18.217 Lecture 30 11/13/2020

The Littlewood-Richardson coeffs. (Cont'd)

last time:

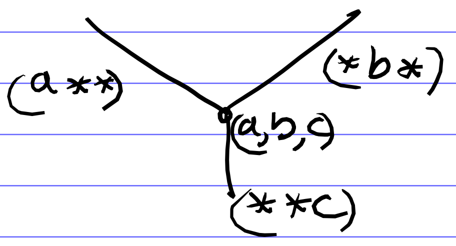
Littlewood-Richardson $\quad\quad\quad\quad$ Berenstein-Zelevinsky

$\{\text{LR-tableaux}\} \overset{?}{\longleftrightarrow} \{\text{BZ-triangles}\}$

$\{\text{KT-honeycombs}\}$ (Knutson-Tao) — with "?" between it and the LR-tableaux, and an arrow to the BZ-triangles

Recall, a Knutson-Tao's honeycomb is "honeycomb-like" picture drawn on the plane $P = \{(x,y,z) \mid x+y+z=0\} \subset \mathbb{R}^3$ with line segments & rays belonging to lines of the 3 types:

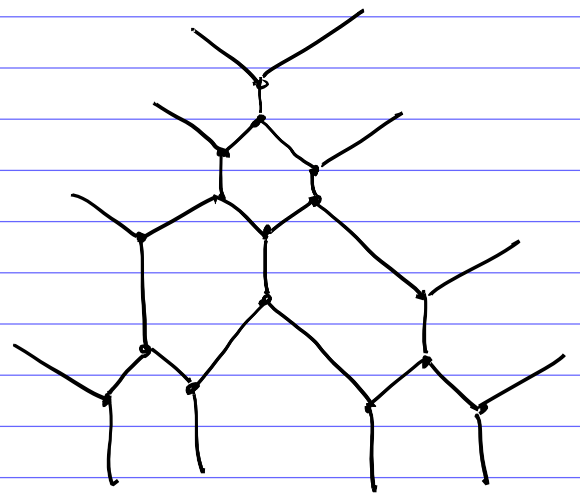
$(a,*,*)$, $(*,b,*)$, $(*,*,c)$.



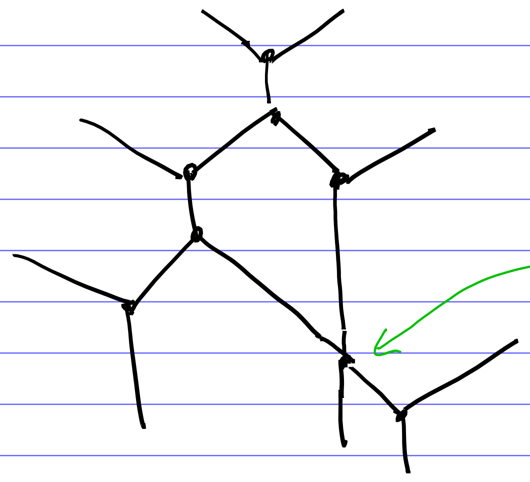
$a+b+c=0$

Ex.

n=4

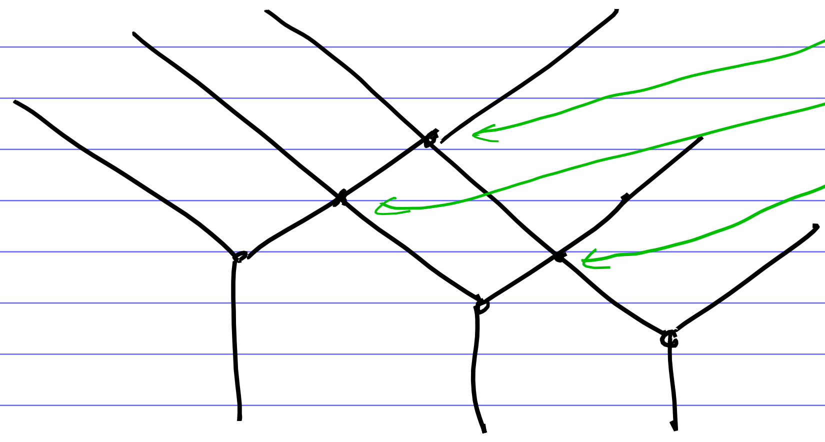


n=3



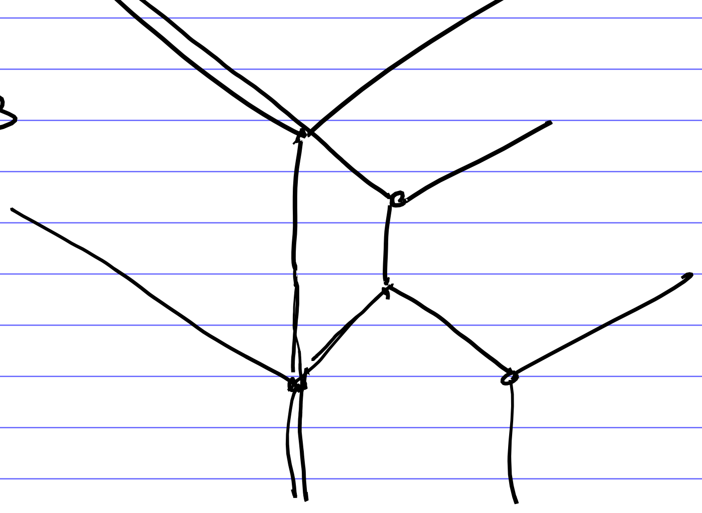
a honeycomb with 1 contracted edge

n=3



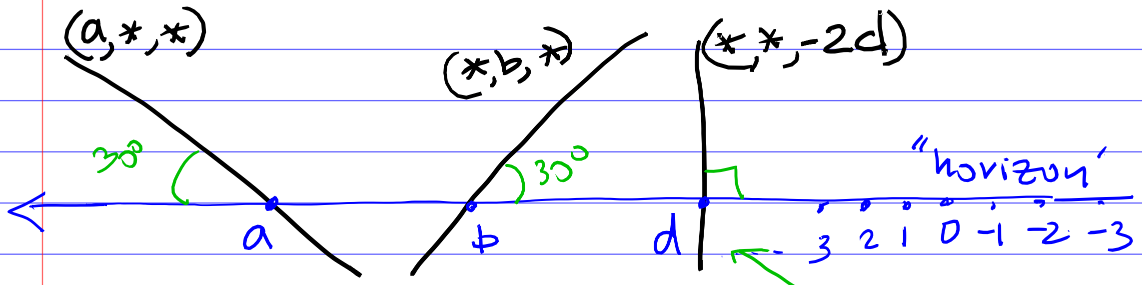
even more degenerate honeycomb with 3 contracted edges

n=3



also a valid honeycomb

To be more explicit

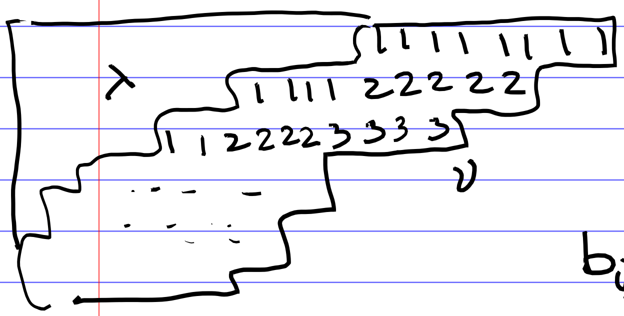


For an integer honeycomb,

$a, b \in \mathbb{Z}$ and $d \in \frac{1}{2}\mathbb{Z}$.

