last time:

- λ/μ skew Young diagram, $|\lambda/\mu| = n$
  $\beta = (\beta_1 \ldots \beta_k)$ composition of n

Young's orthogonal form $\Rightarrow$

$$\chi_{\lambda/\mu}(\beta) = \sum_{T \in SYT(\lambda/\mu)} \prod_{i \in [n] \setminus \{\beta_1, \beta_1+\beta_2, \ldots, \beta_1+\ldots+\beta_k\}} \frac{1}{c_{i+1} - c_i}$$

the value of character of $V_{\lambda/\mu}$ on a permutation with cyclic type β

$c(T) = (c_1 \ldots c_n)$ content vector of T

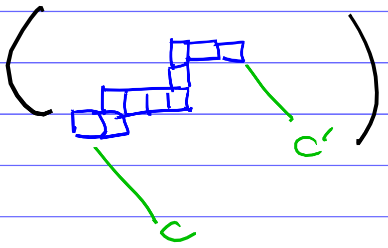
- **Murnaghan - Nakayama Rule**:

$$\chi_{\lambda/\mu}(\beta) = \sum_{\substack{RT \text{ ribbon tableau} \\ \text{of shape } \lambda/\mu \\ \text{and type } \beta}} (-1)^{ht(RT)}$$

- Moreover, we gave a multivariate generalization of the M.-N. rule where we replaced c_i by $-x_i$, where x_i's are variables:

$$\chi^{(x_i)}_{\lambda/\mu} = \sum_{\substack{RT \text{ ribbon tableau} \\ \text{of shape } \lambda/\mu \\ \text{and type } \beta}} wt(RT)$$

$$wt(RT) := \prod_{\text{ribbon in } RT} wt(\text{ribbon})$$

wt() $:= (-1)^{ht} \prod_{i \in \{c, c+1, \ldots, c'-1\}} \frac{1}{x_i - x_{i+1}}$

This M.-N. rule basically reduces to 3 lemmas about certain rational expressions in x_i's.

For $i_1, \dots, i_n \in \mathbb{Z}$, let

$$\langle i_1, \dots, i_n \rangle := \frac{1}{x_{i_1} - x_{i_2}} \frac{1}{x_{i_2} - x_{i_3}} \cdots \frac{1}{x_{i_{n-1}} - x_{i_n}}$$

Lemma 1. A, B non-empty disjoint sequences of int.

$$\sum_{C \in \text{Shuffle}(A,B)} \langle C \rangle = 0$$

all shuffles of A & B

Lemma 2. A, B, {1} non-empty disjoint

$$\sum_{C \in \text{Shuffle}(A,B)} \langle 1, C, 1 \rangle = 0$$

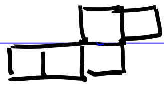
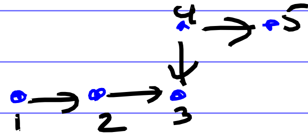
Lemma 3. (Tree relation)

Let T be any tree on vertices $\{1, \dots, n\}$ with directed edges.

$$\sum_{\substack{\sigma = \sigma_1 \dots \sigma_n \text{ permutation of } [n] \\ \sigma^{-1}(i) < \sigma^{-1}(j) \text{ for} \\ \text{any edge } i \to j \text{ in } T}} \langle \sigma_1 \dots \sigma_n \rangle = \prod_{\substack{i \to j \\ \text{edge in } T}} \langle ij \rangle$$

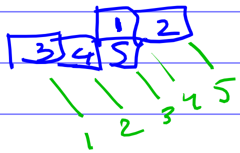
(For the M.N rule we need the "ribbon relation" which is a special case of Lemma 3 when T is a chain.)

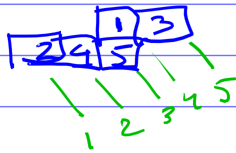
Example

 $\rightsquigarrow$ $T=$ 

$\sigma = (\sigma_1, \ldots, \sigma_n)$ correspond to SYT's of ribbon shape (i.e. linear extensions of the poset given by directed tree) where σ = content vector of tableau.

$$\langle 4,5,1,2,3\rangle + \langle 4,1,5,2,3\rangle + \cdots$$





$$= \langle 12\rangle\langle 23\rangle\langle 43\rangle\langle 45\rangle = (-1)\langle 1,2,3,4,5\rangle$$

wt(ribbon).

We'll prove these lemmas by induction. We need another lemma...

Lemma 4 $A, B, \{1\}$ disjoint

$$\sum_{C \in \mathrm{shuffle}(A,B)} \langle 1, C \rangle = \langle 1, A \rangle \langle 1, B \rangle$$

Example $A = (a_1)$ $B = (b_1, b_2)$

$$\langle 1, a_1, b_1, b_2 \rangle + \langle 1, b_1, a_1, b_2 \rangle + \langle 1, b_1, b_2, a_1 \rangle = \langle 1, a_1 \rangle \langle 1, b_1, b_2 \rangle$$

$$= \frac{1}{x_1 - x_{a_1}} \frac{1}{x_{a_1} - x_{b_1}} \frac{1}{x_{b_1} - x_{b_2}}$$

$$\frac{1}{x_1 - x_{a_1}} \cdot \frac{1}{x_1 - x_{b_1}} \cdot \frac{1}{x_{b_1} - x_{b_2}}$$

Proof of Lemma 4. Induction on $|A| + |B|$.

