18.217 Lecture 29 11/09/2020

Littlewood-Richardson coefficients (Cont'd).

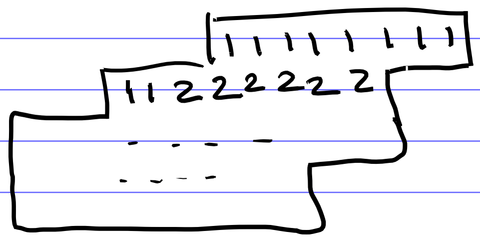
Berenstein-Zelevinsky triangles and Knutson-Tao honeycombs.

Goal: Express the Littlewood-Richardson rule in a more symmetric form.

Fix n.

LR-coefficients $c_{\lambda\mu}^{\nu}$, where λ, μ, ν partitions with at most n parts.

RL-rule: $c_{\lambda\mu}^{\nu} := \#\left\{\begin{array}{l}\text{LR-tableaux of} \\ \text{shape } \nu/\lambda \text{ and} \\ \text{weight } \mu\end{array}\right\}$



i-th row may contain only entries $1, 2, \ldots, i$

Let $a_{ij} = \# j\text{-s in } i^{th}$ row

- a_{ij}'s satisfy the Gelfand-Tsetlin conditions ($\Longleftrightarrow$ this is a valid SSYT)
- they also satisfy the lattice conditions (the reverse reading word is a lattice word)

Main point: All these conditions are given by certain linear inequalities for the a_{ij}.

So we have a certain convex polytope, called the Berenstein-Zelevinsky polytope

$BZ_{\lambda\mu}^{\nu} \subset \mathbb{R}^{\binom{n+1}{2}}$ ← # of a_{ij}'s.

whose # integer lattice points equals $c_{\lambda\mu}^{\nu}$.

(1) <u>Gelfand-Tsetlin conditions</u>:

$$\begin{array}{ccccccccccc}
 & & \lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & & \lambda_n & \\
 & \nu_1 & & b_{21} & & b_{31} & & \cdots & & b_{n1} & \\
 & & \nu_2 & & b_{32} & & b_{42} & \cdots & b_{n2} & & \\
 & & & \nu_3 & & & \cdots & & & & \\
 & & & & & & b_{n\,n-1} & & & & \\
 & & & & & \nu_n & & & & &
\end{array}$$

where

$$b_{ij} := \lambda_i + a_{i1} + a_{i2} + \cdots + a_{ij}$$
$$= \lambda_i + \#\text{ entries} \leq j \text{ in } i^{\text{th}} \text{ row}.$$

(2) <u>weight condition</u>: $\mu_j = \sum_{i=1}^{j} a_{ij}$, for $j = 1, 2, \ldots, n$.

(3) <u>lattice conditions</u>.

(rev. reading word: $1^{a_{11}} 2^{a_{22}} 1^{a_{21}} 3^{a_{33}} 2^{a_{32}} 1^{a_{31}} 4^{a_{44}} 3^{a_{43}} 2^{a_{42}} 1^{a_{41}} \cdots$

$a_{11} \geq a_{22}$

$a_{22} \geq a_{33}$

$a_{11} + a_{21} \geq a_{22} + a_{32}$

$a_{33} \geq a_{44}$

$a_{22} + a_{32} \geq a_{33} + a_{43}$

$a_{11} + a_{21} + a_{31} \geq a_{22} + a_{32} + a_{42}$

etc.

<u>Berenstein-Zelevinsky polytope</u>

$$BZ^{\nu}_{\lambda\mu} := \left\{ (a_{ij}) \in \mathbb{R}^{\binom{n+1}{2}} \;\middle|\; a_{ij}\text{'s satisfy (1), (2), (3)} \right\}$$

Now LR-rule $\Longleftrightarrow$

$$\boxed{c^{\nu}_{\lambda\mu} = \# BZ^{\nu}_{\lambda\mu} \cap \mathbb{Z}^{\binom{n+1}{2}}}$$

polytopal interpretation of the LR-rule.

Berenstein & Zelevinsky described a linear change of variables

$(a_{ij}) \Longleftrightarrow$ (new variables) such that in the new variables the BZ-polytope is given by more symmetric conditions.

These are called <u>BZ-triangles</u>

<u>Knutson-Tao honeycombs</u> are essentially the same thing as BZ-triangles.

We will first describe KT-honeycombs and then show how they are related to BZ-triangles.

Honeycombs

Setup: $\lambda = (\lambda_1, \dots, \lambda_n)$

$\mu = (\mu_1, \dots, \mu_n)$

$\nu = (\nu_1, \dots, \nu_n)$

3 vectors in $\mathbb{Z}^n$ with weakly decreasing entries. (The entries of λ, μ, ν can be negative.)

LR-coefficients $c_{\lambda\mu}^{\nu}$ are defined for such λ, μ, ν assuming that

$$c_{\lambda + k(1,\dots,1),\ \mu + \ell(1,\dots,1)}^{\nu + (k+\ell)(1,\dots,1)} = c_{\lambda\mu}^{\nu}$$

for any $k, \ell \in \mathbb{Z}$.

