recall from last time:

λ/μ a skew Young diagram, $|\lambda/\mu| = n$.

$V_{\lambda/\mu} :=$ linear space with basis

$$\{v_T \mid T \in SYT(\lambda/\mu)\}$$

Young's orthogonal form: The action of S_n on $V_{\lambda/\mu}$ given by

$$R: S_n \to GL(V_{\lambda/\mu})$$
$$s_i \mapsto R_i \quad , \quad i = 1, \dots, n-1$$

$$\boxed{R_i : v_T \mapsto \frac{1}{c_{i+1} - c_i} v_T + \sqrt{1 - \frac{1}{(c_{i+1} - c_i)^2}} \cdot v_{s_i(T)} \quad \text{for } i = 1, 2, \dots, n-1}$$

where $(c_1, \dots, c_n) = c(T)$ the content vector of T, i.e.

c_i = the content of box filled with i in the tableau T.

$s_i(T)$ = the SYT obtained from T by switching i & $i+1$

($s_i(T)$ is not defined if $c_{i+1} - c_i = \pm 1$)

Characters of representations of S_n

$\chi_{\lambda/\mu}$ is the character of $V_{\lambda/\mu}$.

For a composition $\beta = (\beta_1, \ldots, \beta_k)$ of n,

$$R_\beta := (R_1 R_2 \ldots R_{\beta_1 - 1}) \cdot (R_{\beta_1+1} \cdots R_{\beta_1+\beta_2-1}) \cdots$$

skip these terms

$$= R_1 R_2 \ldots \widehat{R}_{\beta_1} \ldots \widehat{R}_{\beta_1+\beta_2} \ldots \widehat{R}_{\beta_1+\beta_2+\beta_3} \ldots R_{n-1}$$

$$\in GL(V_{\lambda/\mu}).$$

(Note $(s_1 \ldots s_{\beta_1-1})(s_{\beta_1+1} \ldots s_{\beta_1+\beta_2-1}) \ldots$

cycles of sizes $\beta_1, \beta_2, \ldots, \beta_k$

$= (1\,2 \ldots \beta_1)(\beta_1+1 \ldots \beta_1+\beta_2) \ldots (\beta_1+\ldots+\beta_{k-1}+1, \ldots, n)$

is a permutation in S_n of the cyclic type β.)

Character values:

$$\chi_{\lambda/\mu}(\beta) := tr(R_\beta).$$

trace of the operator R_β acting on $V_{\lambda/\mu}$

(Note that $\chi_{\lambda/\mu}(\beta)$ does not change if we permute parts of β, because a character is constant on conjugacy classes.)

Murnaghan – Nakayama Rule

$$\boxed{\chi_{\lambda/\mu}(\beta) = \sum_{\substack{RT \text{ ribbon tableau} \\ \text{of shape } \lambda/\mu \\ \text{and type } \beta}} (-1)^{ht(RT)}}$$

the sizes of ribbons are $\beta_1, \beta_2, \ldots$

Example: A ribbon tableau

$$RT = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 3 & 3 & 4 & 6 \\ 3 & 3 & 4 & 4 & \\ 5 & 5 & & & \\ 5 & & & & \end{array}$$

shape $\lambda = (5,5,5,4,2,1)$

type $\beta = (6,5,4,3,3,1)$

height $ht(RT) = 2+1+1+1+1+0$
$= 6$

(height of a ribbon =
$=$ # rows -1.)

Observation Only diagonal terms

in $R_i : \delta_T \mapsto \frac{1}{c_{i+1}-c_i}\,\delta_T + \sqrt{\cdots}\,\delta_{s_i(T)}$

make a contribution to

$$tr(R_\beta) := \sum_{T \in SYT(\lambda/\mu)} (\text{coeff. of } \delta_T \text{ in } R_\beta(\delta_T))$$

So $\chi_{\lambda/\mu}(\beta) \overset{\text{def}}{:=}$

$$= \sum_{T \in SYT(\lambda/\mu)} \prod_{i \in [n] \setminus \{\beta_1, \beta_1+\beta_2, \ldots, \beta_1+\cdots+\beta_\ell\}} \frac{1}{c_{i+1}-c_i}$$

$c(T) = (c_1, \ldots, c_n)$
content vector of T

Example

$\lambda/\mu = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \multicolumn{1}{|c|}{\ } \\ \cline{1-1}\end{array}$, $\beta = (4)$

$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array}$ $\qquad$ $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}$ $\qquad$ $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1}\end{array}$

$c(T) = (0,1,2,-1)$ $\qquad$ $c(T) = (0,1,-1,2)$ $\qquad$ $c(T) = (0,-1,1,2)$

$$\chi_\lambda(\beta) := \frac{1}{(1-0)(2-1)(-1-2)}$$

$$+ \frac{1}{(1-0)(-1-1)(2-(-1))}$$

$$+ \frac{1}{((-1)-0)(1-(-1))(2-1)}$$

$$= -\frac{1}{3} - \frac{1}{6} - \frac{1}{2} = -1.$$

M-N Rule: $\chi_{31}((4)) = (-1)^1$

only 1 ribbon tableau $RT = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1}\end{array}$

with $ht(RT) = -1$.