Base: $A = B = \emptyset$

LHS $= \langle 1 \rangle = 1$, RHS $= \langle 1 \rangle \langle 1 \rangle = 1$

(by convension, $\mathrm{shuffle}(\emptyset, \emptyset)$ consists of 1 element $\emptyset$; and $\prod_{\text{empty set}} = 1$)

Induction Step: Let

$A = (a_1, A')$ $B = (b_1, B')$

$$\sum_{C \in \mathrm{shuffle}(A,B)} \langle 1, C \rangle =$$

any shuffle of A & B starts with a_1 or b_1

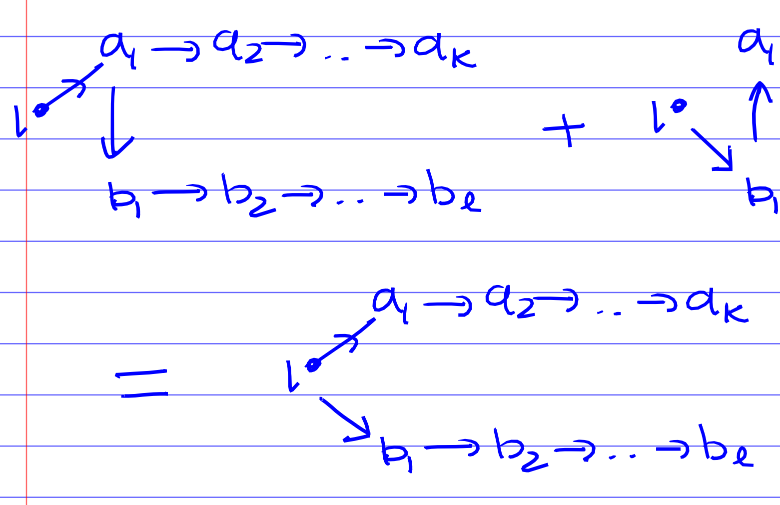
$$= \sum_{C' \in \mathrm{shuffle}(A', B)} \langle 1, a_1 \rangle \langle a_1 C' \rangle + \sum_{C'' \in \mathrm{shuffle}(A, B')} \langle 1, b_1 \rangle \langle b_1, C'' \rangle$$

$$\overset{\text{ind. hypothesis}}{=} \langle 1, a_1 \rangle \langle a_1 A' \rangle \langle a_1 B \rangle + \langle 1, b_1 \rangle \langle b_1 A \rangle \langle b_1 B' \rangle$$

$$= \left(\langle 1, a_1, b_1 \rangle + \langle 1, b_1, a_1 \rangle \right) \langle A \rangle \langle B \rangle$$

$$= \langle 1\, a_1 \rangle \langle 1, b_1 \rangle \langle A \rangle \langle B \rangle$$

$$= \langle 1, A \rangle \langle 1, B \rangle, \text{ as needed}$$



Here an edge $i \to j$ represents a term $\langle i j \rangle := \frac{1}{x_i - x_j}$

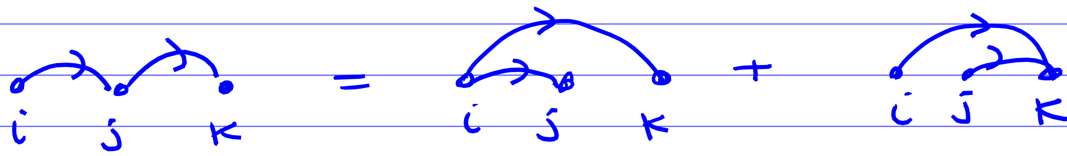
∎

follows from $x_i - x_k = (x_i - x_j) + (x_j - x_k)$.

Key Relation:

$$\frac{1}{x_i - x_j} \cdot \frac{1}{x_j - x_k} = \frac{1}{x_i - x_k} \cdot \frac{1}{x_i - x_j} + \frac{1}{x_j - x_k} \cdot \frac{1}{x_i - x_k}$$

$$\langle ij \rangle \cdot \langle jk \rangle = \langle ik \rangle \langle ij \rangle + \langle jk \rangle \langle ik \rangle$$



Proof of Lemma 1

$A = (a_1 \; A')$

$B = (b_1 \; , B')$

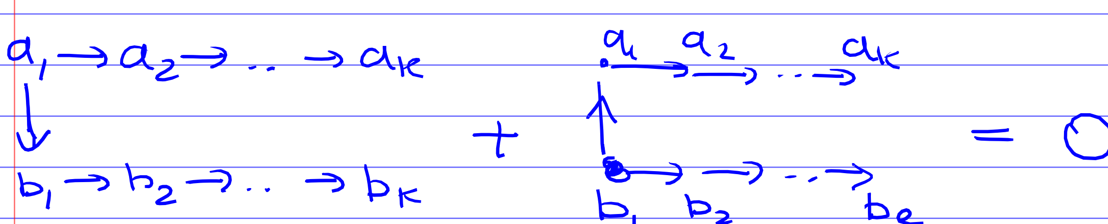
$$\sum_{C \in \text{shuffle}(A,B)} \langle C \rangle$$

$$= \sum_{C' \in \text{shuffle}(A', B)} \langle a_1 C' \rangle + \sum_{C'' \in \text{shuffle}(A, B')} \langle b_1 C'' \rangle$$

by Lemma 4

$$= \langle a_1 A' \rangle \langle a_1 B \rangle + \langle b_1 A \rangle \langle b_1 B' \rangle$$

$$= \big(\langle a_1 b_1 \rangle + \langle b_1 a_1 \rangle \big) \langle A \rangle \cdot \langle B \rangle = 0$$



$\square$

Proof of Lemma 3 (tree relation)

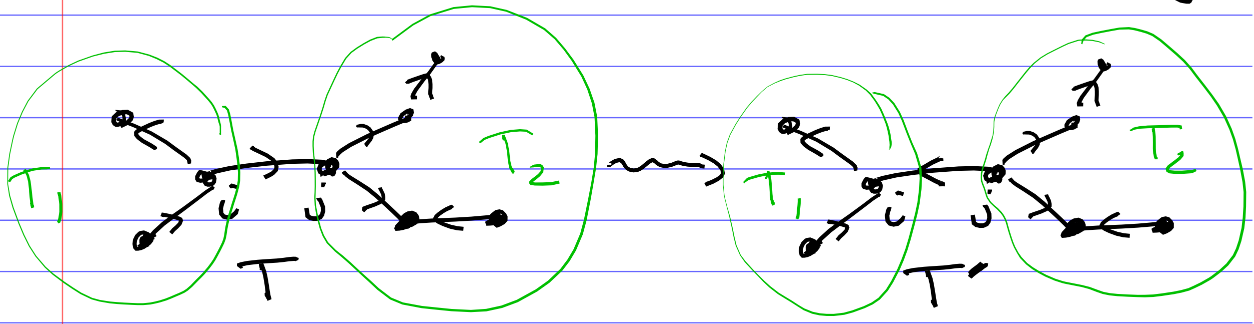
Induction on n (# vertices in a tree).