Let $\nu^* := (-\nu_n, -\nu_{n-1}, \dots, -\nu_1)$

Define

$$\boxed{\tilde{c}_{\lambda\mu\nu} := c_{\lambda\mu}^{\nu^*}}$$

Here we should have

$$\sum_i \lambda_i + \sum_i \mu_i + \sum_i \nu_i = 0$$

We mentioned earlier that $\tilde{c}_{\lambda\mu\nu}$'s are invariant under all 6 permutations of λ, μ, ν.

We will work inside the plane

$$P = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$$

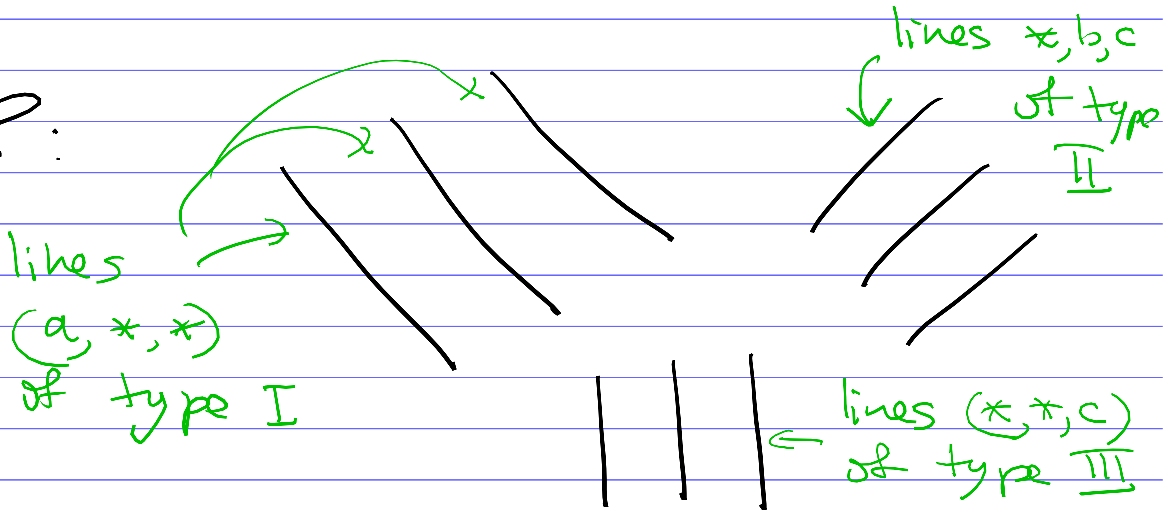
We'll consider 3 types of lines in P.

(I) $(a,*,*) := \{(x,y,z) \in P \mid x = a\}$

(II) $(*,b,*) := \{(x,y,z) \in P \mid y = b\}$

(III) $(*,*,c) := \{(x,y,z) \in P \mid z = c\}$

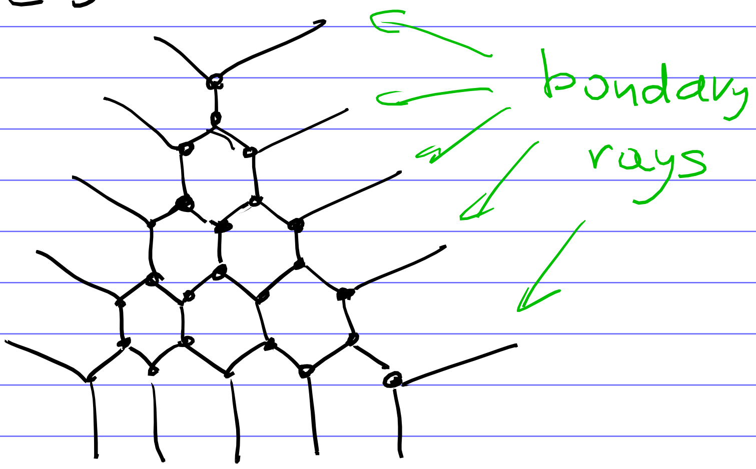
P:



The <u>honeycomb graph</u> H_n

<u>Example.</u> $n=5$

$H_5 =$

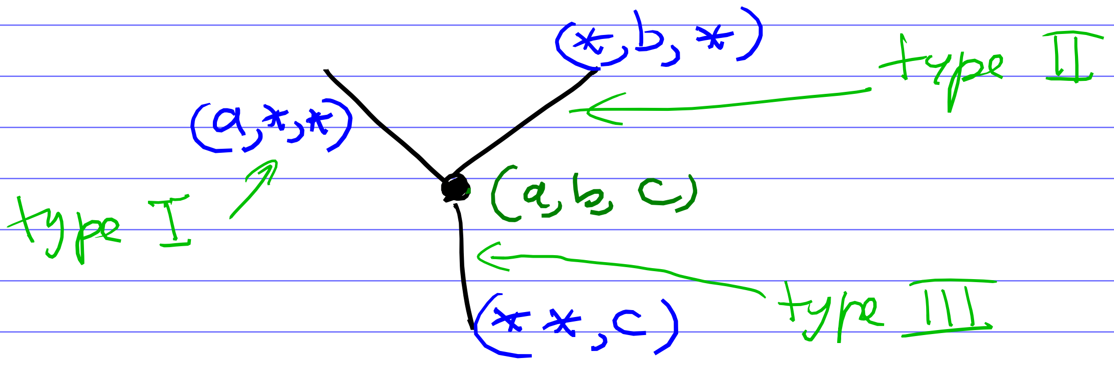


H_n has $3n$ <u>boundary</u> <u>"rays"</u> (edges with only 1 vertex)