Fix n, $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n)$, $\nu = (\nu_1, \dots, \nu_n)$ partitions

$c_{\lambda\mu}^{\nu} = \#\{$LR-tableaux of shape ν/λ and weight $\mu\}$

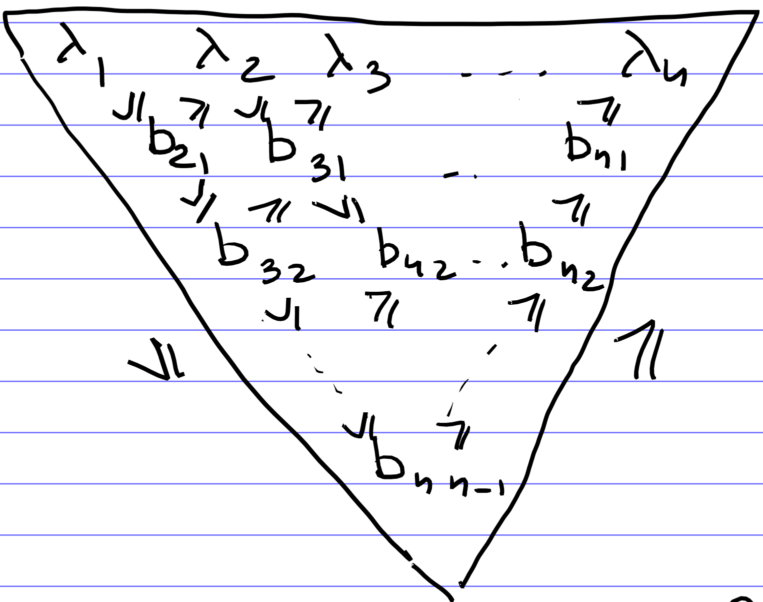


$a_{ij} :=$ #j-s in i^{th} row

$b_{ij} := \lambda_i + a_{i1} + \dots + a_{ij}$

$= \lambda_i + \#\{$entries $\le j$ in i^{th} row$\}$

Then we obtain a GT-pattern



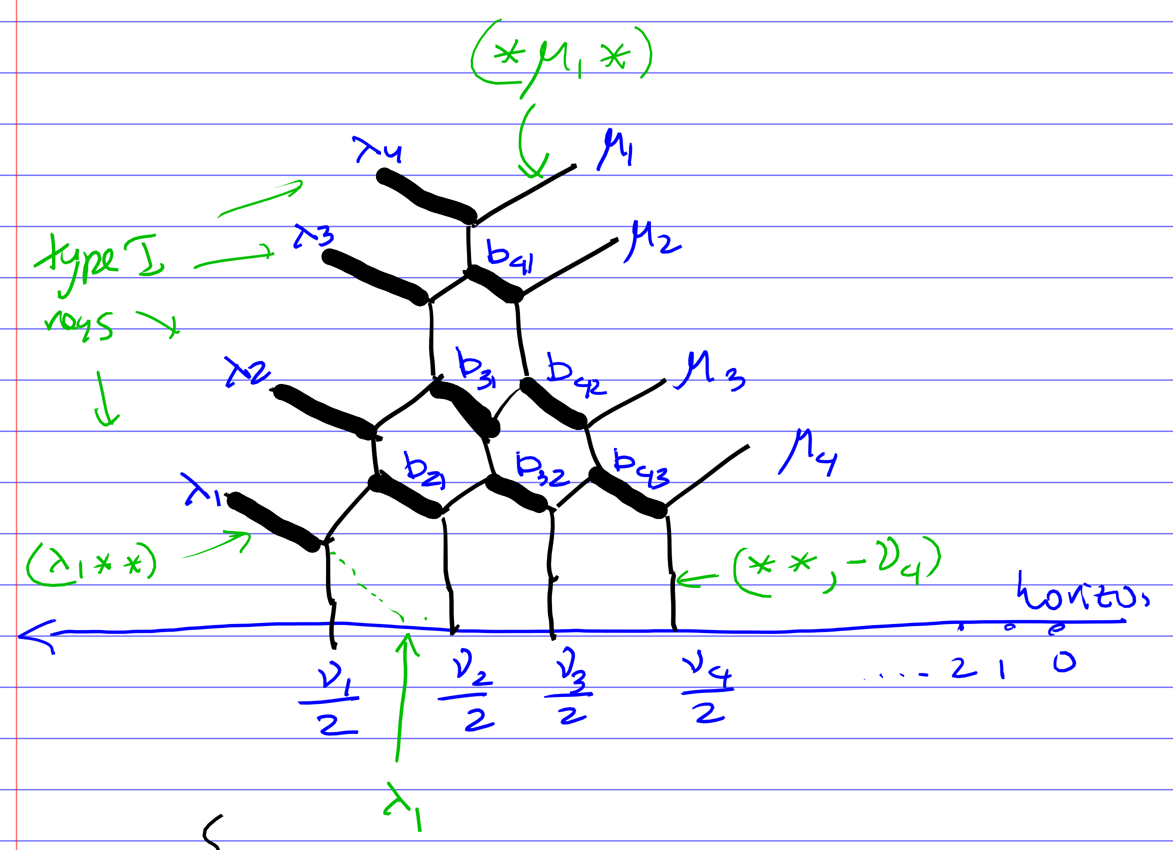
$\{$LR-tableaux$\} \overset{b_{ij}}{\longleftrightarrow} \{$such GT-patterns satisfying some additional linear (in)equalities$\}$

GT-conditions + weight conditions + lattice word conditions

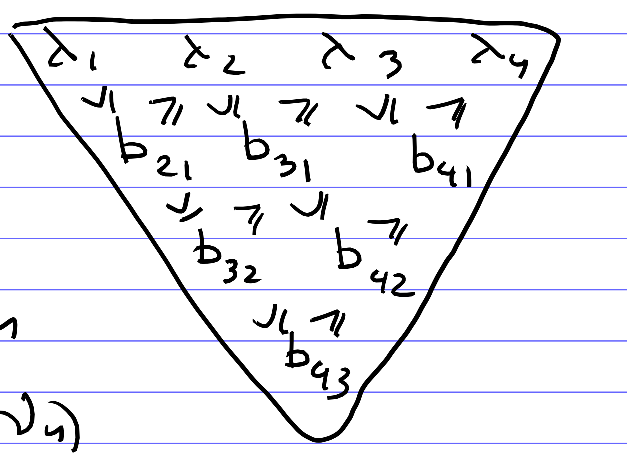
Bijection: $\left\{\begin{array}{l}\text{integer honeycombs} \\ \text{with boundary rays} \\ \text{type I}: \lambda_1 \dots \lambda_n \\ \text{type II}: \mu_1 \dots \mu_n \\ \text{type III}: -\nu_n, -\nu_{n-1}, \dots, -\nu_1\end{array}\right\}$

$\Big\updownarrow$ bij.