Need to show

$$\frac{1}{x_i - x_j}$$

$$\underbrace{\sum_{\sigma \leftrightarrow \text{lin. ext. of } T} \langle \sigma \rangle}_{\text{LHS}} = \underbrace{\prod_{\substack{i \to j \\ \text{edges of } T}} \langle ij \rangle}_{\text{RHS}}$$

Let T' is obtained from T by reversing direction of one edge $\overset{\to}{i\ j}$



Clearly, $(\text{RHS for } T) = -(\text{RHS for } T')$

$(\text{LHS for } T) + (\text{LHS for } T')$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma^{-1}(a) < \sigma^{-1}(b) \text{ for} \\ \text{all edges } a \to b \text{ of} \\ \text{the forest } F := T - (\text{edge } i \to j)}} \langle \sigma \rangle$$

lin. extension of F

Since forest F has 2 connected components T_1 & T_2

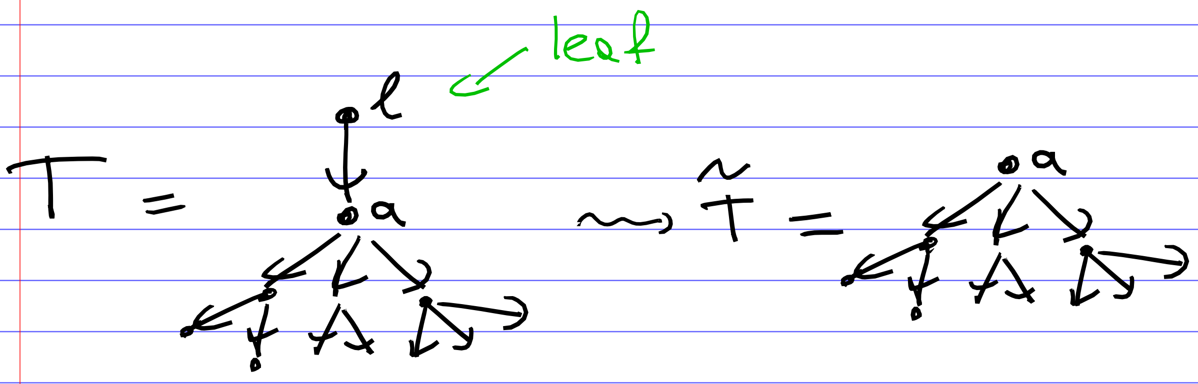
$$= \sum_{\substack{A \text{ lin. ext. of } T_1 \\ B \text{ lin. ext. of } T_2}} \sum_{C \in \text{Shuffle}(A,B)} \langle C \rangle$$

$= 0$ (by lemma 1).

So $(\text{LHS for } T) = -(\text{LHS for } T')$.

So when we reverse directions of edges in T, both sides of the needed equality simultaneously change signs.

Let us pick a leaf ℓ of T and direct all edges away from ℓ



any lin. ext. of T starts as $\ell, a, \ldots$

So LHS for $T = \langle \ell\, a \rangle \cdot$ LHS for $\tilde{T}$,

$\tilde{T} = T \setminus$ leaf ℓ

Clearly, RHS for $T = \langle \ell\, a \rangle$ RHS for $\tilde{T}$

By induction, needed equality holds for $\tilde{T}$. Done. $\square$

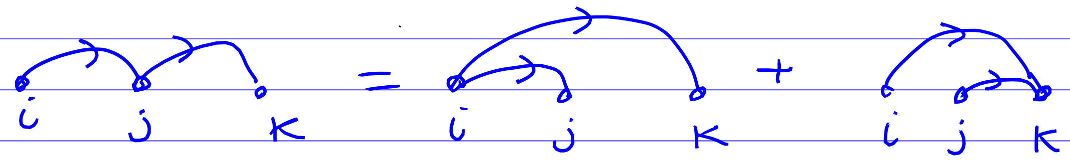
We proved all lemmas, except Lem.2.

Exercise. Prove lemma 2:

$$\sum_{C \in \mathrm{shuffle}(A,B)} \langle 1, C, 1 \rangle = 0$$

for $|A|, |B| \geq 1$.

More on the "Key Relation":



$\langle ij\rangle \langle jk\rangle = \langle ik\rangle \langle ij\rangle + \langle jk\rangle \langle ik\rangle$

