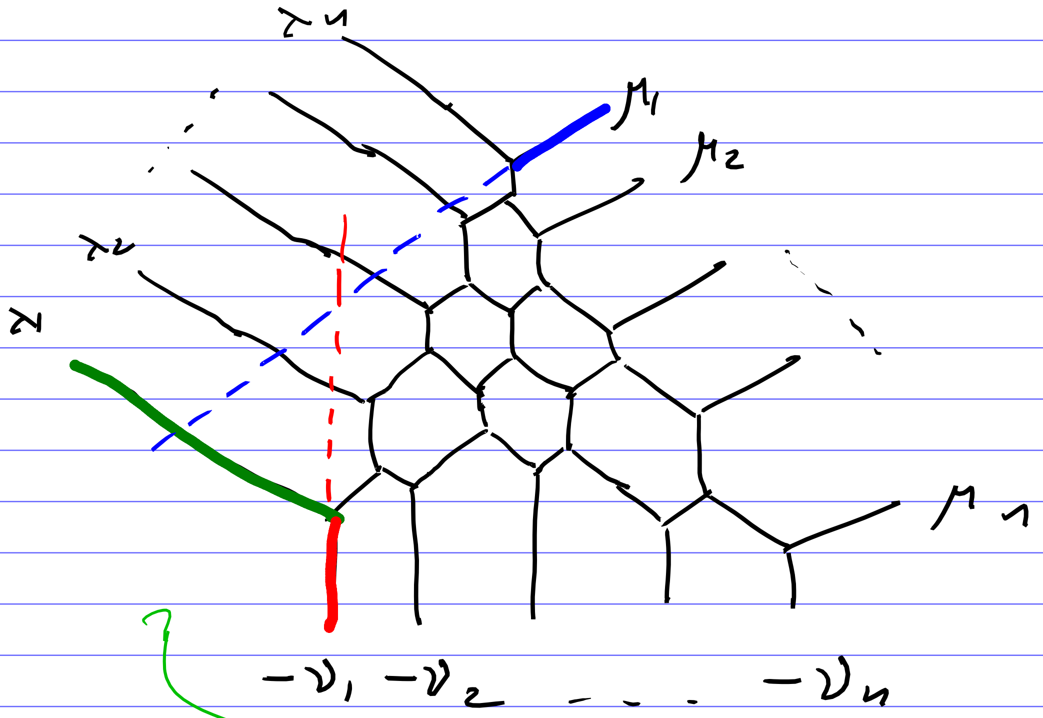


$a+b+c=0$

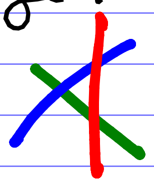




if $a+b+c<0$ if $a+b+c>0$



$\lambda_1 + \mu_1 \geq \nu_1 \iff$ these 3 line are arranged like this

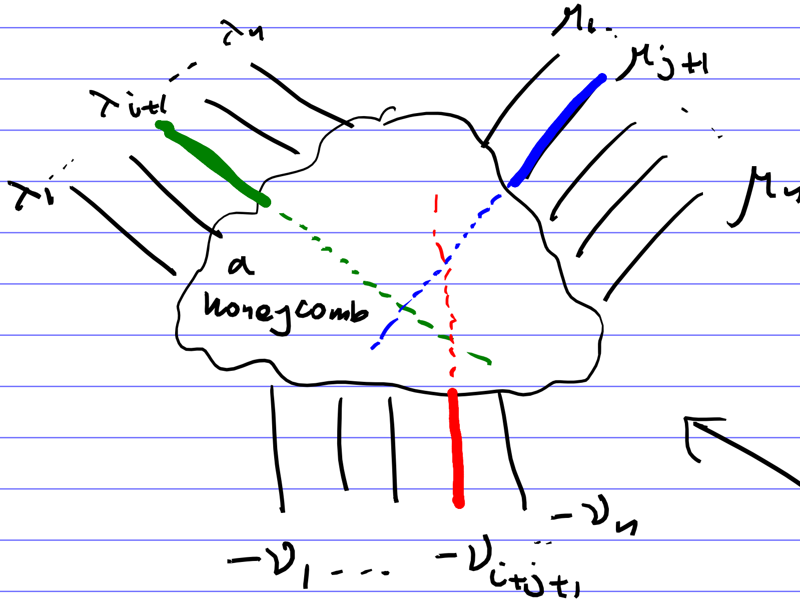


this is easy to see by looking at the honeycomb.

