Stembridge's concise proof of the Littlewood-Richardson rule

(J. R. Stembridge, Electr. J. Combin., 2002)

last time: $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu \, s_\nu$

$$s_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda \, s_\nu$$

$c_{\lambda\mu}^\nu$ — the Littlewood-Richardson coefficients.

Fix n.

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ a partition with at most n parts, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

$\rho := (n-1, n-2, \ldots, 0)$ the "staircase" partition

For any vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and

$$a_\alpha := \det [\underbrace{x_i^{\alpha_j}}_{n\times n \text{ matrix}}] = \sum_{w \in S_n} (-1)^{\ell(w)} x^{w(\alpha)}$$

ν/μ any skew Young diagram (It is allowed to have $> n$ rows)

$SSYT(\nu/\mu, n)$ - the set of semi-standard Young tableaux of shape ν/μ with entries $\in \{1, 2, \ldots, n\}$.

For $T \in SSYT(\mu/\nu, n)$, $\text{weight}(T) = (\beta_1, \beta_2, \ldots, \beta_n)$ the weight of T.

$\beta_i := \#\{i\text{'s in } T\}$

(Note: μ, ν might have $> n$ parts, but weight(T) is an n vector.)

(Skew) Schur polynomials:

$$s_{\mu/\nu} = s_{\mu/\nu}(x_1, \ldots, x_n) := \sum_{T \in SSYT(\mu/\nu, n)} x^{\text{weight}(T)}$$

In todays lecture, $s_{\mu/\nu}$ is
- a polynomial in n variables
- defined combinatorially as the sum over SSYT's.

For $T \in SSYT(\mu/\nu, n)$,

let $T_{\geq j}$ be the subtableau of T in columns $\geq j$.

Similarly, $T_{<j}$ is —ıı— in columns $<j$.

Ex.

$$T = \begin{array}{ccc|cccc} & & 1 & 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 & 4 & \\ 4 & 4 & 4 & 4 & 5 & 5 & \end{array}$$

$$\begin{array}{ccc} & & 1 \\ 1 & 2 & 2 \\ 4 & 4 & 4 \end{array} \qquad\qquad \begin{array}{cccc} 1 & 2 & 3 & 3 \\ 3 & 3 & 4 & \\ 4 & 5 & 5 & \end{array}$$

$$T_{<j} \qquad\qquad\qquad T_{\geq j}$$

<u>Theorem</u>. λ, μ/ν as above (λ is a partition w/ $\leq n$ parts, μ & ν arbitary partitions s.t. $\mu \supseteq \nu$.) Then

$$(*) \quad a_{\lambda+\rho} \cdot s_{\mu/\nu} = \sum_T a_{\lambda + \text{weight}(T) + \rho},$$

where the sum is over $T \in SSYT(\mu/\nu, n)$ s.t.

$$(**) \quad \lambda + \text{weight}(T_{\geq j}) \text{ is a partition } \forall j \geq 1.$$

Before we prove it, let's give some corollaries of this theorem.

Specialize: $\lambda = \emptyset$, $\nu = \emptyset$

we get

$$a_\rho \cdot s_\mu = \sum_T a_{\mathrm{weight}(T)+\rho}$$
$$= a_{\mu+\rho}.$$

Indeed, in this case there is a unique tableau T satisfying (**) namely,

$$T = \begin{array}{|ccccc|} \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & \\ 3 & 3 & 3 & 3 & \\ 4 & 4 & & & \\ \hline \end{array}$$

(**): in this part
$\#1's \geq \#2's \geq \#3's \geq \dots$

Corollary I. $$s_\mu = \frac{a_{\mu+\rho}}{a_\rho}.$$

combinatorially defined Schur polynomial

"classical" definition of Schur polynomial.

Corollary 2

$$s_\lambda \cdot s_{\mu/\nu} = \sum_{\substack{T \text{ satisfying} \\ (**)}} s_{\lambda + \mathrm{weight}(T)}$$

This is Zelevinsky's extension of the LR-rule.