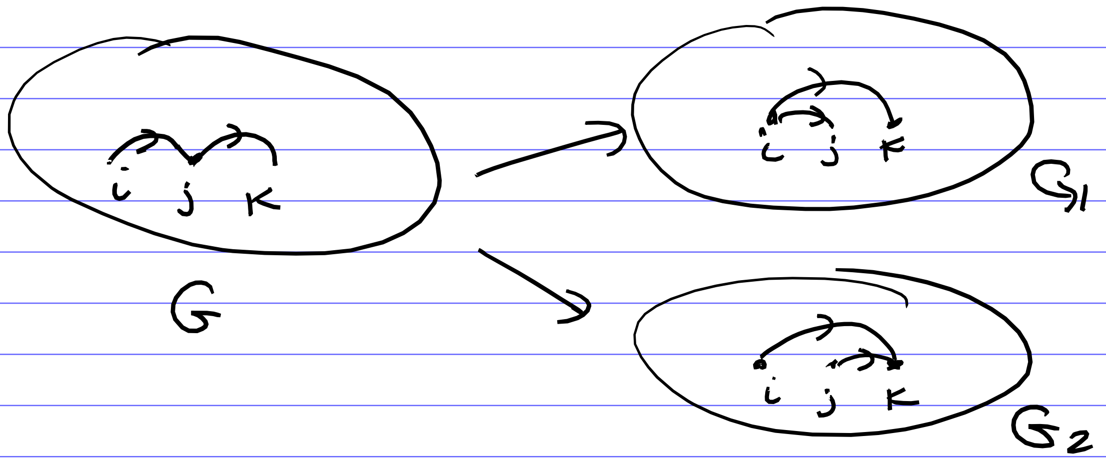
Let's play the following game on graphs:

- Start with a graph G on vertices $1, \dots, n$ with edges directed as $i \to j$ for $i<j$

- If we can find 2 edges e_1, e_2 in G s.t. $i \xrightarrow{e_1} j \xrightarrow{e_2} k$

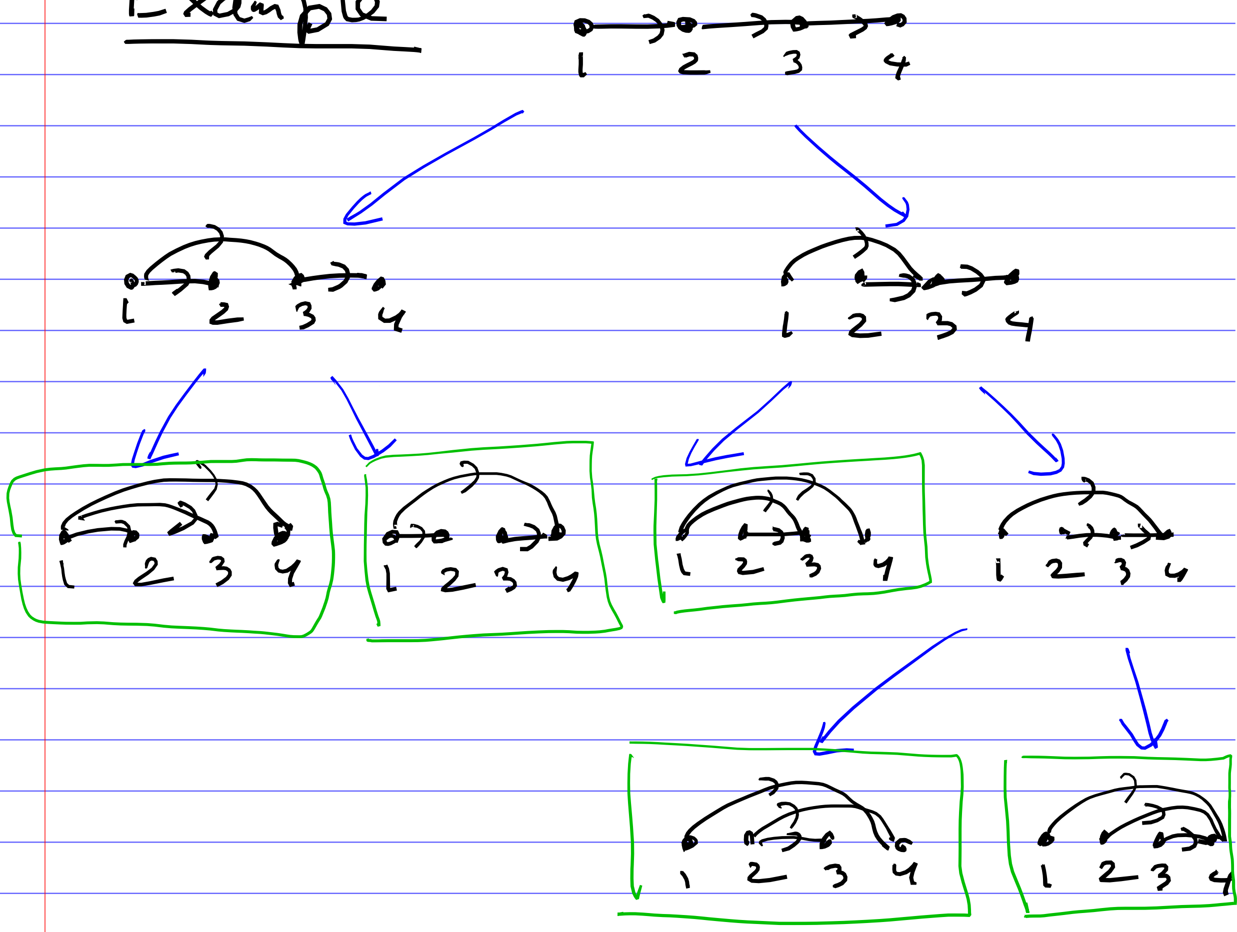
Then replace G by the sum of two graphs G_1 & G_2 (in the space of formal linear combinations of graphs) obtained from G by replacing the edges e_1 & e_2 with

e_1 & $e_3 = i \to k$ or e_2 & e_3



- Then apply the same operations to graphs G_1 & G_2 and continue untill cannot find a pair of edges $\bullet\to\bullet\to\bullet$