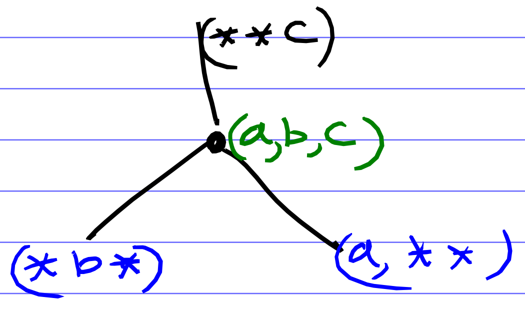
Def. A <u>honeycomb</u> is a drawing of the honeycomb graph H_n on the plane P (i.e. a map $f: \{\text{vertices of } H\} \to P$) s.t.

- each edge & each boundary ray is drawn as a line segment or a ray on one of the lines of the form $(a,*,*)$, $(*,b,*)$, or $(*,*,c)$

- For each vertex of H_n, the 3 adjacent edges are drawn as line segments of the 3 types:

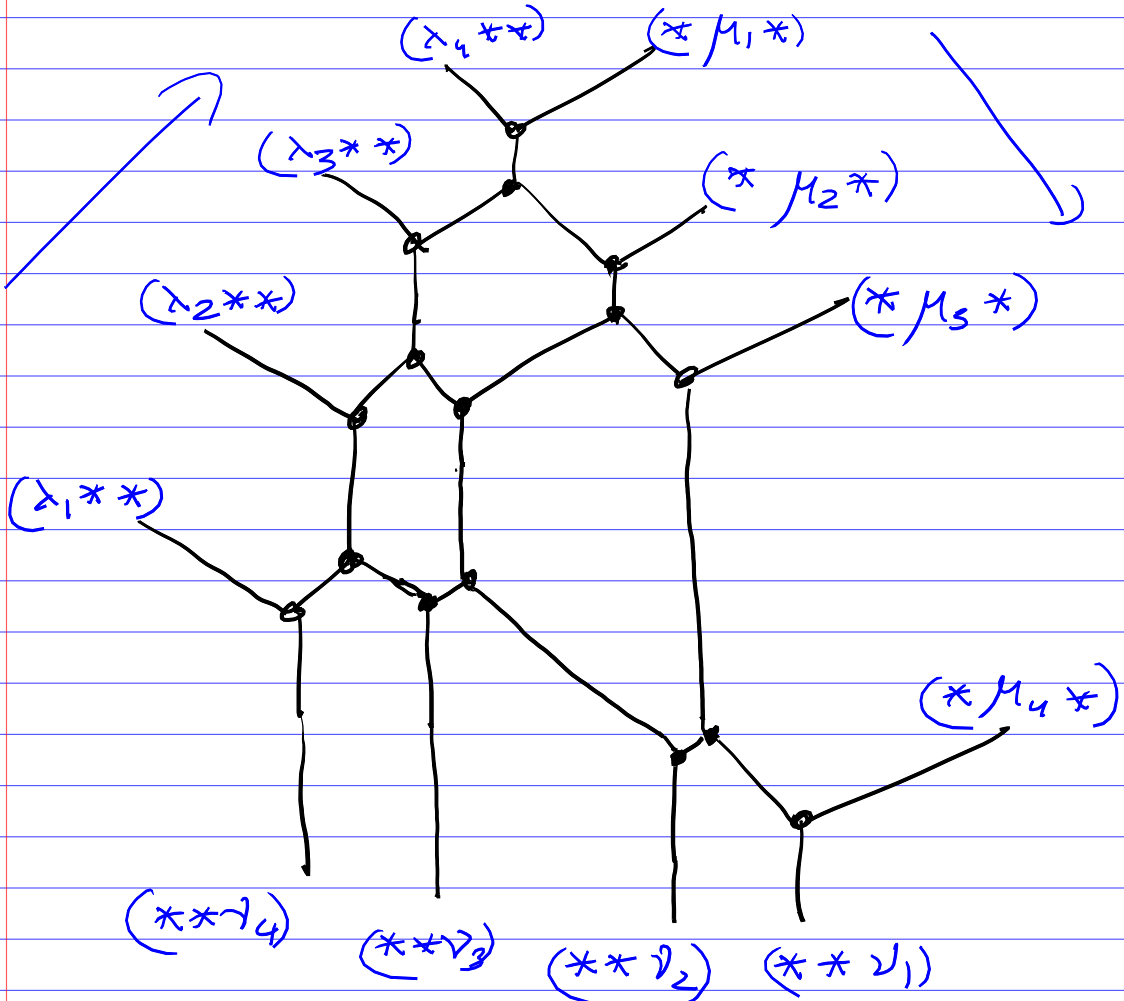


or

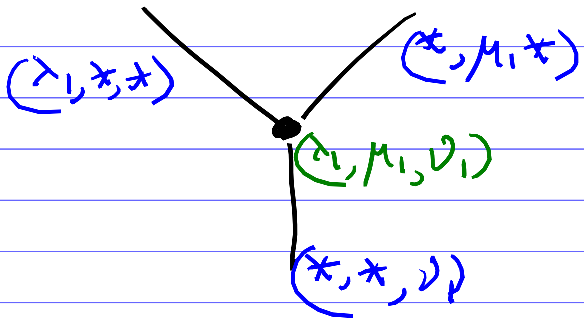