$\left\{\begin{array}{l}\text{LR-tableaux of shape } \nu/\lambda \\ \text{and weight } \mu\end{array}\right\}$

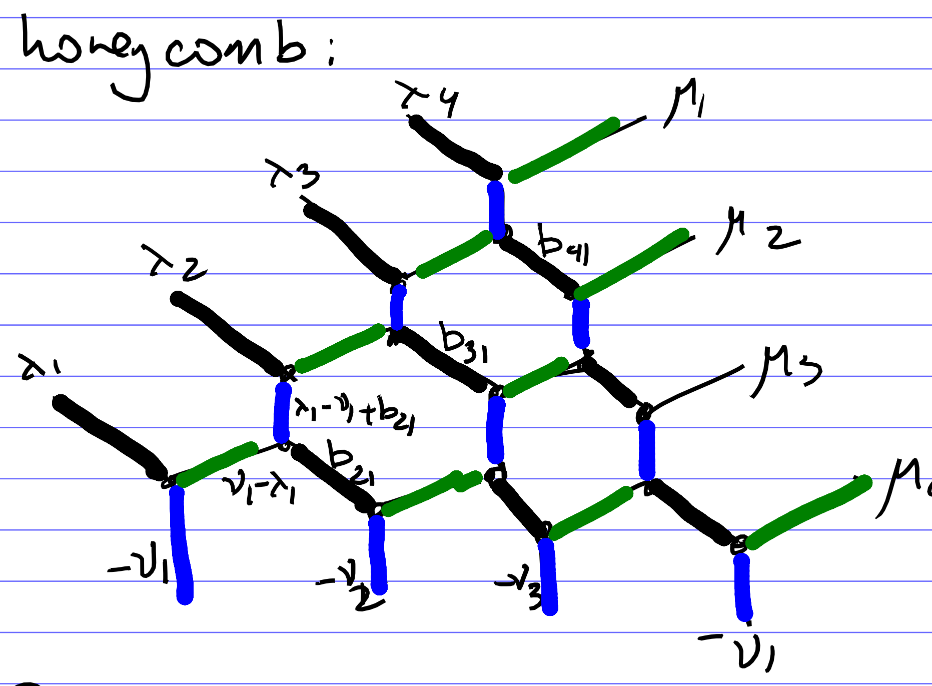


GT-pattern:

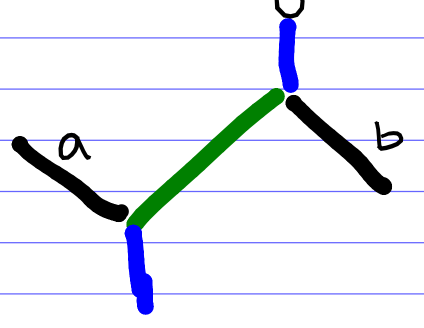


This GT-pattern & vector $(\nu_1 \dots \nu_n)$ uniquely determine the honeycomb.

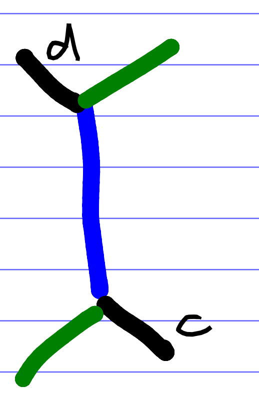
Indeed, we can recover (from λ, ν, (b_{ij})) the coordinates of all line segments in the honeycomb:



GT-interlacing conditions $\Longleftrightarrow$ all line segments of type II & III have "lengths" ≥ 0.



"length" of green edge $= a - b \geq 0$

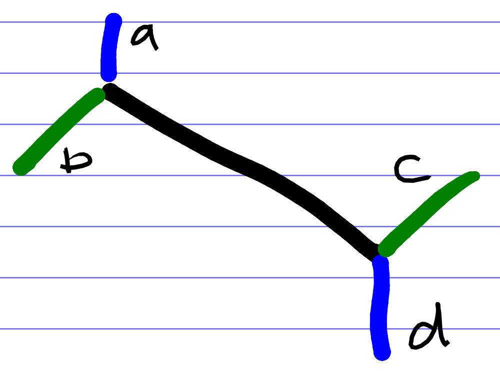


"length" of blue edge $= c - d \geq 0$

In order to have a valid honeycomb, in addition to this GT-conditions we need to require:

(*) The coordinates of the boundary rays of type II, which can be expressed in terms of λ, ν, b_{ij}'s are $\mu_1 \dots \mu_n$

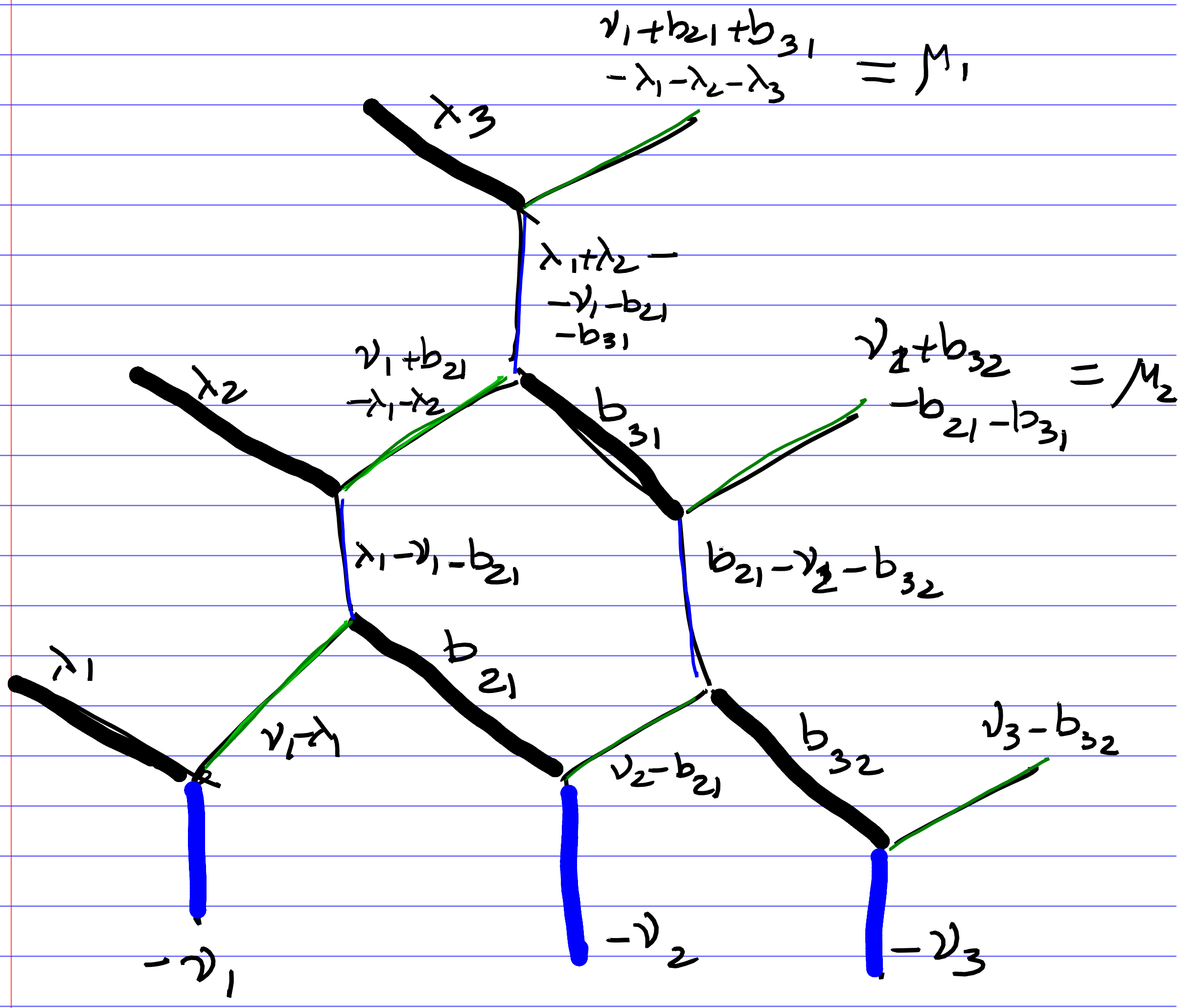
(**) The "lengths" of line segments of type I are ≥ 0



$d - a = b - c \geq 0$

Exercise: Check that (*) & (**) are exactly the "weight condition" and "lattice word conditions" that we need to add to GT-conditions to get LR-tableaux.