Proof. Divide both sides of $(*)$ by a_ρ & use Corollary 1. □

Specialize $\lambda = \emptyset$

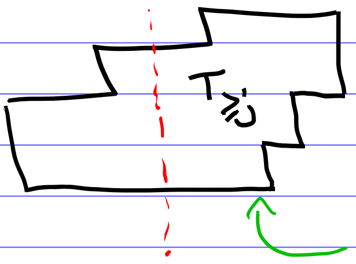
Corollary 3.

$$s_{\mu/\nu} = \sum_{\gamma} \# \left\{ \begin{array}{l} T \text{ satisfying} \\ (**) \text{ s.t.} \\ \mathrm{weight}(T) = \gamma \end{array} \right\} s_\gamma$$

Exercise. Prove that these tableaux are exactly the Littlewood-Richardson tableau of shape μ/ν and weight γ.

So this Corollary 3 $\Leftrightarrow$ the classical formulation of the LR-rule.

Conditions on T in Corollary 3:

$T =$ 

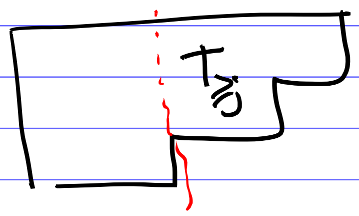
SSYT's of shape μ/ν and weight γ s.t. weight $(T_{\geq j})$ is a partition.

This condition is easier (to formulate) than the def. of LR-tableaux in terms of lattice paths.
(No need to consider the reverse reading word of T.)

In Cor. 2, specialize $\nu = \emptyset$.

Corollary 4.

$$s_\lambda \cdot s_\mu = \sum_\gamma \# \left\{ \begin{array}{c} \text{SSYT's} \\ \text{of shape } \mu \\ \text{and weight } \gamma \\ \text{satisfying } (**) \end{array} \right\} s_\gamma$$



shape $= \mu$
weight $= \gamma$

weight $(T_{\geq j}) + \lambda$ is a partition $\forall j$

Exercise. Show bijectively that

$\#$ such tableaux T =

= $\#$ LR-tableaux of shape γ/μ and weight λ.

So Corollary 2 extends both forms of LR-rule for coefficients in the expansions:

$$s_{\mu/\nu} = \sum_\gamma c^\mu_{\nu\gamma} \, s_\gamma$$

$$s_\lambda \cdot s_\mu = \sum_\gamma c^\gamma_{\lambda\mu} \, s_\gamma$$

Stembridge's proof of Thm. is based on the involution principle.

Need: $a_{\lambda+\rho} \cdot s_{\mu/\nu} = \sum_{\substack{T \\ \text{satisfying } (**)}}' a_{\lambda + \text{weight}(T) + \rho}$

Main ingredient: A sign reversing involution constructed using Bender-Knuth involutions.

Stembridge remarks that essentially the same construction was early given by Berenstein-Zelevinsky in the language of Gelfand-Tsetlin patterns.

Recall that the Bender-Knuth involution σ_k is an involution on the set of SSYT's s.t.

- if $\text{weight}(T) = \beta$, then $\text{weight}(\sigma_k(T)) = s_k(\beta)$

switch β_k & β_{k+1}

- the tableau $\sigma_k(T)$ is obtained from T by changing some entries k to $k+1$, and vise versa.

The Bender-Knuth involutions σ_k show that the Kostka numbers

$$K_{\mu/\nu, \beta} := \#\left\{ \begin{array}{l} \text{SSYT's of shape } \mu/\nu \\ \text{and weight } \beta \end{array} \right\}$$

do not change when we permute entries of $\beta = (\beta_1, \dots, \beta_n)$.

Proof of Theorem.

We have $a_{\lambda+\rho} \cdot s_{\mu/\nu} \overset{\text{def}}{=}$

$$\overset{\text{def}}{=} \sum_{w \in S_n} \sum_{T \in SSYT(\mu/\nu, n)} (-1)^{\ell(w)} x^{w(\lambda+\rho) + \mathrm{weight}(T)}$$

$$= \sum_{w \in S_n} \sum_{T \in SSYT(\mu/\nu, n)} (-1)^{\ell(w)} x^{w(\lambda+\rho+\mathrm{weight}(T))}$$

because Kostka #'s $K_{\mu/\nu, \beta}$ are S_n-invariant w.r.t. permutations of entries in β

$$= \sum_{\substack{T \\ \cap \\ SSYT(\mu/\nu, n)}} \sum_{w \in S_n} (-1)^{\ell(w)} x^{w(\lambda+\rho+\mathrm{wt}(T))}$$

$$= \sum_{T \in SSYT(\mu/\nu, n)} a_{\lambda+\mathrm{weight}(T)+\rho}$$

$$\overset{?}{=} \sum_{\substack{T \in SSYT(\mu/\nu, n) \\ \text{satisfying } (**)}} a_{\lambda+\mathrm{weight}(T)+\rho}$$

we need to show this

We'll construct a sign reversing involution on the set of tableaux $T \in SSYT(\mu/\nu, n)$ s.t. condition $(**)$ fails.