- The drawings of the edges cannot intersect each other in P (except at their vertices) (But we allow "collapsed edges" when both vertices of an edge map to the same point in P.)
- The positions of boundary rays are given by entries of the vectors λ, μ, ν.

Example $n=4$.

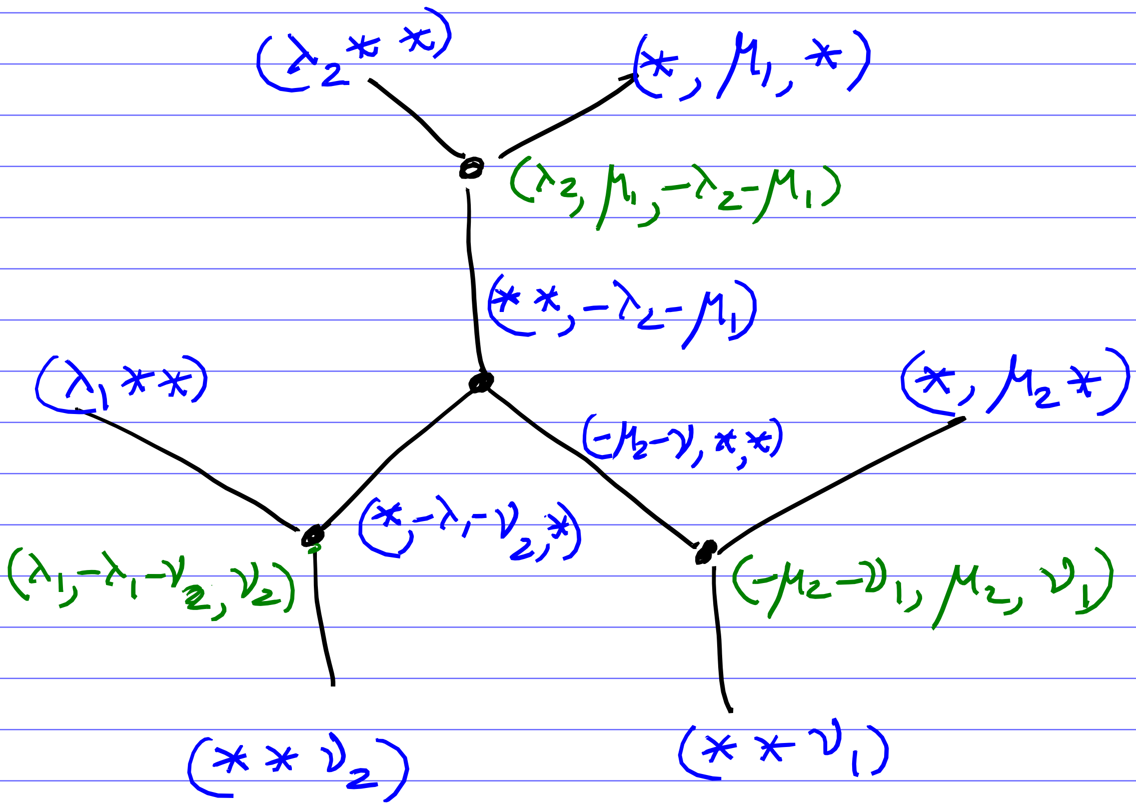


For $n=1$, there is only 1 honeycomb:



if $\lambda_1 + \mu_1 + \nu_1 = 0$.

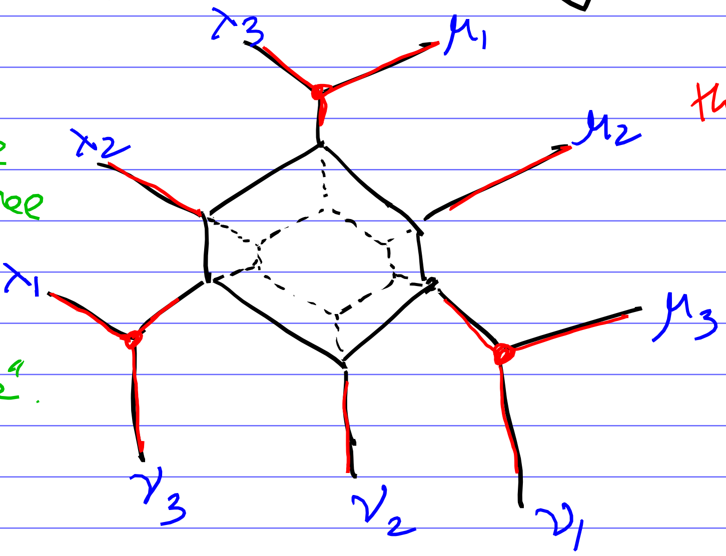
For $n=2$ (and given λ, μ, ν) there is at most 1 honeycomb.