Example $n=3$



Conditions for a valid honeycomb:

$$\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \geq \; \geq & & \geq \; \geq & \\ & b_{21} & & b_{31} & \\ & & \geq \; \geq & & \\ & & b_{32} & & \end{array}$$

GT interlacing conditions

(*) $\mu_1 = \nu_1 + b_{21} + b_{31} - \lambda_1 - \lambda_2 - \lambda_3$

$\mu_2 = \nu_2 + b_{32} - b_{21} - b_{31}$

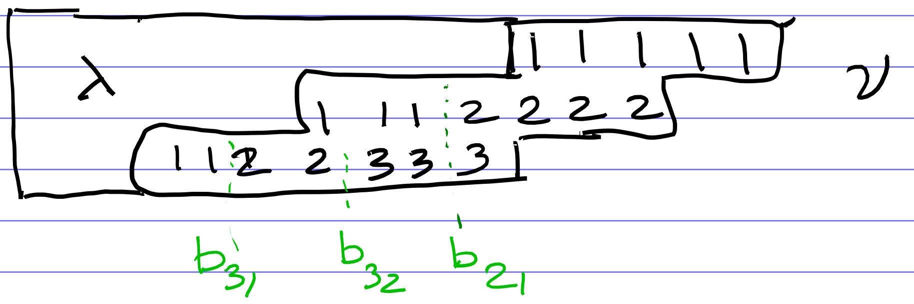
$\mu_3 = \nu_3 - b_{32}$

(**) $\nu_1 - \lambda_1 \geq \nu_2 - b_{21}$

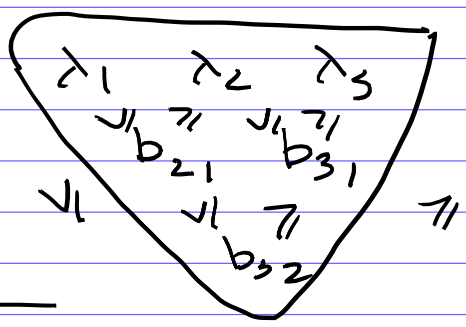
$$\nu_1 + b_{21} - \lambda_1 - \lambda_2 \geq \nu_2 + b_{32} - b_{21} - b_{31}$$

$$\nu_2 - b_{21} \geq \nu_3 - b_{31}$$

These are exactly the conditions needed to get a valid LR-tableau:



interlacing conditions:



(*) $\iff$ weight conditions.

$$\# 1's = (\nu_1 - \lambda_1) + (b_{21} - \lambda_2) + (b_{31} - \lambda_3) = \mu_1$$

$$\# 2's = (\nu_2 - b_{21}) + (b_{32} - b_{31}) = \mu_2$$

$$\# 3's = \nu_3 - b_{32} = \mu_2$$

(**) $\iff$ lattice word conditions

rev. reading word =

$$\underbrace{1 \dots 1}_{\nu_1 - \lambda_1} \; \underbrace{2 \dots 2}_{\nu_2 - b_{21}} \; \underbrace{1 \dots 1}_{b_{21} - \lambda_2}$$

$$\underbrace{3 \dots 3}_{\nu_3 - b_{32}} \; \underbrace{2 \dots 2}_{b_{32} - b_{31}} \; \underbrace{1 \dots 1}_{b_{31} - \lambda_3}$$

conditions: $\nu_1 - \lambda_1 \geq \nu_2 - b_{21}$

$\nu_1 - b_{21} \geq \nu_3 - b_{32}$

$$(\nu_1 - \lambda_1) + (b_{21} - \lambda_2) \geq (\nu_2 - b_{21}) + (b_{32} - b_{31}).$$

So we proved the "honeycomb LR-rule"

Theorem $c_{\lambda\mu}^{\nu}$ = # integer honeycombs with boundary rays given by the entries of λ, μ, ν (as shown above).

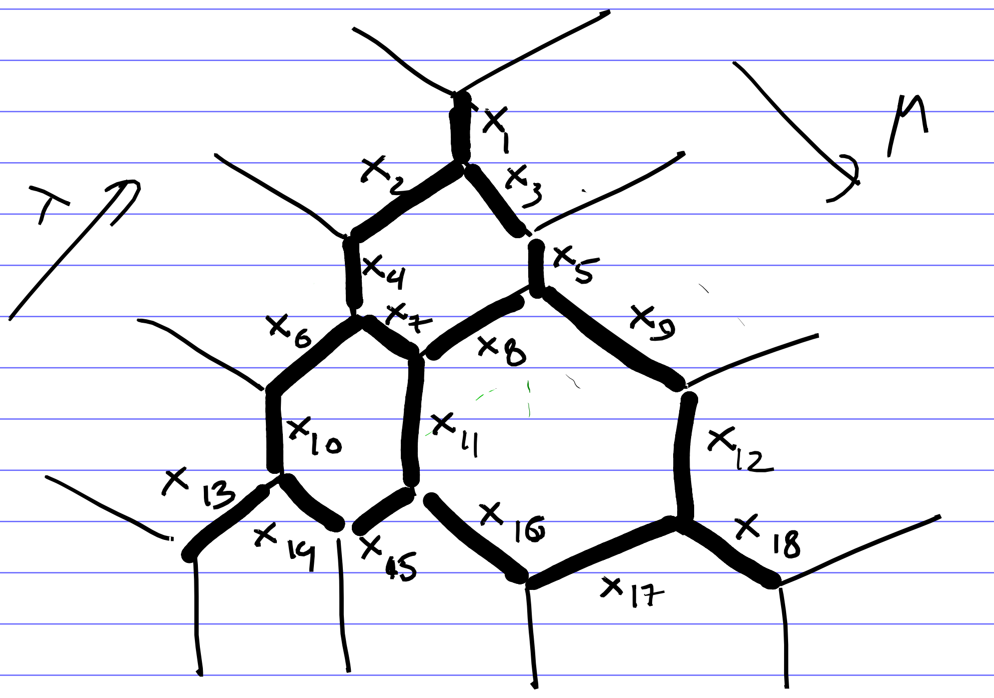
Bijection: $\left\{ \begin{array}{c} \text{(integer)} \\ \text{honeycombs} \end{array} \right\}$

$\updownarrow$

$\left\{ \begin{array}{c} \text{(integer) Berenstein-} \\ \text{Zelevinsky triangles} \end{array} \right\}$