Exercise Prove Weyl's inequality using honeycombs, i.e., show that

$$\lambda_{i+1} + \mu_{j+1} \geq \nu_{i+j+1}$$

for $i, j \geq 0$, $i+j < n$



You need to show that these 3 lines are arranged like this

Let's us now give all inequalities defining the Klyachko cone.

<u>Theorem</u>. The Klyachko cone $\text{Klyachko}(n) \subset \mathbb{R}^{3n}$ is given as follows:

$$(\lambda, \mu, \nu) \in \text{Klyachko}(n) \text{ iff}$$

- $\sum_i \lambda_i + \sum_i \mu_i = \sum_i \nu_i$
- $\lambda_1 \geq \ldots \geq \lambda_n$
  $\mu_1 \geq \ldots \geq \mu_n$
  $\nu_1 \geq \ldots \geq \nu_n$
- $\boxed{\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \geq \sum_{k \in K} \nu_k}$

for any triple of subsets $I, J, K \subset [n]$

$\# I = \# J = \# K = r$

$r \in \{1, \ldots, n-1\}$ such that the Littlewood-Richardson coefficient $c_{\gamma(I), \gamma(J)}^{\gamma(K)} \neq 0$

where $\gamma(I) := (i_r - r, i_{r-1} - (r-1), \ldots, i_1 - 1)$ for $I = \{i_1 < \ldots < i_r\}$

the partition associated with subset I.

This gives answers to the following 2 questions:

- $\exists$ Hermitian $n \times n$ matrices $A + B = C$ with eigenvalues $(\lambda_1, \dots \lambda_n)$, $(\mu_1 \dots \mu_n)$ $(\nu_1 \dots \nu_n)$ iff $(\lambda, \mu, \nu) \in$ Klyachko(n)

- $c_{\lambda\mu}^{\nu} \neq 0$ iff $(\lambda, \mu, \nu) \in$ Klyachko$(n) \cap \mathbb{Z}^{3 \cdot n}$.

Horn conjectured these inequalities for eigenvalues of Hermitian matrices.
Klyachko proved that they are necessary & sufficient.
Knutson-Tao proved that this describes all $c_{\lambda\mu}^{\nu} \neq 0$.

Thm says that non-zero LR-coefficients are described in terms of non-zero LR-coeffs. ???

For any partitions λ, μ, ν with $\leq n$ parts (that can be arbitrarily large) in order to figure out whether $c_{\lambda\mu}^{\nu} \neq 0$ we need to know whether $c_{\tilde{\lambda}\tilde{\mu}}^{\tilde{\nu}} \neq 0$ for $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \subseteq r \times (n-r)$ for some $r \in \{1, .., n-1\}$.

There are finitely many such $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$'s and all of them have $\lneq n$ parts.

So Theorem gives a <u>recursive</u> description of triples of partitions λ, μ, ν with $c_{\lambda\mu}^{\nu} \neq 0$.

<u>Open Problem.</u> Is there a non-recursive description of such triples of partitions?

An application of honeycombs:

The PRV conjecture

(after Parthasarathy, Ranga Rao, Varadarajan)

Proved by Kumar and Mathieu.

For type A.