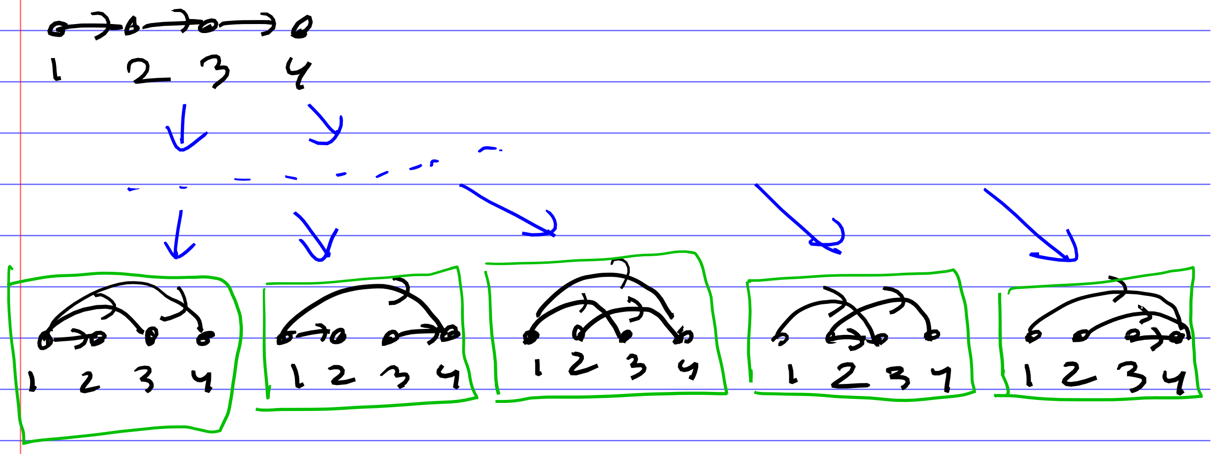
Example



Thus we get the identity:

$$\langle 1\ 2\ 3\ 4 \rangle = \sum_{\substack{5 \text{ endpoints } T \\ \text{of this game}}} \langle T \rangle$$

There are many ways to play this game. Another way to play it produces:



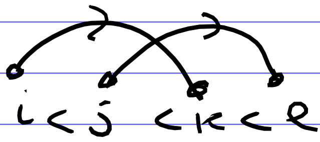
But the number of end-points is the same...

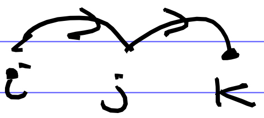
Theorem For any initial graph G, any way to play the game produces the same number of end-points.

Example $G = 1 \to 2 \to 3 \to \cdots \to n$

Theorem. (1) # end-points in this case is the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$

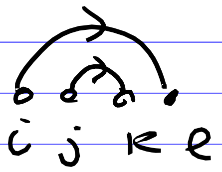
(2) One way to play the game produces the sum over **non-crossing alternating trees** on $[n]$.

- non-crossing: no 

- alternating: no 

(every vertex is either a source or a sink)

(3) Another way to play the game produces the sum over **non-nesting alternating trees** on $[n]$

- non-nesting: no 

So we obtain 2 identities:

$$\langle 1,2,\ldots,n\rangle = \sum_{\substack{T \text{ non-crossing} \\ \text{alternating} \\ \text{tree on } [n]}} \langle T\rangle$$

$$= \sum_{\substack{\tilde{T} \text{ non-nesting} \\ \text{alternating tree} \\ \text{on } [n]}} \langle \tilde{T}\rangle$$

where $\langle T\rangle := \prod_{\substack{i\to j \text{ edge} \\ \text{of } T}} \langle ij\rangle$.

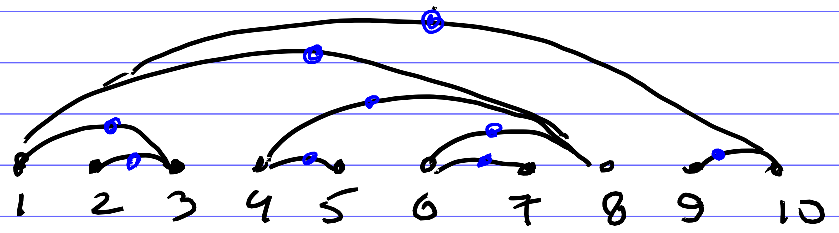
Bijection: $\left\{\begin{array}{l}\text{non-crossing} \\ \text{alternating trees} \\ \text{on } [n]\end{array}\right\}$

$\updownarrow$ bij

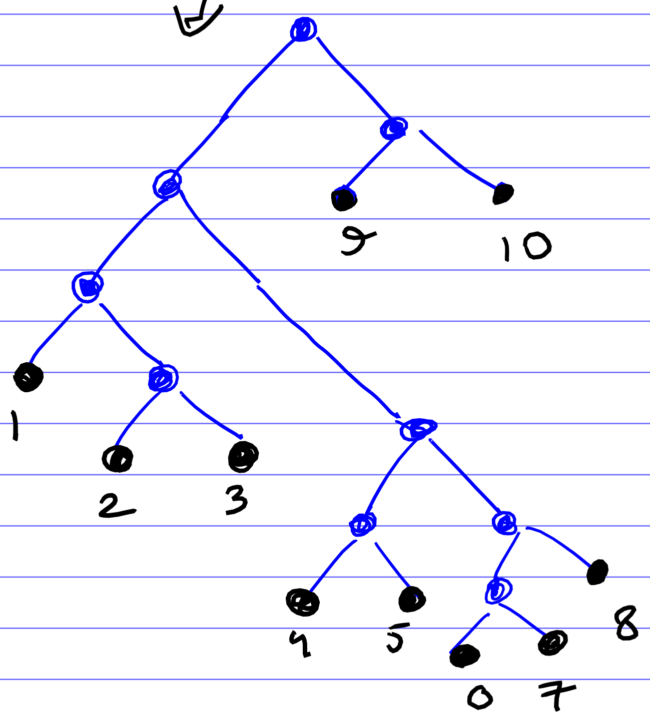
$\left\{\begin{array}{l}\text{complete plane} \\ \text{binary trees} \\ \text{with } n \text{ leaves}\end{array}\right\}$

it is well known that # such binary trees is the Catalan number C_n

Ex.



a non-crossing alternating treee



complete binary tree (every non-leaf vertex has exactly 2 children)

A more non-trivial result...