A multivariate generalization of Murnaghan - Nakayama Rule.

$x = (x_i)_{i \in \mathbb{Z}}$ collection of variables x_i

Define the x-characters

$$\chi^x_{\lambda/\mu}(\beta) :=$$

$$:= \sum_{T \in SYT(\lambda/\mu)} \prod_{i \in [n] \setminus \{\beta_1, \beta_1+\beta_2, \ldots, \beta_1+\ldots+\beta_k\}} \frac{1}{x_{c_i(T)} - x_{c_{i+1}(T)}}$$

basically the same expression as for $\chi_{\lambda/\mu}(\beta)$ except that we use arbitrary variables x_i instead of contents.

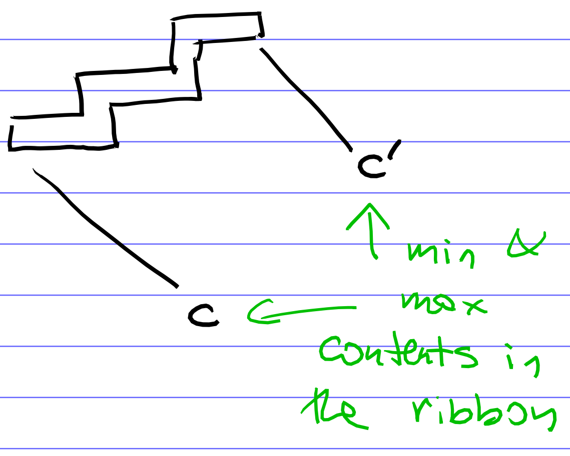
Theorem.

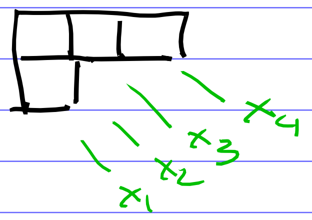
$$\boxed{\chi^x_{\lambda/\mu}(\beta) = \sum_{\substack{RT \text{ ribbon tableau} \\ \text{of shape } \lambda/\mu \\ \text{and type } \beta}} wt(RT)}$$

where $wt(RT) := \prod_{\text{ribbon} \in RT} wt(\text{ribbon})$

$$wt(\text{ribbon}) := (-1)^{ht} \prod_{i \in \{c, c+1, \ldots, c'-1\}} \frac{1}{x_i - x_{i+1}}$$

for a ribbon



Example $\lambda/\mu =$  $\beta = (4)$

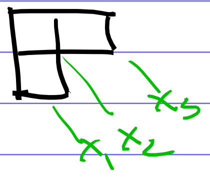
$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1}\end{array}$ $\qquad \dfrac{1}{(x_2-x_3)(x_3-x_4)(x_4-x_1)}$

$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & & \\ \cline{1-1}\end{array}$ $\qquad + \dfrac{1}{(x_2-x_3)(x_3-x_1)(x_1-x_4)}$

$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & & \\ \cline{1-1}\end{array}$ $\qquad + \dfrac{1}{(x_2-x_1)(x_1-x_3)(x_3-x_4)}$

$$\overset{\text{theorem}}{=} (-1)\frac{1}{(x_1-x_2)(x_2-x_3)(x_3-x_4)}.$$

Remark $\chi^{\times}_{\lambda/\mu}(\beta)$ changes when we permute the entries of β.

Example $\lambda/\mu =$  $\beta = (2,1)$ and $\beta = (1,2)$

$$\chi^{\times}_{\boxplus}((2,1)) := \frac{1}{x_2-x_3} + \frac{1}{x_2-x_1}$$

SYTs: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$ $\qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$

$$\overset{\text{thm}}{=} \frac{1}{x_2-x_3} + \frac{(-1)}{x_1-x_2}$$

ribbon tableaux: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$ $\qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \cline{2-2} 1 & \\ \cline{1-1}\end{array}$

$$\chi^{\times}_{\boxplus}(1,2) := \frac{1}{x_3-x_1} + \frac{1}{x_1-x_3}$$

SYTs: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$ $\qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$

$\overset{\text{thm}}{=} 0$ there is no ribbon tableaux of type $(1,2)$

Proof of generalized M-N rule...