But for $n = 3$ (and given λ, μ, ν) we might have many honeycombs

this picture has 1 degree of freedom

the hexagon can "breathe".

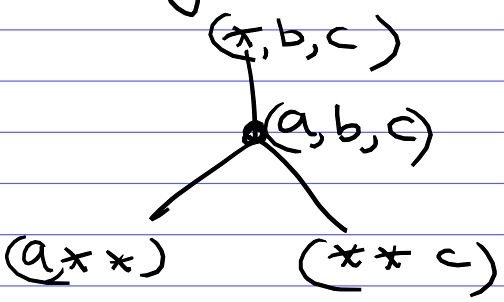


the positions of boundary rays are fixed by λ, μ, ν

The hexagon can "breathe" (i.e. become smaller or larger) without changing the fixed positions of the boundary rays.

In this case, we have a 1-dimensional space (line segment) of honeycombs.

Def. A honeycomb is integer if all its vertices & edges have integer coordinates:



$a, b, c \in \mathbb{Z}$.

Honeycomb Version of Littlewood-Richardson Rule.

Theorem (Knutson-Tao, cf. Berenstein-Zelevinsky)

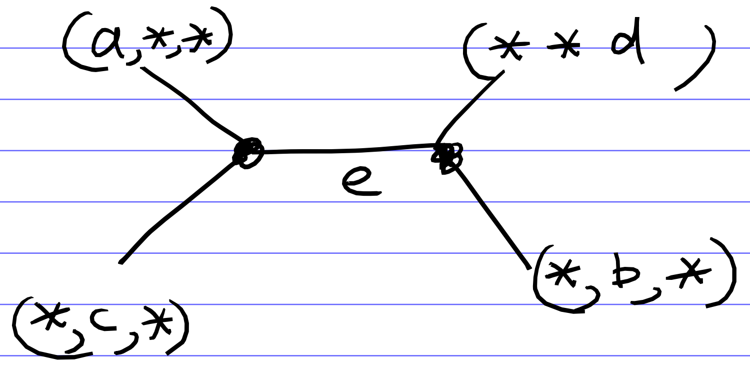
$\tilde{c}_{\lambda\mu\nu}$ $(= c_{\lambda\mu}^{\nu^*})$ equals the number of integer honeycombs with fixed positions of the boundary rays given by λ_i's, μ_i's, and ν_i's.

Let's us explicitly express this theorem in coordinates:

Def. The *length* of an edge e in a honeycomb

$:= \frac{1}{\sqrt{2}}$ Eucledian length of e.

Ex.



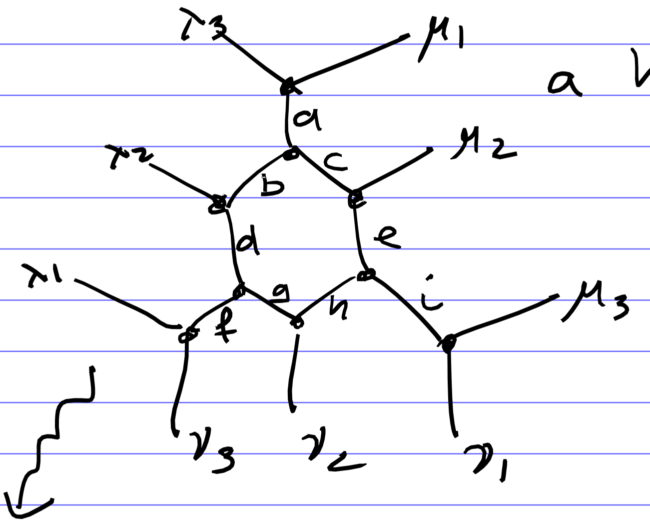
$$\text{length}(e) = a - b = c - d.$$

Clearly, the array of lengths of all edges in a honeycomb uniquely determines the honeycomb.

Berenstein-Zelevinsky triangles

are arrays of lengths of edges in honeycombs.

Ex. $n=3$



a honeycomb

$$\begin{array}{ccccccc} & & & a & & & \\ & & b & & c & & \\ & d & & & & e & \\ f & & g & & h & & i \end{array}$$

a BZ-triangle:

Conditions : • $a, b, c, \ldots, i \in \mathbb{Z}_{\geq 0}$

• (boundary conditions) :

$$\lambda_1 - \lambda_2 = f + d$$

$$\lambda_2 - \lambda_3 = b + a$$

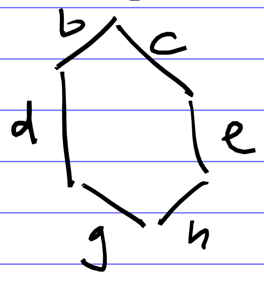
$$\mu_1 - \mu_2 = a + c$$

$$\mu_2 - \mu_3 = e + i$$

$$\nu_1 - \nu_2 = i + h$$

$$\nu_2 - \nu_3 = g + f$$

• (hexagon condition)