Theorem For $G = K_n$, the game produces
$C_{n-1} \cdot C_{n-2} \cdot \ldots \cdot C_1$ end-points

This theorem is equivalent to Chan-Robbin-Yuen conjecture, which was proved analytically by Zeiberger in 1998.

[D. Zeilberger] Proof of a conjecture of Chan-Robbins-Yuen.

Open problem (for 22 years)
Find a combinatorial proof of this theorem.

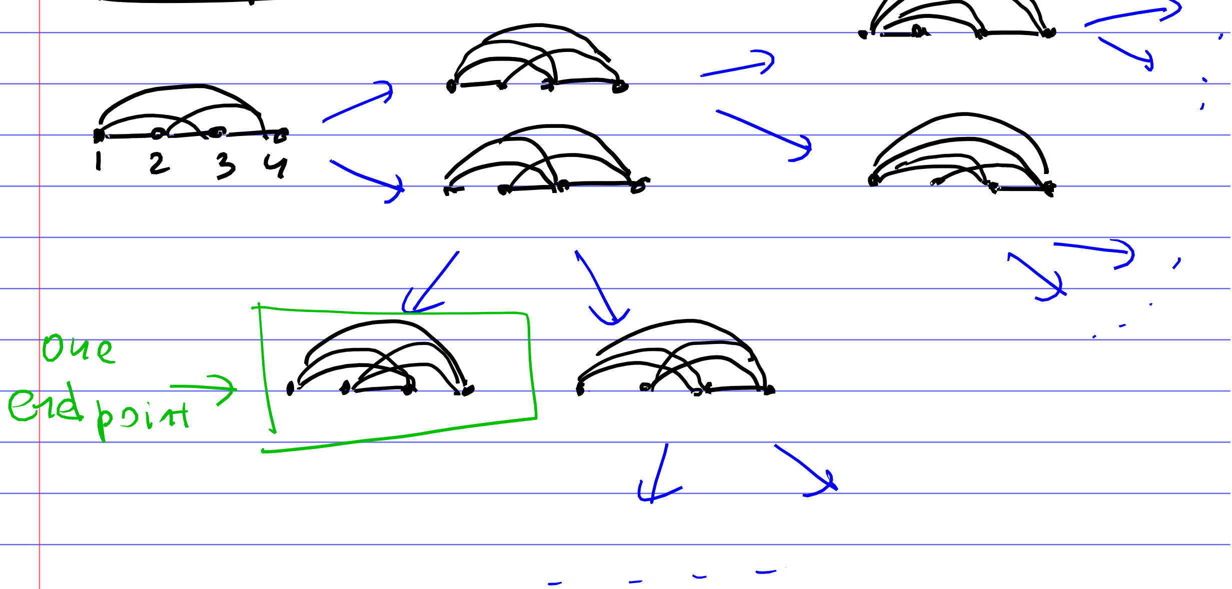
This stuff is closely related to Kostant's partition function & flow polytopes that we discussed before...

The Above Theorem $\Longleftrightarrow$

Theorem $K(1, 2, \ldots, n-1, -\binom{n}{2}) =$
$= C_1 C_2 \cdots C_{n-1}$.

the value of Kostant's partition function $\overset{\text{def}}{=}$ # number of ways to express this vector as a non-negative integer linear combination of vectors $e_i - e_j$, $1 \leq i < j \leq n$.

Example $G = K_4$



this game will have $C_3 \cdot C_2 = 10$ end-points no matter how you play it.

This stuff is also related to root polytopes

The root polytope

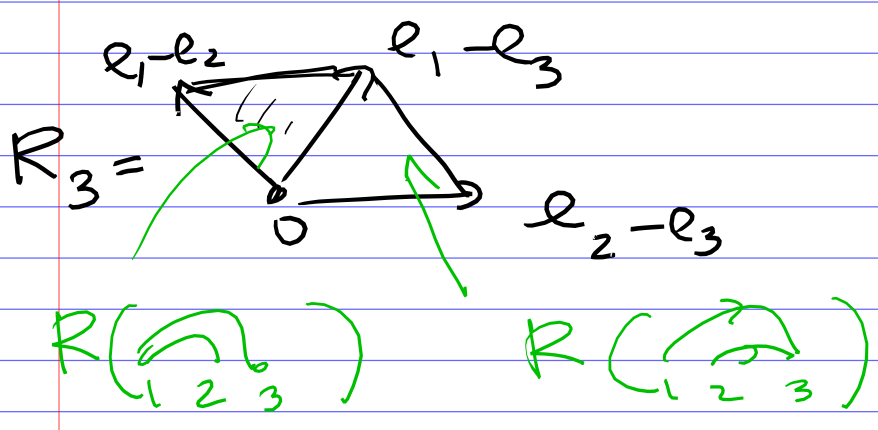
$R_n := \mathrm{conv}(0,\ e_i - e_j,\ 1 \le i < j \le n)$

For a tree T on $[n]$
(with edges directed as $i \to j$, $i<j$)

$R(T) := R_n \cap \langle$ positive span of $e_i - e_j$ for edges $i \to j$ in $T \rangle$

$R_n = R(1 \to 2 \to 3 \cdots \to n)$

Ex. $n = 3$



So $R(1 \to 2 \to 3) =$

$= R(1 \to 3,\ 1 \to 2) \cup R(1 \to 3,\ 2 \to 3)$

a union of two polytopes with disjoint interior

More generally,

$R(i \to j \to k) = R(i \to k,\ i \to j) \cup R(i \to k,\ j \to k)$

So playing the game on trees is equivalent to subdividing root polytopes into smaller pieces.