It is not hard to deduce the thm from its special case for $\beta = (n)$.

For a sequence $(i_1, \dots, i_n)$ of integers, denote

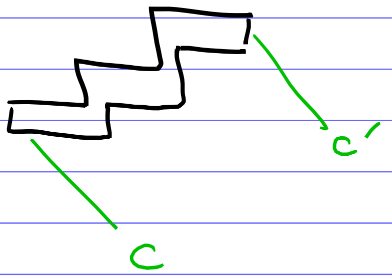
$$\langle i_1, \dots, i_n \rangle := \frac{1}{(x_{i_1} - x_{i_2})} \cdot \frac{1}{(x_{i_2} - x_{i_3})} \cdots \frac{1}{(x_{i_{n-1}} - x_{i_n})}.$$

Theorem

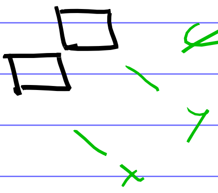
$$\sum_{T \in SYT(\lambda/\mu)} \langle c(T) \rangle = \begin{cases} wt(\text{ribbon}) & \text{if } \lambda/\mu \text{ is a ribbon} \\ 0 & \text{otherwise} \end{cases}$$

$$wt(\text{ribbon}) := (-1)^{ht} \langle c, c+1, \dots, c' \rangle = (-1)^{ht} \prod_{i=c}^{c'-1} \frac{1}{x_i - x_{i+1}}$$

for ribbon =

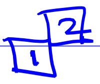
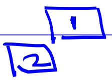


Examples:

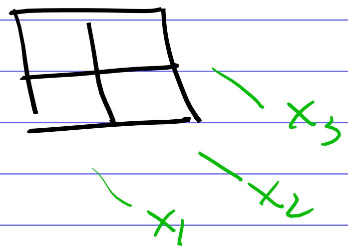
$\lambda/\mu =$ 

a disconnected skew diagram

$$\chi^{(x,y)}_{\lambda/\mu}((2)) = \frac{1}{x-y} + \frac{1}{y-x} = 0$$

Thm says that we will always get 0 if λ/μ is disconnected.

$\lambda/\mu =$ 

$$\chi^{x}_{\lambda/\mu}((4)) =$$

$$= \frac{1}{(x_2-x_3)(x_3-x_1)(x_1-x_2)} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

$$+ \frac{1}{(x_2-x_1)(x_1-x_3)(x_3-x_2)} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

$$= 0$$

Thm says that we will always get 0 if at least 1 diagonal of λ/μ has ≥ 2 boxes.

How to prove this?

2 shuffle lemmas:

$A = (a_1, \dots, a_k)$ $\quad B = (b_1, \dots, b_\ell)$
two disjoint sequences of integers.

$\mathrm{shuffle}(A, B) \overset{\text{def}}{:=}$ the set of all shuffles of A, B

Example: $\mathrm{Shuffle}((a_1, a_2), (b_1, b_2))$

$$= \{(a_1\, a_2\, b_1\, b_2), (a_1\, b_1\, a_2\, b_2), (a_1\, b_1\, b_2\, a_2), (b_1\, a_1\, a_2\, b_2), (b_1\, a_1\, b_2\, a_2), (b_1\, b_2\, a_1\, a_2)\}$$

$\mathrm{Shuffle}(A, B)$ contains $\binom{k+\ell}{\ell}$ sequences.

Lemma 1. $\displaystyle\sum_{C \in \mathrm{shuffle}(A,B)} \langle C \rangle = 0$

Lemma 2. $A, B, \{1\}$ disjoint

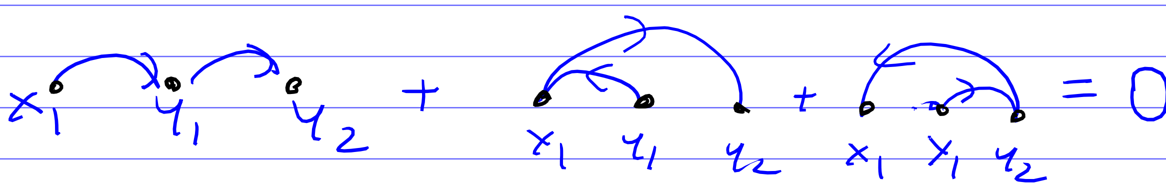
$$\sum_{C \in \mathrm{Shuffle}(A,B)} \langle 1, C, 1 \rangle = 0$$