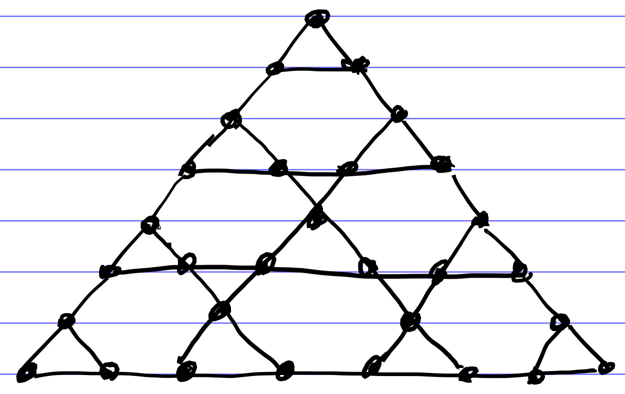


$\exists$ a hexagon with such edge length:

$$\Longleftrightarrow \begin{cases} b+c = g+h \\ c+e = d+g \\ e+h = b+d \end{cases}$$

For general n, we have the following construction of BZ-triangles.

Berenstein - Zelevinsky triangles

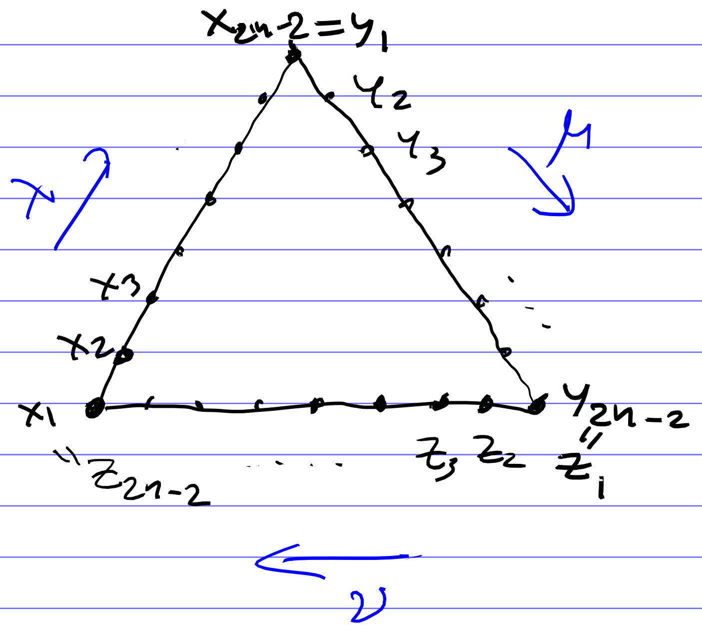


this particular triangle is for $n=5$

- A triangle with $2(n-1)$ points on each side.
- Draw lines (of 3 possible directions $\parallel$ sides of the triangle) through every second point on the sides of the triangle), as shown above
- For all intersection points p of these lines (inside the triangle) we have variables x_p.

Conditions on these variables:

- nonnegativity: $\boxed{x_p \geq 0}$
- boundary conditions:



$l_i := \lambda_i - \lambda_{i+1}$

$m_i := \mu_i - \mu_{i+1}$

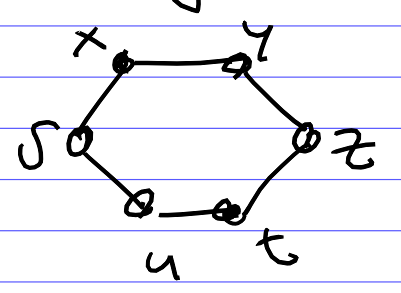
$n_i := \nu_i - \mu_{i+1}$

for $i=1,\dots,n-1$

$$\begin{cases} x_1+x_2 = l_1, \quad x_3+x_4 = l_2, \dots \\ y_1+y_2 = m_1, \quad y_3+y_4 = m_2, \dots \\ z_1+z_2 = n_1, \quad z_3+z_4 = n_2, \dots \end{cases}$$

- hexagon conditions:

$\forall$ hexagon we have



$$\begin{cases} x+y = t+u \\ y+z = u+s \\ z+t = s+x \end{cases}$$

Def A BZ triangle (x_p) is integer, if all variables $x_p \in \mathbb{Z}_{\geq 0}$.

BZ-version of LR-rule

Theorem (Berenstein-Zelevinsky)

$$\tilde{c}_{\lambda\mu\nu} = \# \left\{ \begin{array}{l} \text{integer BZ-triangles} \\ \text{with boundary cond.} \\ \text{given by } \lambda, \mu, \nu \end{array} \right\}$$

Def. Let $\widetilde{BZ}_{\lambda\mu\nu} \subset \mathbb{R}^{N \sim n^2}$ the polytope of $\mathbb{R}$-valued BZ-triangles.

Previous Thm $\Longleftrightarrow$

$$\tilde{c}_{\lambda\mu\nu} = \#(\widetilde{BZ}_{\lambda\mu\nu} \cap \mathbb{Z}^N).$$

How to prove all these Thms?