(Cancel the Bad Guys)

(**) fails $\iff$ $\exists\, j$ s.t.

$\lambda + \text{weight}(T_{\geq j})$ is <u>not</u> a partition

$\iff \exists\, j$ & k s.t.

k^{th} & $(k+1)^{st}$ entries of the weight of $T_{\geq j}$

$$\lambda_k + \text{weight}_k(T_{\geq j}) < \lambda_{k+1} + \text{weight}_{k+1}(T_{\geq j}).$$

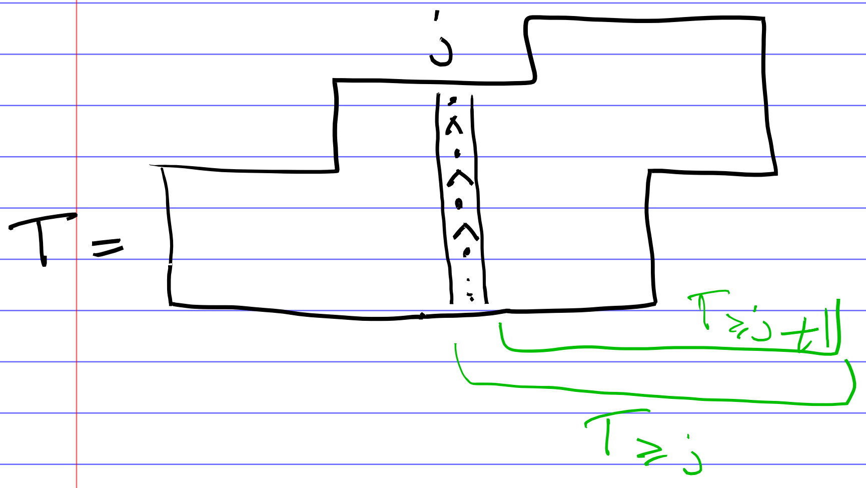
Among all such pairs k, j, choose one that <u>maximizes</u> j, and among those, choose the smallest k.

So

$\lambda + \text{weight}(T_{\geq j+1})$ <u>is</u> a partition

but

$\lambda + \text{weight}(T_{\geq j})$ is <u>not</u> a partition



The $(j+1)^{st}$ column contains at most 1 entry k and at most 1 entry $k+1$.

So $\text{weight}_k(T_{\geq j}) - \text{weight}_{k+1}(T_{\geq j})$ can increase or decrease by at most 1 if we increase j by 1.