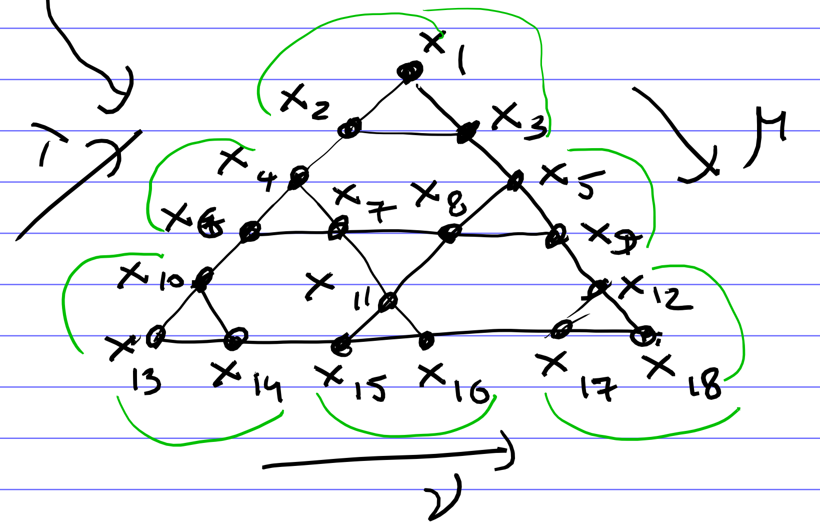
a honeycomb $\longmapsto$ the array of "lengths" of all its line segments

Example $n=4$



$\nu^* = (-\nu_4, \dots, -\nu_1)$

$x_1, x_2, \dots, x_{16} \in \mathbb{Z}_{\geq 0}$ are edge lengths in the honeycomb



Corresponding BZ-triangle

nonnegativity: $x_1, \dots, x_{18} \in \mathbb{Z}_{\geq 0}$

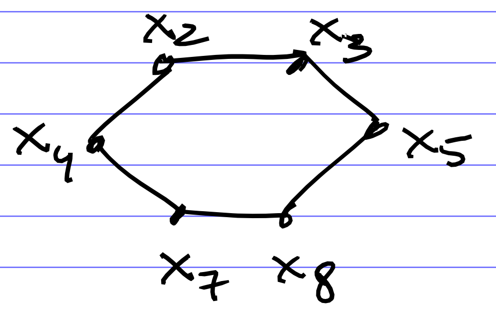
boundary conditions:

$$x_{13} + x_{10} = \lambda_1 - \lambda_2$$
$$x_6 + x_4 = \lambda_2 - \lambda_3$$
$$x_2 + x_1 = \lambda_3 - \lambda_4$$

$$x_1 + x_3 = \mu_1 - \mu_2$$
$$x_5 + x_9 = \mu_2 - \mu_3$$
$$x_{12} + x_{18} = \mu_3 - \mu_4$$

$$x_{13} + x_{14} = \nu_1 - \nu_2$$
$$x_{15} + x_{16} = \nu_2 - \nu_3$$
$$x_{17} + x_{18} = \nu_3 - \nu_4$$

hexagon conditions:



$$x_2 + x_3 = x_7 + x_8$$
$$x_3 + x_5 = x_7 + x_4$$
$$x_5 + x_8 = x_4 + x_2$$

& similar conditions for 2 other hexagons inside the BZ-triangle.

So we proved the BZ-version of LR-rule.

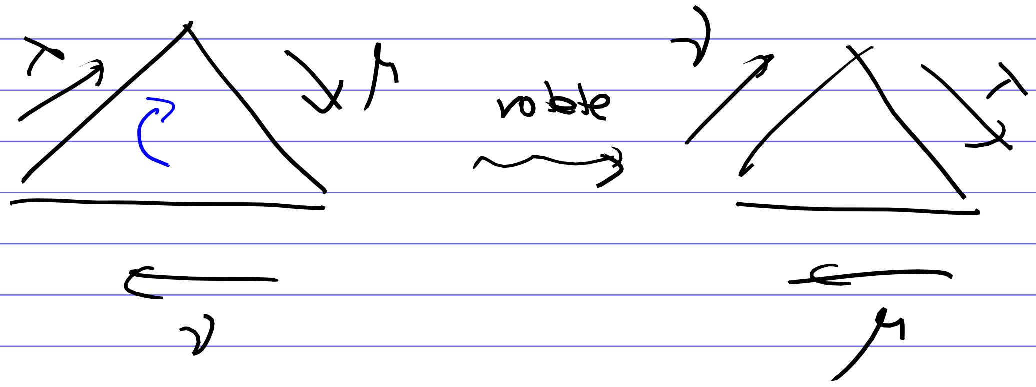
<u>Theorem</u> $c_{\lambda\mu}^{\nu}$ = # integer BZ triangles with boundary conditions given by λ, μ, ν.

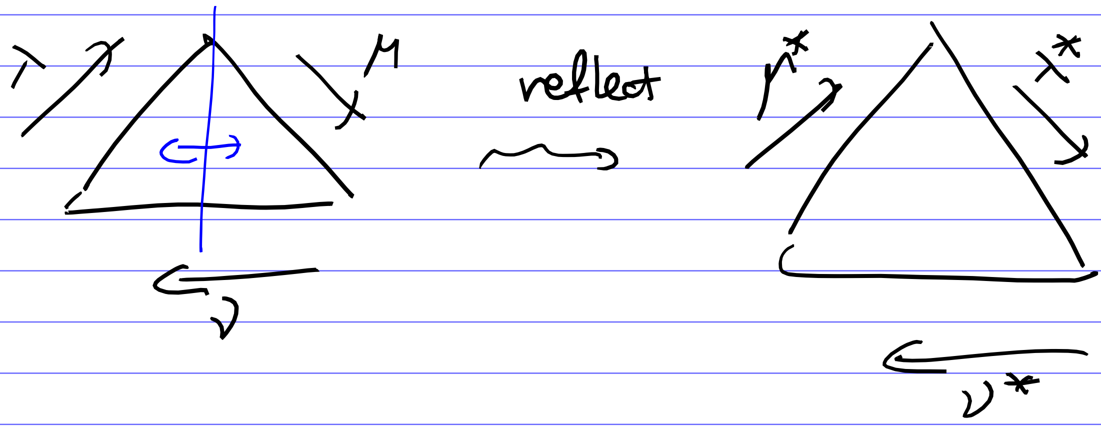
<u>Symmetries</u> of LR-coeffs.

$$\tilde{c}_{\lambda\mu\nu} = c_{\lambda\mu}^{\nu^*}$$

$$\nu = (\nu_1 \dots \nu_n), \quad \nu^* = (-\nu_n, \dots, -\nu_1)$$

We can <u>rotate</u> & <u>reflect</u> BZ-triangles (or honeycombs)





So the BZ-version (or honeycomb version) of the LR-rule explains the following symmetries:

<u>Corollary</u> (1) $\tilde{c}_{\lambda\mu\nu} = \tilde{c}_{\mu\nu\lambda} = \tilde{c}_{\nu\lambda\mu}$.

(2) $\tilde{c}_{\lambda\mu\nu} = \tilde{c}_{\mu^*\lambda^*\nu^*}$.

Q: How about the "most obvious" symmetry $\boxed{c^{\nu}_{\lambda\mu} = c^{\nu}_{\mu\lambda}}$

($\Leftrightarrow$ the <u>commutativity</u> of products of Schur functions)

$$s_\lambda \cdot s_\mu = s_\mu \cdot s_\lambda = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu$$

Actually, this is the <u>most nontrivial</u> symmetry of LR-coeffs.