Claim

The classical LR-rule

$\Updownarrow$

The LR-rule in the form of GT-patterns + the lattice conditions (which we gave in the beginning of the lecture)

all these equivalences are trivial, except this one $\rightarrow$ $\Updownarrow$?

The BZ-version of LR-rule

$\Updownarrow$

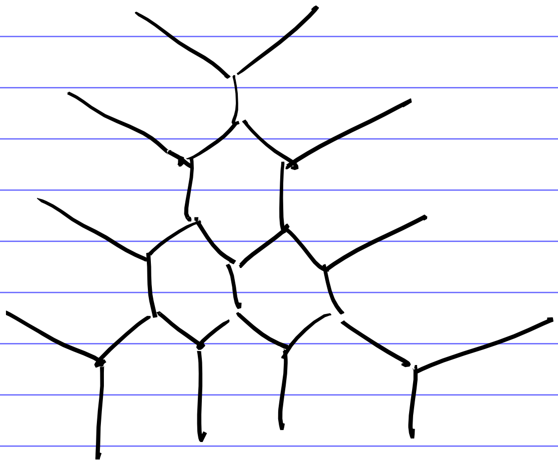
The honeycomb version of LR-rule.

Theorem. The polytope $BZ_{\lambda\mu}^{\nu^*}$ is related to the polytope $\widetilde{BZ}_{\lambda\mu\nu}$ by a linear (integer point preserving) change of variables.

$$(a_{ij}) \longleftrightarrow (x_p)$$

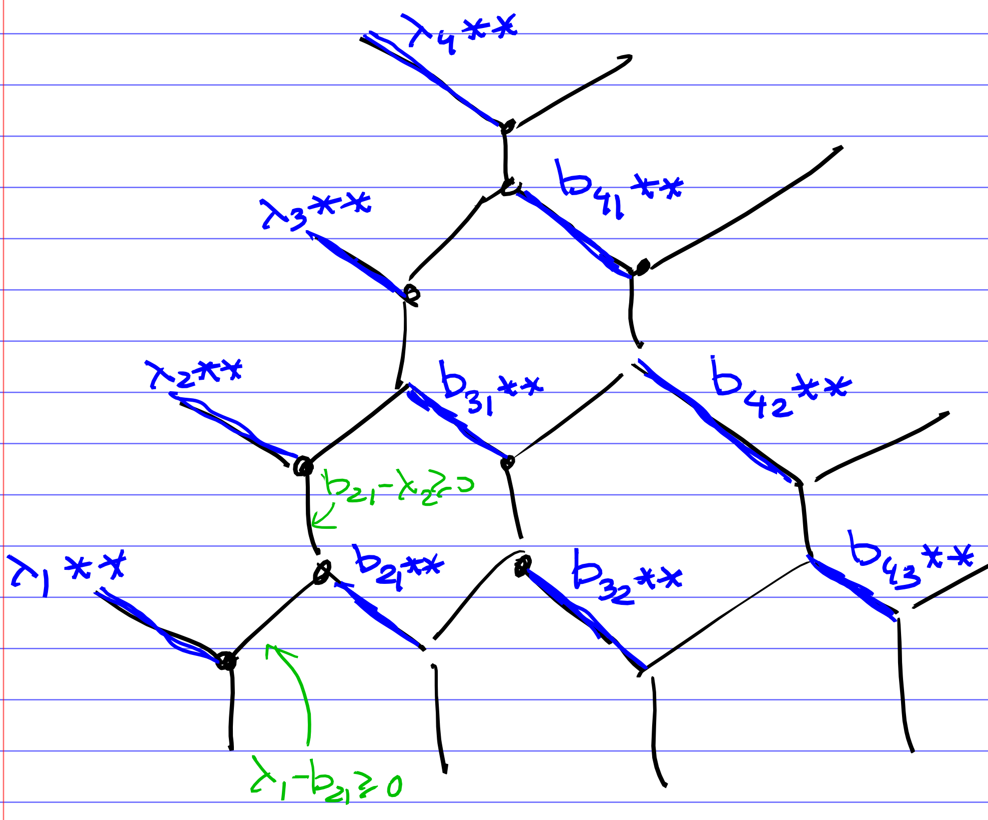
$$BZ_{\lambda\mu}^{\nu^*} \simeq BZ_{\lambda\mu\nu}.$$

Let us explicitly describe this change of variables using honeycombs.