Theorem Let

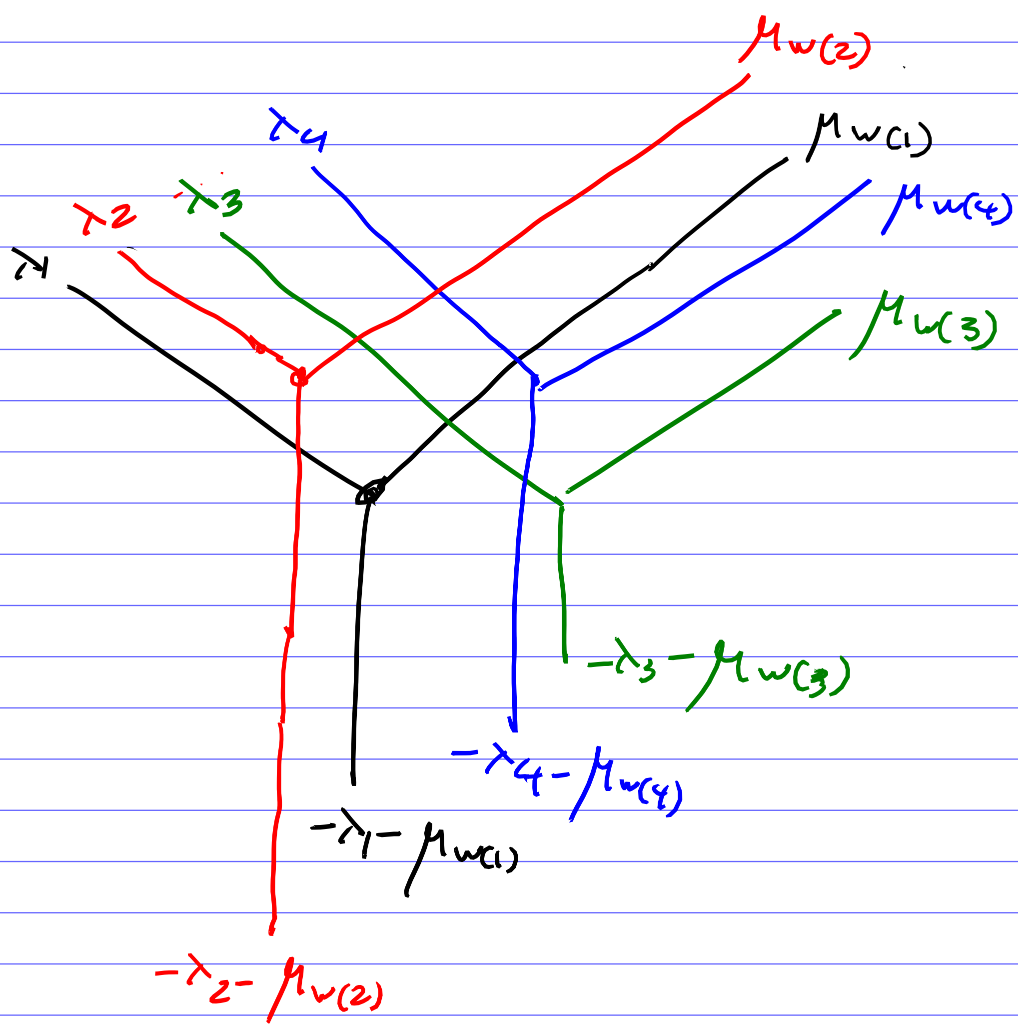
$\lambda = (\lambda_1 \dots \lambda_n)$, $\mu = (\mu_1 \dots \mu_n)$ be two partitions, and $w \in S_n$

ν - weakly decreasing rearrangement of parts of $(\lambda_1 + \mu_{w(1)}, \lambda_2 + \mu_{w(2)}, \dots, \lambda_n + \mu_{w(n)})$.

Then $c_{\lambda\mu}^{\nu} \neq 0$.

Let's explain PRV conjecture using honeycombs:

<u>Proof</u>. Consider the honeycomb obtained by a union of several "Y"s.



This is an integer honeycomb.

So $c_{\lambda\mu}^{\nu} \neq 0$. $\square$

Special case:

$\lambda = \mu = (n, n-1, \ldots, 1)$

PRV vectors = "sums of 2 permutations"

$(1 + w(1), 2 + w(2), \ldots, n + w(n))$

(rearranged in decreasing order)

for all $w \in S_n$

Problem: Give a combinatorial description of PRV vectors.