We have

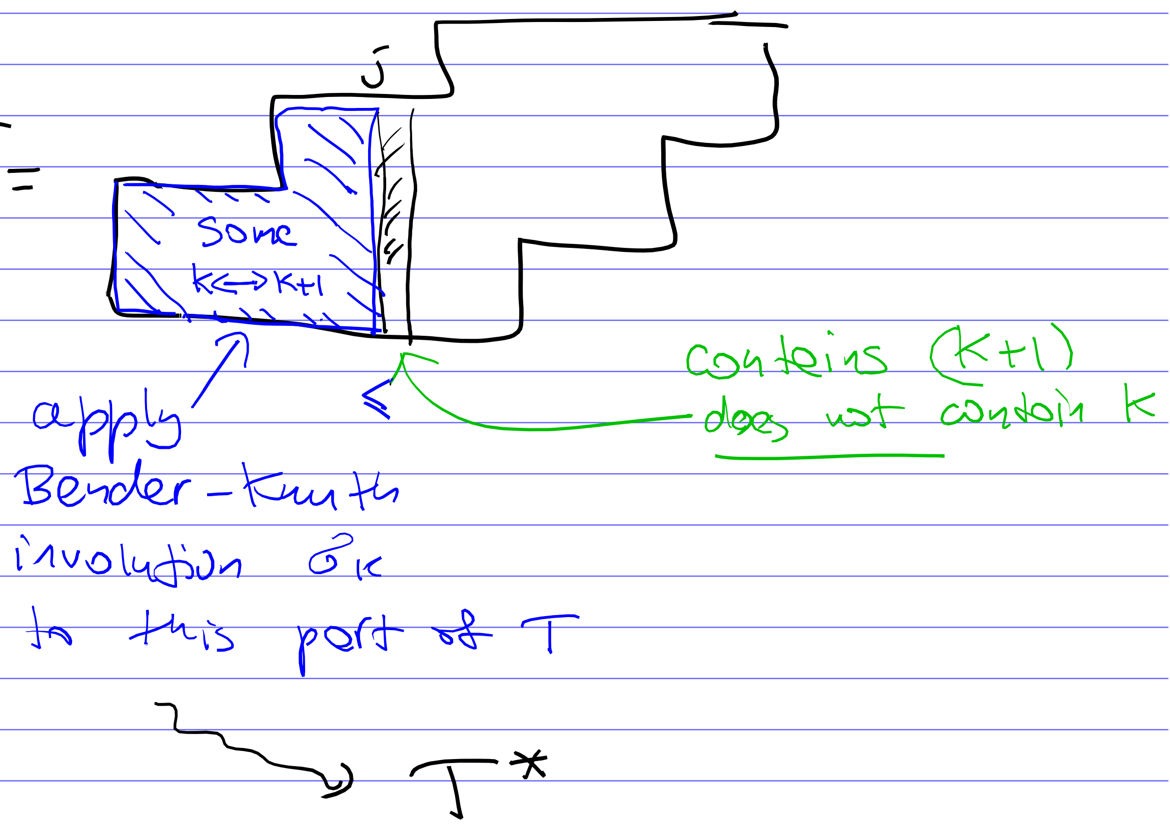
$$\begin{cases} \lambda_k + \mathrm{weight}_k(T_{\geq j+1}) \geq \lambda_{k+1} + \mathrm{weight}_{k+1}(T_{\geq j+1}) \\ \lambda_k + \mathrm{weight}_k(T_{\geq j}) < \lambda_{k+1} + \mathrm{weight}_{k+1}(T_{\geq j}) \end{cases}$$

$$\Downarrow$$

$$\lambda_k + \mathrm{weight}_k(T_{\geq j}) + 1 = \lambda_{k+1} + \mathrm{weight}_{k+1}(T_{\geq j})$$

and $(j)^{st}$ column of T contains $(k+1)$ and does not contain k.

Let T^* be the tableaux obtained from T by applying the Bender-Knuth involution σ_k to the subtableau $T_{<j}$, leaving the reminder unchanged

$T =$ 

Since j^{th} column does not contain k's, T^* is a semistandard Young tableaux.

We have

$$\begin{cases} \bullet\ \mathrm{weight}(T^*_{<j}) \overset{\text{by properties of Bender-Knuth inv}}{=} s_k(\mathrm{weight}\, T_{<j}) \\ \bullet\ \lambda_k + \mathrm{weight}_k(\underset{\substack{\| \\ T^*_{\geq j}}}{T_{\geq j}}) + 1 = \lambda_{k+1} + \mathrm{weight}_{k+1}(T_{\geq j}) \end{cases}$$

$$\Downarrow$$

$$\lambda + \mathrm{weight}(T^*) + \rho = s_k(\lambda + \mathrm{weight}(T) + \rho)$$

$$\Updownarrow$$

$$a_{\lambda + \mathrm{weight}(T^*) + \rho} = -a_{\lambda + \mathrm{weight}(T) + \rho}$$

because $a_{\dots} = \det(\dots)$ is anti symmetric w.r.t. permutation of columns

So contributions of T & T^* cancel each other. We get

$$a_{\lambda+\rho} \cdot s_{\mu/\nu} = \sum_{\substack{T \text{ any SSYT} \\ \text{of shape } \mu/\nu}} a_{\lambda + \mathrm{weight}(T) + \rho}$$

$$= \sum_{\substack{T:\ \text{SSYT} \\ \text{of shape } \mu/\nu \\ \text{satisfying } (**)}} a_{\lambda + \mathrm{weight}(T) + \rho},$$

as needed. Q.E.D.