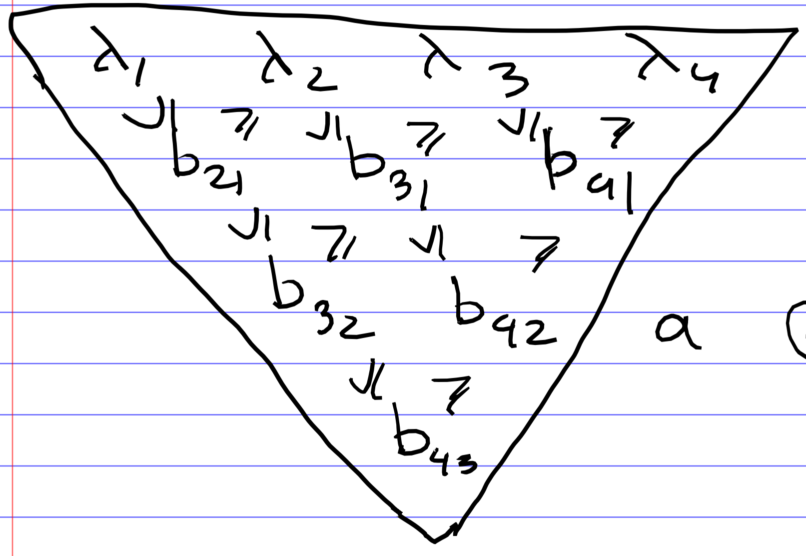


We can see both kinds of variables in this honeycomb picture

- b_{ij}'s $(= \lambda_i + a_{i1} + a_{i2} + \dots + a_{ij})$ are the coordinates all lines of type (I) (containing the edges of a honeycomb)
- BZ-variables x_p are the edge lengths in a honeycomb.



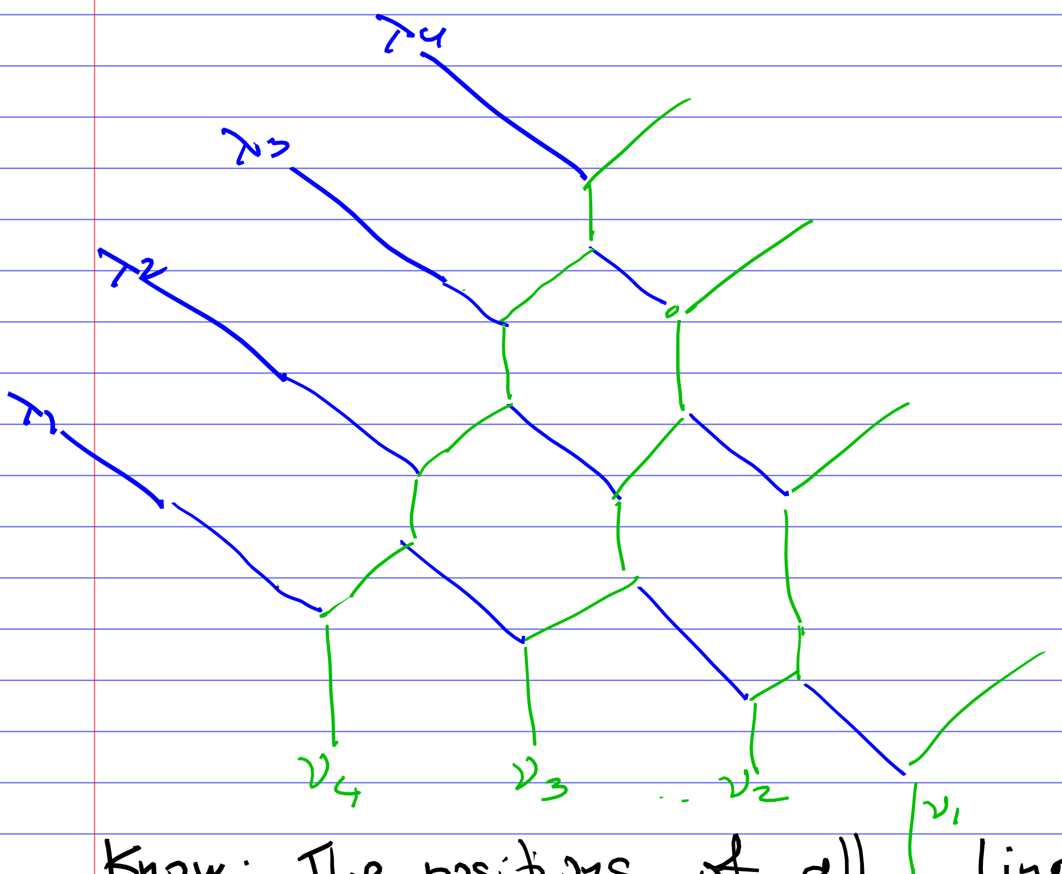
a honeycomb



a GT-pattern

Notice the GT-condition
$\Longleftrightarrow$ all "edge length" coordinates of edges of types (II) & III are ≥ 0.

Also notice that if we know the positions of all edges of type I in a honeycomb & also know the position of boundary rays (given by λ, μ, ν), we can linearly express all other coordinates.



Know: The positions of all lines containing the blue edges (type I)
- Position of boundary rays of type III (given by ν_i's)

Then we can reconstruct the positions of all edges in the honeycomb.

We need to check that we get a valid honeycomb.

We already got the correct positions of the boundary rays for λ_i's & ν_i's

- "Edge length" coord ≥ 0 for edges of types II & III ($\Leftrightarrow$ GT interlacing conditions)

In addition to this, we need the correct positions of boundary rays for μ_i's & the non-negativity of "edge length" coord for edges of type I

<u>Claim</u> These are exactly the "weight conditions" and the "lattice word" conditions that we need to impose on the GT pattern

$$\begin{array}{ccccc} \lambda_1 & \lambda_2 & \cdots & & \lambda_n \\ & b_{21} & b_{31} & \cdots & \\ & & b_{32} & \cdots & \\ & & & b_{nn-1} & \end{array}$$