Theorem [GGP]

There are 2 triangulations of the root polytope R_n into unit simplices

$$R_n = \bigcup_{\substack{T \text{ non-crossing} \\ \text{alternating tree on } [n]}} R(T)$$

the "non-crossing" triangulation

$$= \bigcup_{\substack{\tilde{T} \text{ non-nesting} \\ \text{alternating tree} \\ \text{on } [n]}} R(\tilde{T})$$

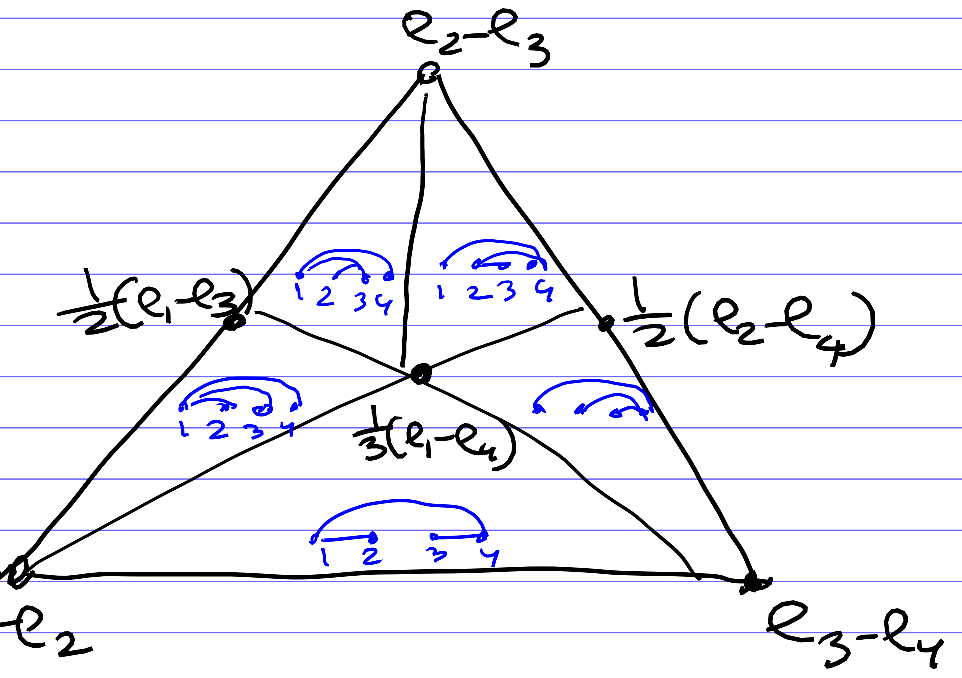
the "non-nesting" triangulation

In particular,

$$\mathrm{Vol}(R_n) = \frac{1}{(n-1)!} C_{n-1}.$$

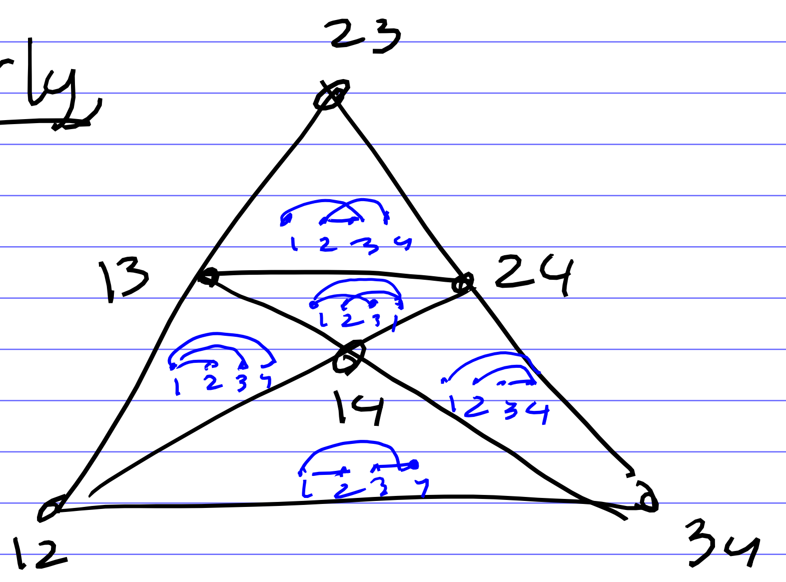
Example $n = 4$,

R_4 is 3-dimensional, but we will represent it by the 2-dim section with the affine plane passing through the points $e_1 - e_2$, $e_2 - e_3$, $e_3 - e_4$



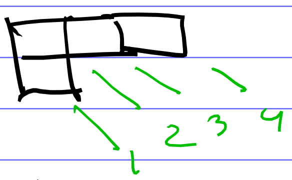
the non-crossing triangulation of R_4

Similarly,



the non-nesting triangulation of R_4

Example Ribbon relation

for 

3 SYTs:

1	2	3
4		

1	2	4
3		

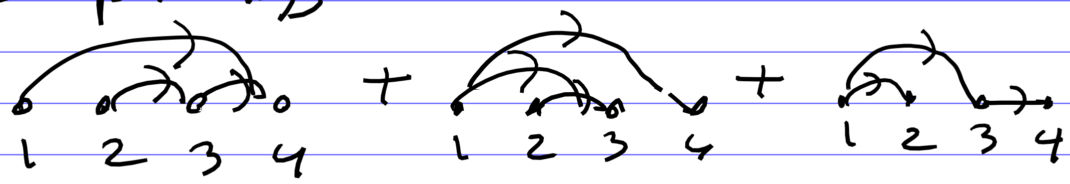
1	3	4
2		

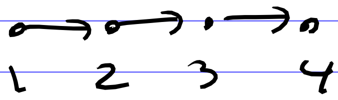
$$\langle 2,3,4,1\rangle + \langle 2,3,1,4\rangle + \langle 2,1,3,4\rangle$$