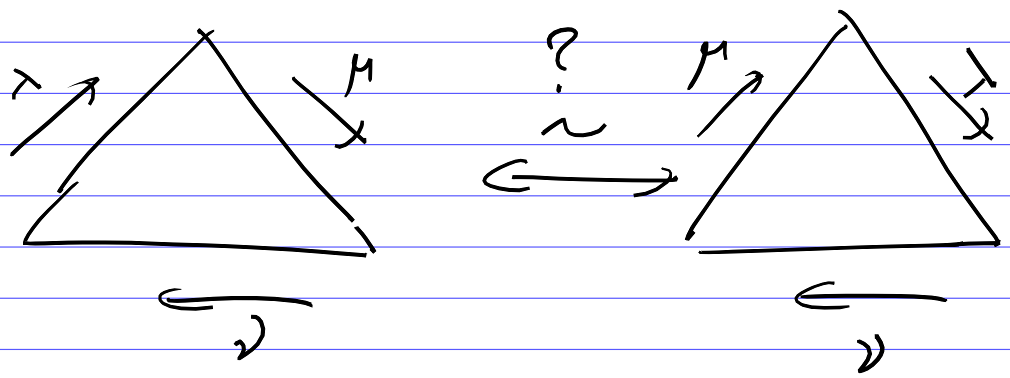
$$\left\{\begin{array}{c}\text{BZ-triangles}\\ \text{for } \lambda, \mu, \nu\end{array}\right\} \xleftrightarrow{\text{bij.}} \left\{\begin{array}{c}\text{BZ-triangles}\\ \text{for } \mu, \lambda, \nu\end{array}\right\}$$

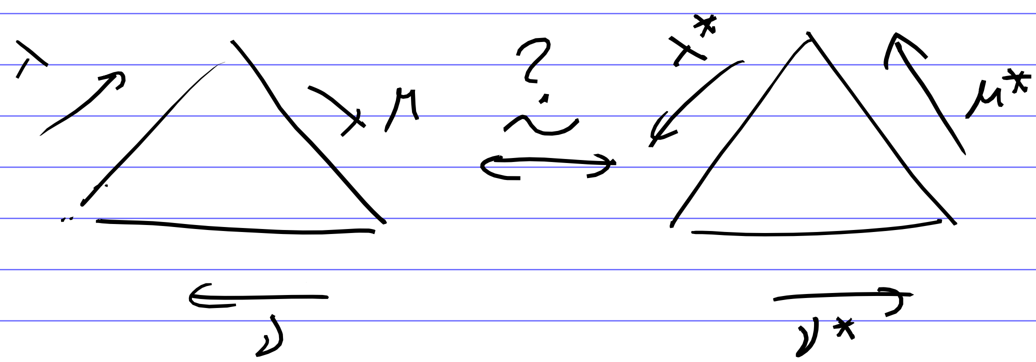
a certain nontrivial piecewise linear transformation of BZ-patterns.

Non-trivial symmetry

$$\tilde{c}_{\lambda\mu\nu} \overset{?}{=} \tilde{c}_{\mu\lambda\nu}$$

$$\tilde{c}_{\lambda\mu\nu} \overset{?}{=} \tilde{c}_{\lambda^*\mu^*\nu^*}$$





How about the symmetry

$$c_{\lambda\mu}^{\nu} = c_{\lambda'\mu'}^{\nu'} \quad ?$$

$\lambda \leftrightarrow \lambda'$ the conjugate partition

Knutson-Tao-Woodward puzzles

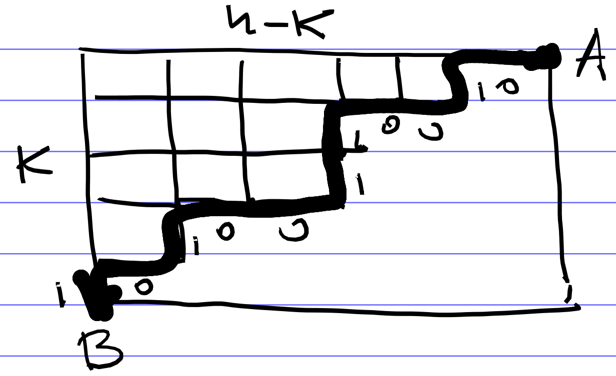
Fix $n \geq k \geq 0$.

(Now we are thinking about LR-coefficients in terms of the Grassmannian $Gr(k,n)$).

$\lambda, \mu, \nu \subset k \times (n-k)$

$\lambda \leftrightarrow$ 01-vector with k 1's and $(n-k)$ 0's

Ex

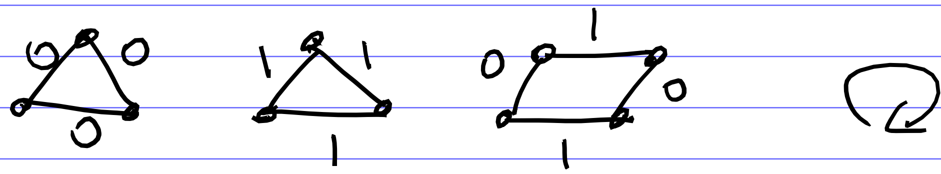


path from A to B

$\lambda = (5,3,3,1) \mapsto (0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1)$

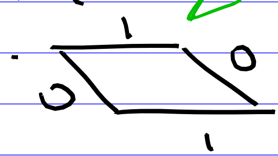
$\lambda, \mu, \nu \rightsquigarrow 3$ 01-vectors

A <u>puzzle</u> is a tiling of an equilateral triangle with sides of length n by the following <u>puzzle pieces</u>:



and all their rotations

(but not reflections, we <u>don't</u> have the puzzle piece



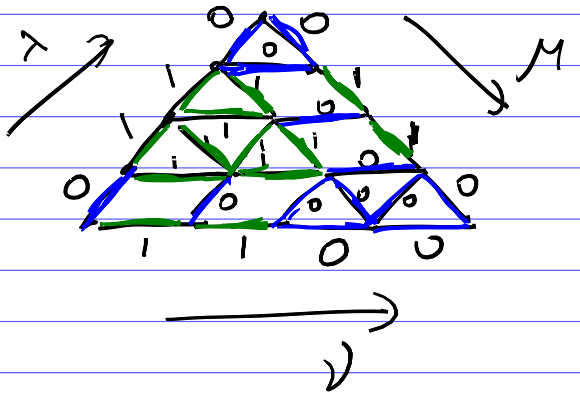
)

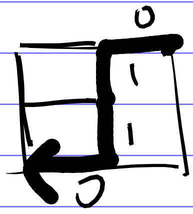
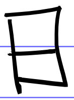
not allowed

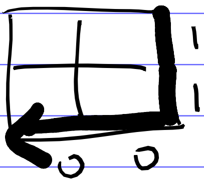
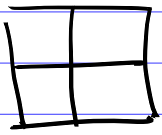
such that any two puzzle pieces sharing a side have the same labelling of the side.

The 01-vectors appearing on the sides of the big triangle correspond to λ, μ, ν.