As we mentioned, Berenstein-Zelevinsky constructed basically the same involution using the language of Gelfand-Tsetlin patterns...

Littlewood-Richardson rule via Gelfand-Tsetlin patterns.

Recall Classical LR-rule:

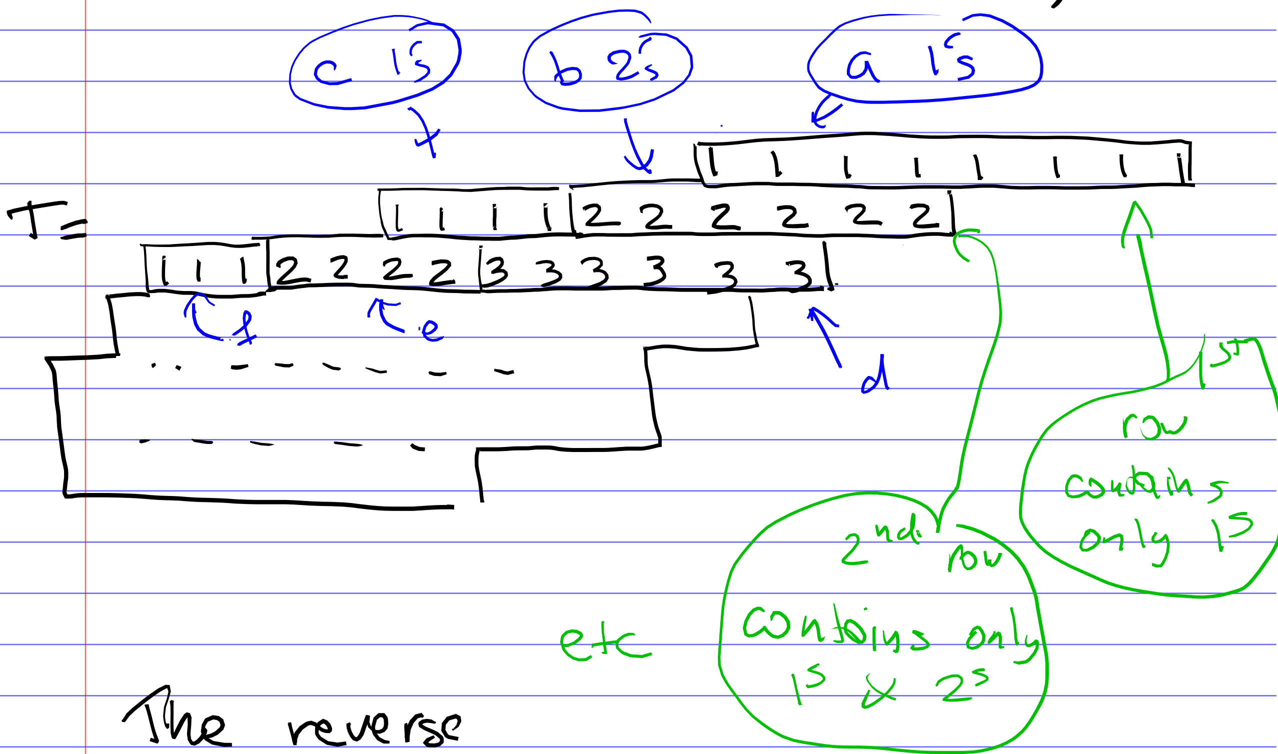
$$c^{\lambda}_{\mu\nu} = \#\{\text{LR-tableaux of shape } \lambda/\mu \ \& \text{ weight } \nu\}.$$

LR-tableaux are SSYT's whose reverse reading word is a lattice word.

Assume that $\lambda = (\lambda_1, .., \lambda_n)$
$\mu = (\mu_1, .., \mu_n)$, $\nu = (\nu_1 .. \nu_n)$
(i.e., λ, μ, ν have $\leq n$ parts)

A LR-tableau of shape λ/μ.