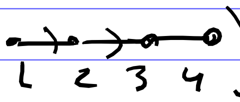
$$= - \langle 1,2,3,4\rangle$$

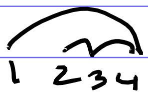
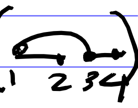
$(-1)^{4+}$

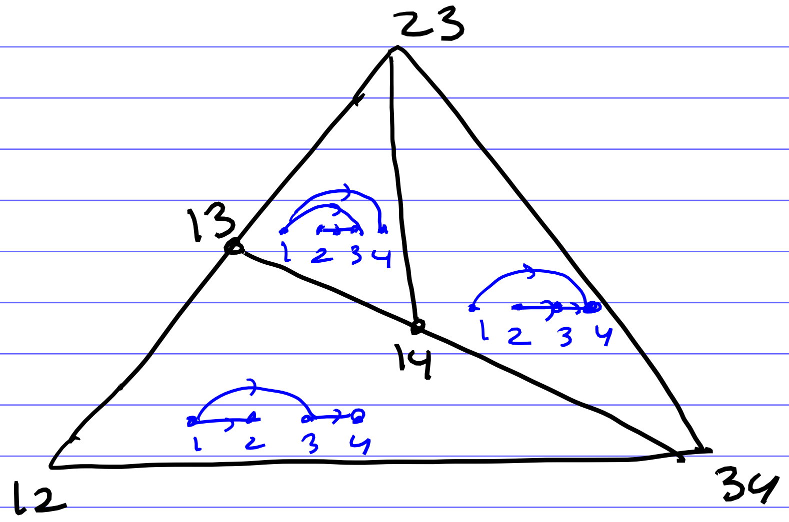
Graphically,



= 

Geometrically, $R_4 = R($  $)$

$= R($  $) \cup R($  $) \cup R($  $)$



Orlik-Terao Algebra

(for the braid arrangement of type A_{n-1})

$OT_n :=$ the algebra of rational functions on $\mathbb{C}^n$ generated by $\langle ij \rangle = \frac{1}{x_i - x_j}$ for $1 \le i \ne j \le n$.

Equiv. $OT_n =$ the commutative algebra over $\mathbb{C}$ with generators $\langle ij \rangle$, $i, j \in [n]$, $i \ne j$, and relations:

- $\langle ij \rangle = - \langle ji \rangle$
- $\langle ij \rangle \langle jk \rangle = \langle ik \rangle \langle ij \rangle + \langle jk \rangle \langle ik \rangle$

Basically, all relations that we discussed today are relation in the Orlik-Terao algebra OT_n.

Clearly, the symmetric group S_n acts on OT_n by permutations of the coordinates x_i.

$$\underset{S_n}{\overset{w}{\rotatebox{90}{$\in$}}} : \langle ij \rangle \longmapsto \langle w(i)\; w(j) \rangle$$

Some interesting S_n-invariant subspaces of OT_n

- <u>Tree space</u>

T_n = the space spanned by $\langle T \rangle := \prod_{(i,j)\text{ edge of } T} \langle ij \rangle$

T is a tree on $[n]$

- <u>Forest space</u>

F_n = the space spanned by $\langle F \rangle := \prod_{(i,j)\text{ edge of } F} \langle ij \rangle$

F is a forest on $[n]$

<u>Clearly</u> the forest space is graded by # edges in forests.

$$F_n = F_n^0 \oplus F_n^1 \oplus \dots \oplus F_n^{n-k}$$

F_n^k the space spanned by forests on $[n]$ with k edges.

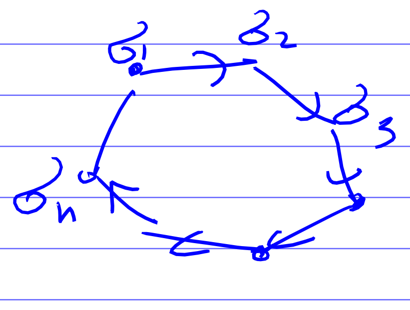
Clearly $T_n = F_n^{n-1}$.

- Parke-Taylor space

PT_n = the subspace of OT_n spanned by Parke-Taylor factors

$PT(\sigma) = \langle \sigma_1 \sigma_2 \dots \sigma_n \sigma_1 \rangle$

$\sigma = \sigma_1 \dots \sigma_n$ is a permutation of $1, 2, \dots, n$.



Each of these spaces

T_n, F_n, F_n^k, PT_n

is not just a space by a representation of S_n.

<u>Problem</u> Investigate these representation. Find their dimensions, characters, decomp. into irreducibles, etc.

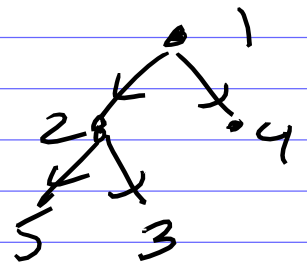
<u>Theorem</u>. • $\dim T_n = (n-1)!$

• $\dim F_n = n!$

• $F_n \simeq$ the regular representation of S_n

• 2 linear bases of T_n ($\&$ F_n):

- all <u>increasing</u> trees (or forests),



oriented away from a root at 1, the labels inrease as we go away from the root.

- all chains $\langle 1, i_1, i_2, \ldots i_{n-1} \rangle$

$i_1, \ldots, i_{n-1}$ a permutation of $2, 3, \ldots, n$.

Example: F_3

$F_3^0 \cong V_{(3)}$ identity rep of S_3

dim $= 1$

$F_3^1 \cong V_{(1,1,1)} \oplus V_{(2,1)}$ dim $= 3$

$F_3^2 = T_2 \cong V_{(2,1)}$ dim $= 2$

Theorem The Hilbert polynomial of F_n is

$$\sum_{k=0}^{n-1} \dim F_n^k \, t^k =$$

$$= (1+t)(1+2t) \cdots (1+(n-1)t).$$