Examples $|A| = 1$, $|B| = 2$

Lemma 1 $\Rightarrow$

$$\Rightarrow \frac{1}{(x_1 - y_1)(y_1 - y_2)} + \frac{1}{(y_1 - x_1)(x_1 - y_2)} + \frac{1}{(y_1 - y_2)(y_2 - x_1)}$$

$$= 0$$



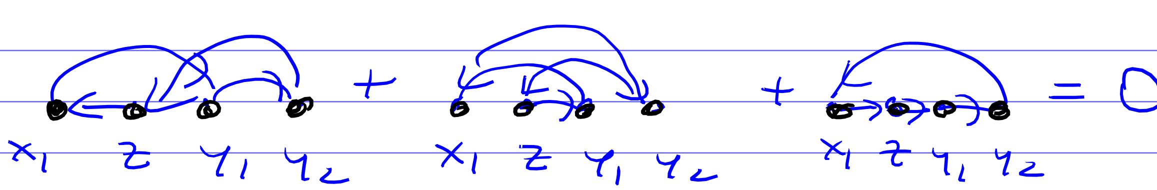
Lemma 2 $\Rightarrow$

$$\frac{1}{(z - x_1)(x_1 - y_1)(y_1 - y_2)(y_2 - z)}$$

$$+ \frac{1}{(z - y_1)(y_1 - x_1)(x_1 - y_2)(y_2 - z)}$$

$$+ \frac{1}{(z - y_1)(y_1 - y_2)(y_2 - x_1)(x_1 - z}$$

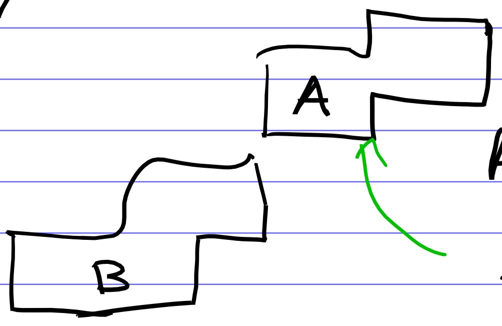
$$= 0$$



Lemma 1 $\Rightarrow$ $\chi^{\times}_{\lambda/\mu}((n)) = 0$

if λ/μ is disconnected:

λ/μ



A = the content vector of some SYT for

B = the content vectors of some SYT for

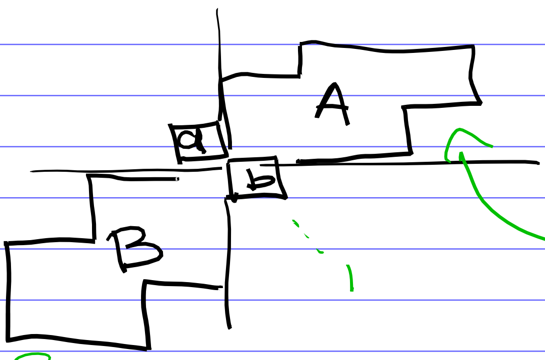
All SYT's of shape λ/μ obtained by shuffles of such A's & B's

$$\chi^{\times}_{\lambda/\mu}((n)) = \sum_{A,B} \Big(\sum_{C \in \text{Shuffle}(A,B)} \langle C \rangle \Big)$$

$$= 0$$

Lemma 2 $\Rightarrow$

$\chi^{\times}_{\lambda/\mu}((n)) = 0$ if some diag. of λ/μ contains ≥ 2 boxes:



A some content vector for

B - some content vector for

this is the part of λ/μ filled by entries $i \in \{a, a+1, \dots, b\}$.

Any shuffle of A, B produce a valid SYT.

$\chi^{\times}_{\lambda/\mu}((n)) =$

$$= \sum_{A,B} \Big(\sum_{C \in \text{Shuffle}(A,B)} \langle 1, C, 1 \rangle \Big) \left(\begin{array}{l}\text{possibly}\\ \text{some}\\ \text{other}\\ \text{factors}\end{array}\right)$$

$$= 0.$$

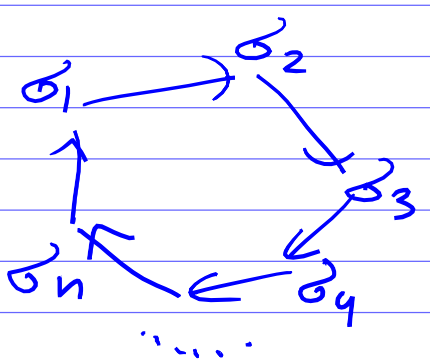
Such expressions appeared in physics of <u>scattering amplitudes</u> under the name