Examples $\quad n=4, \quad k=2$



$\lambda = \mu =$  $=$ 

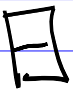
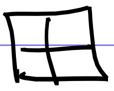
$\nu =$  $=$ 

Theorem (Knutson–Tao–Woodward)

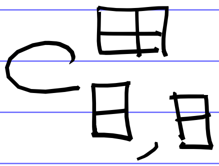
For $\lambda, \mu, \nu \subseteq k \times (n-k)$

$c_{\lambda\mu}^{\nu}$ = # puzzles with sides given by λ, μ, ν (as shown above)

Example For $n=4, \quad k=2$

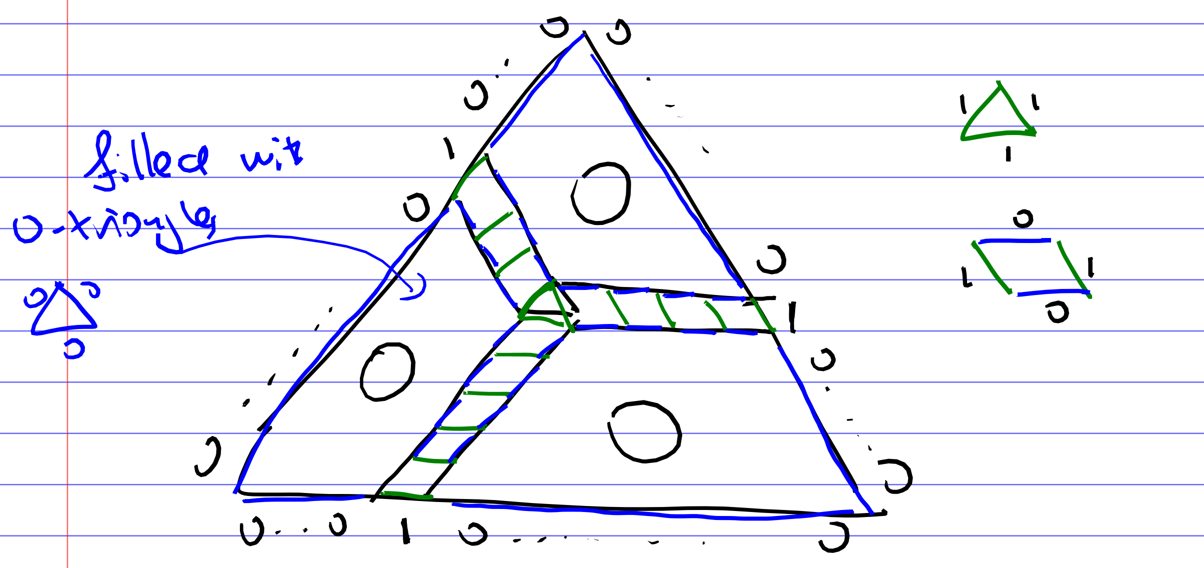
$\lambda = \mu =$ , $\nu =$ 

there is only 1 puzzle with such sides (shown above).

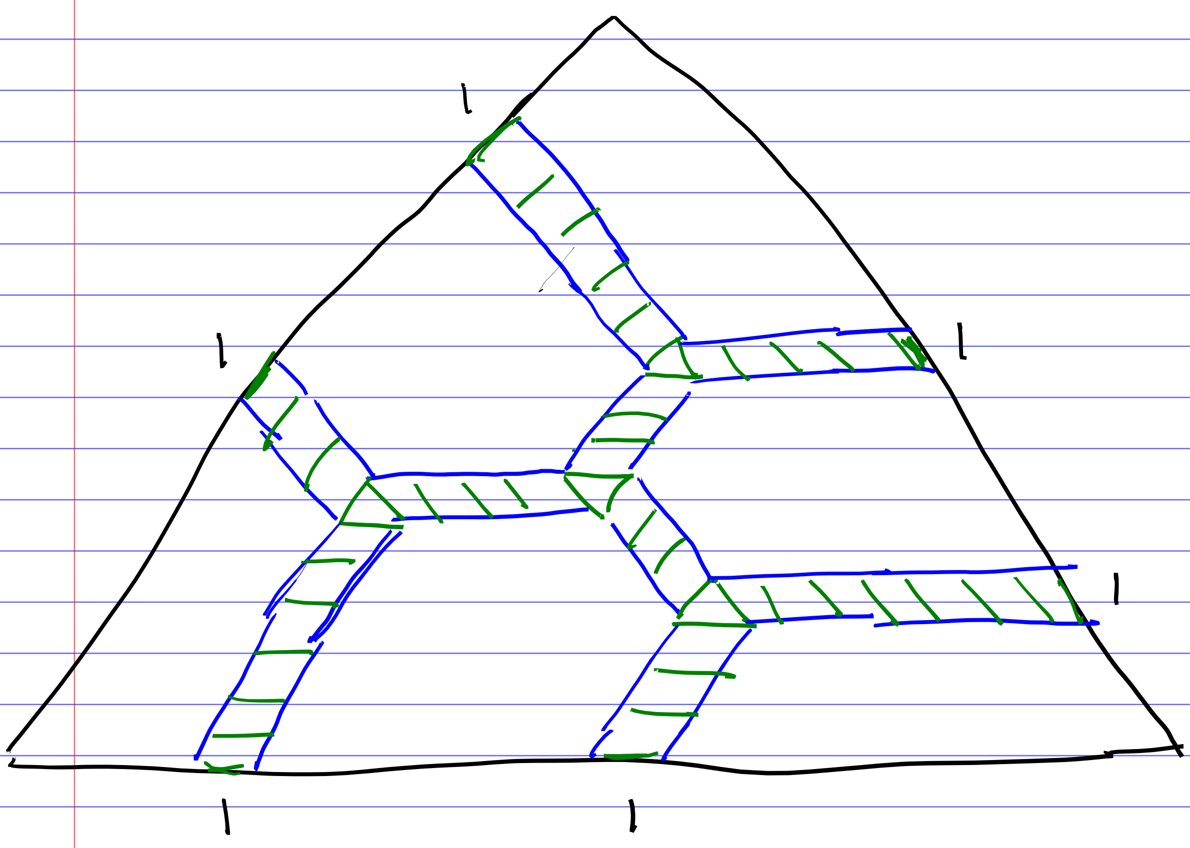
So  $= 1$

Proof Puzzles are in bijection with honeycombs....

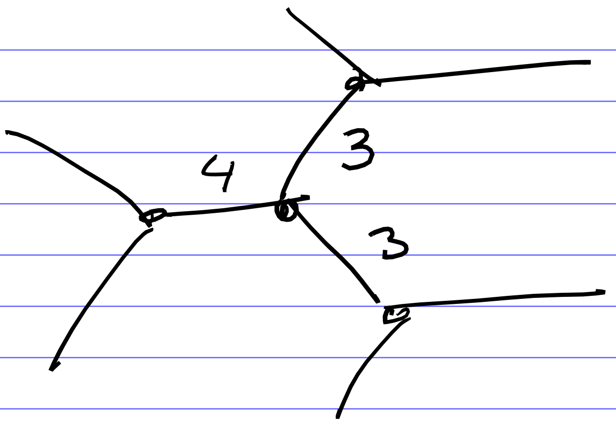
Examples. For $k=1$ every puzzle looks like:



For $k=2$, every puzzle looks like:



This puzzle corresponds to the honeycomb:

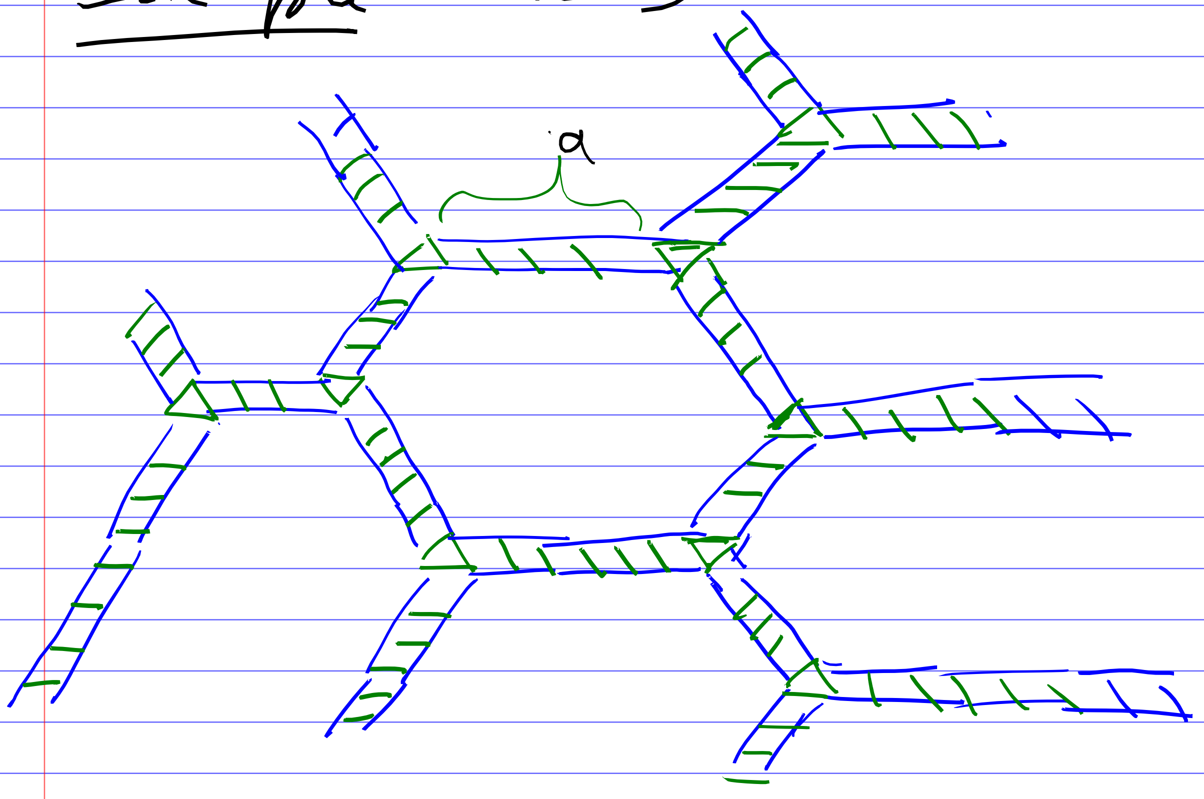


edge lengths in the honeycomb = # rhombuses in the ribbons between little triangles

Basically, a puzzle is a honeycomb with "thickened" edges.

Puzzles = "Ribbonized" honeycombs

Example $k=3$

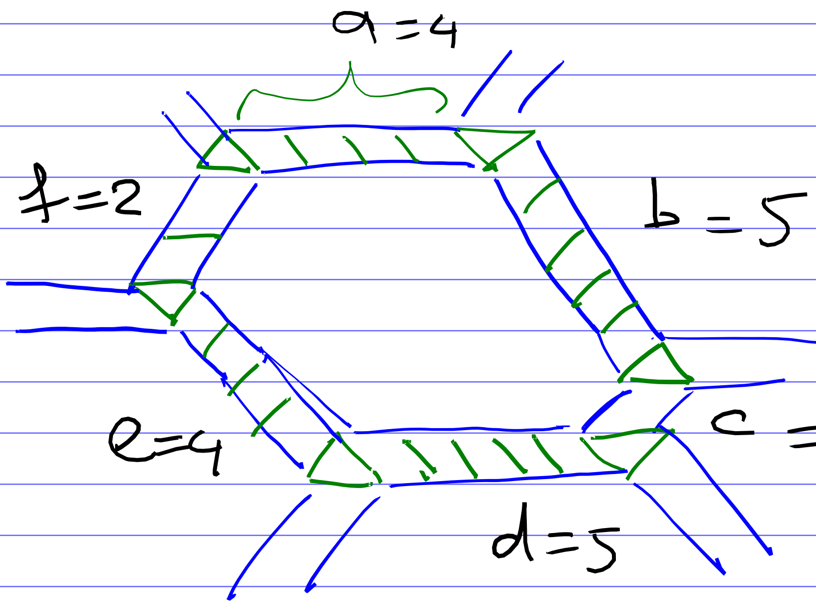


The entries of the corr. Berenstein-Zelevinsky triangle are "lengths" of strips of rhombuses between little green triangles

(:= # rhombuses in the strips)

Theorem. This gives a bijection between puzzles (for given λ, μ, ν) and BZ-triangles.

Proof We get the same hexagon relations for "ribbonized honeycombs"



We have
$$\begin{cases} a+b = d+e \\ b+c = e+f \\ c+d = f+a \end{cases}$$

the same hexagon relations as in BZ-triangles

{BZ-triangles}

bij ↔ {honeycombs}

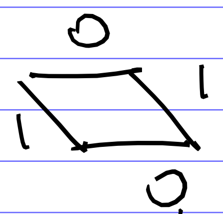
↔ {puzzles, or "ribbonized honeycombs"}

□

New symmetries that we can see in puzzles...

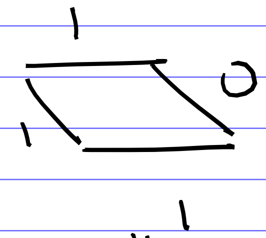
Let P be a puzzle.

If we switch $0 \leftrightarrow 1$ we will not get a valid puzzle



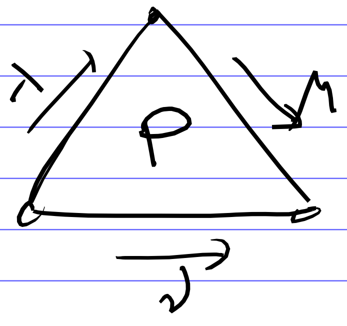
is allowed

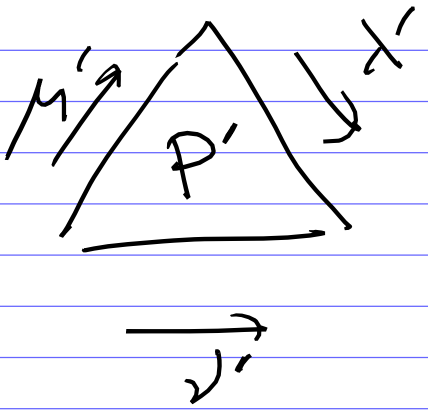
but



is not allowed

Let P' be the puzzle obtained from P by switching $0 \leftrightarrow 1$ and reflecting it w.r.t. vertical axis





The puzzle LR-rule implies the following symmetry

Theorem

$$c^{\nu}_{\lambda\mu} = c^{\nu'}_{\mu'\lambda'}$$

The puzzle LR-rule is the most symmetric version of LR rule. It explains all symmetries of the LR coefficients $c_{\lambda\mu}^{\nu}$ (that we mentioned earlier) <u>except</u> the commutation symmetry $c_{\lambda\mu}^{\nu} = c_{\mu\lambda}^{\nu}$.