$T =$



The reverse reading word of T is

$$\underbrace{1 \dots 1}_{a}\ \underbrace{2 \dots 2}_{b}\ \underbrace{1 \dots 1}_{c}\ \underbrace{3 \dots 3}_{d}\ \underbrace{2 \dots 2}_{e}\ \underbrace{1 \dots 1}_{f}\ \dots$$

lattice conditions:

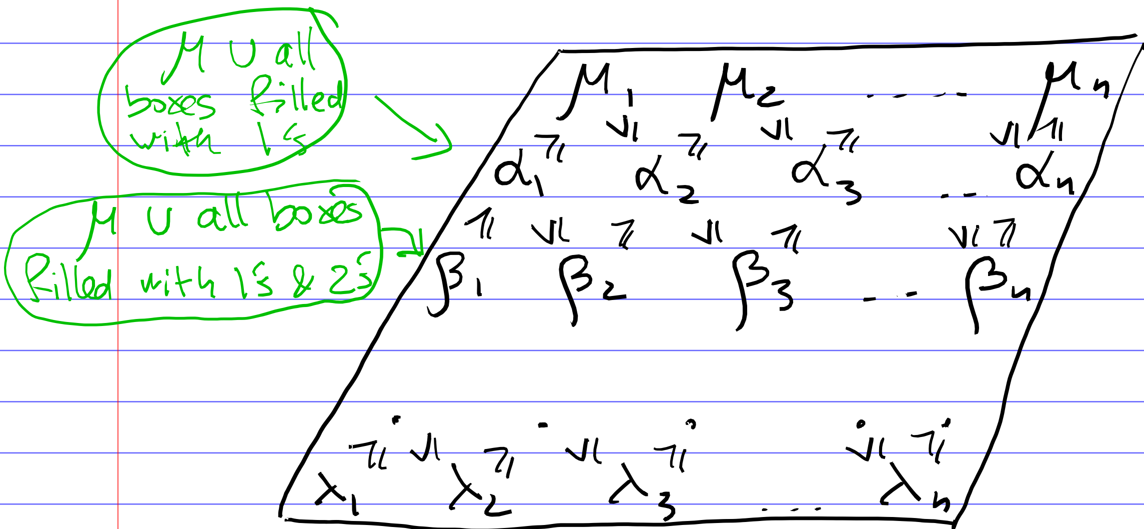
$$a \geq b$$

$$b \geq d$$

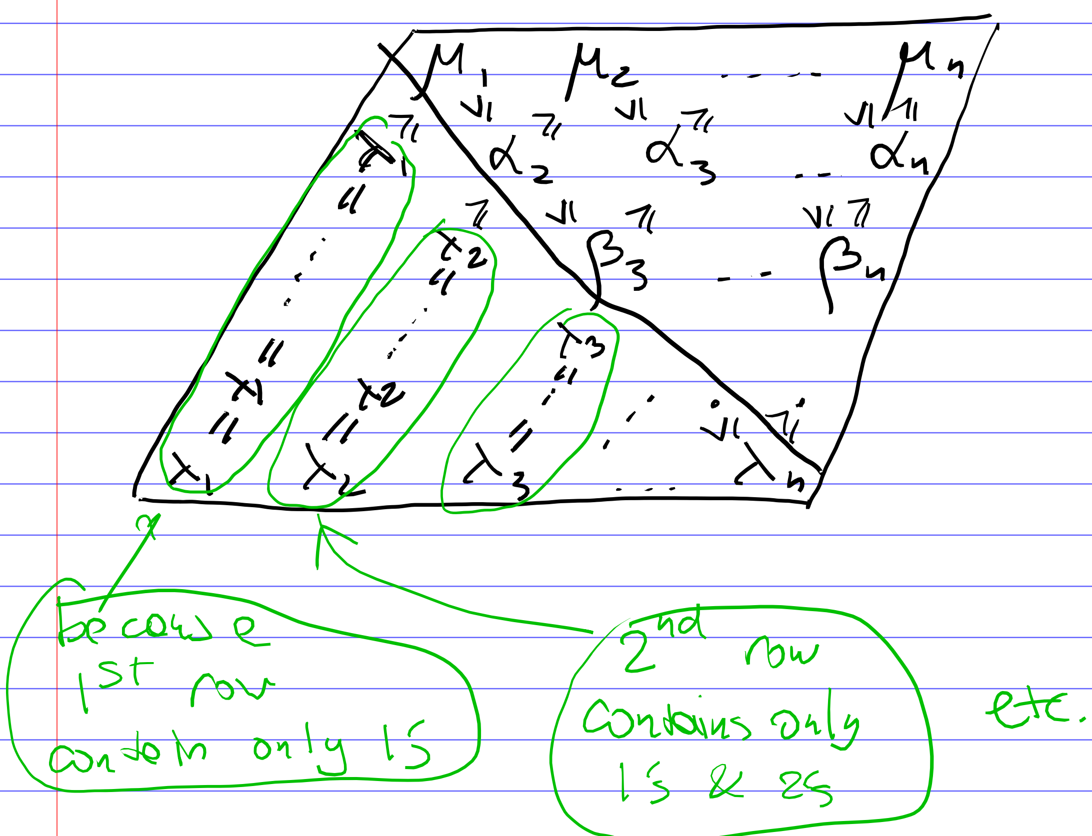
$$a + c \geq b + e$$

etc.

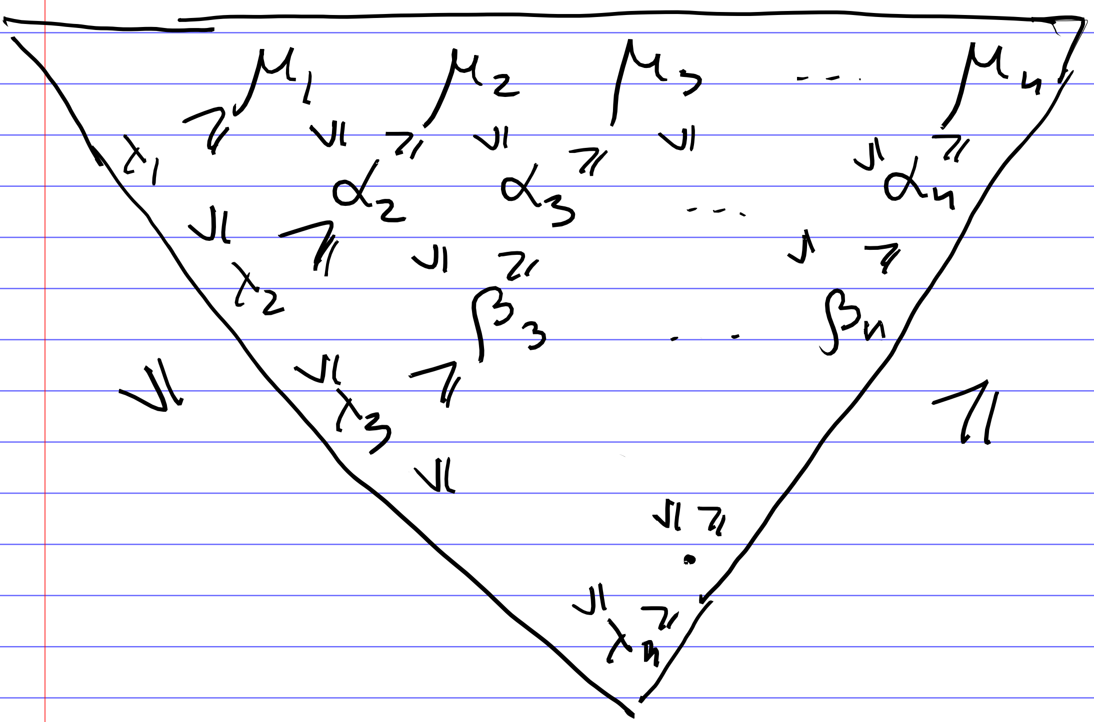
Let's convert T into Gelfand-Tsetlin-like patterns:



A half of this rhombic pattern is "frozen":



The "interesting part" of the pattern has the triangular form



- GT- conditions: rows are interlaced
- lattice word conditions:

$$a = \lambda_1 - \mu_1 \geq b = \lambda_2 - \alpha_2$$

$$b = \lambda_2 - \alpha_2 \geq d = \lambda_3 - \beta_3$$

$$a + c = (\lambda_1 - \mu_1) + (\alpha_2 - \mu_2) \geq b + e = (\lambda_2 - \alpha_2) + (\beta_3 - \alpha_3)$$

etc.

Exercise. Figure out the general form of these inequalities.

So $c^{\lambda}_{\mu\nu} =$

= # GT- patterns (as above) with some additional linear inequalities for certain lin. combinations of entries.