<u>Parke-Taylor</u> <u>factors</u>

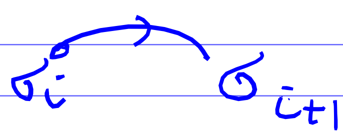
$\sigma = \sigma_1, \dots, \sigma_n$ a permutation of $[n]$

$$PT(\sigma) := \frac{1}{x_{\sigma_1} - x_{\sigma_2}} \cdot \frac{1}{x_{\sigma_2} - x_{\sigma_3}} \cdots$$

$$\cdots \frac{1}{x_{\sigma_{n-1}} - x_{\sigma_n}} \cdot \frac{1}{x_{\sigma_n} - x_{\sigma_1}}$$



edges



represent factors $\frac{1}{x_{\sigma_i} - x_{\sigma_{i+1}}}$

$$PT(\sigma) := \langle \sigma_1 \sigma_2 \dots \sigma_n \sigma_1 \rangle$$

Physicists proved

Kleiss – Kuijf Relations

for Parke–Taylor factors:

$A = (a_1, \dots, a_k)$, $B = (b_1, \dots, b_\ell)$

two disjoint sequences s.t.

$$A \cup B = \{2, \dots, n-1\}$$

Kleiss – Kuijf:

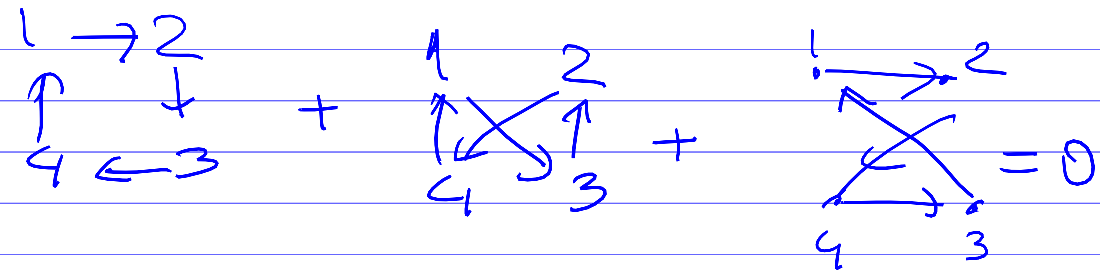
$$\boxed{PT(1, A, n, B) = (-1)^{|B|} \sum_{C \in \mathrm{shuffle}(A, B^T)} PT(1, C, n)}$$

$$B^T := (b_\ell, b_{\ell-1}, \dots, b_1)$$

Example: $n = 4$, $A = (2)$, $B = (3)$

$$PT(1,2,4,3) =$$

$$= -\big(PT(1,2,3,4) + PT(1,3,2,4)\big).$$



an edge $i \to j$ represents factor

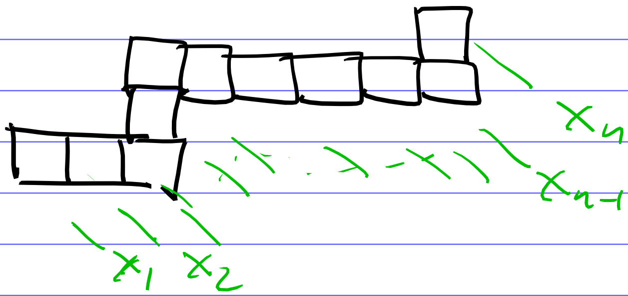
$$\frac{1}{x_i - x_j}$$

Kleiss - Kuijf rels. $\Rightarrow$

Lemma 2 $\Rightarrow$ Lemma 1.

Lemma 3 (Ribbon Relation)

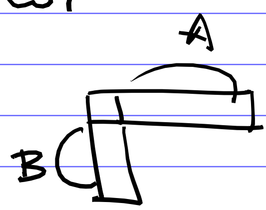
"Ribbon" = any ribbon with n boxes



(Assume that contents in Ribbon $\in \{1, \dots, n\}$)

$$\boxed{\sum_{T \in SYT(\text{Ribbon})} \langle c(T) \rangle = (-1)^{n+} \prod_{i=1}^{n-1} \frac{1}{x_i - x_{i+1}}}$$

In fact, Lemma 3 for

Ribbon = hook = 

$\Leftrightarrow$ Kleiss - Kuijf rels.

More general "Tree relation"

T any tree on vertices $1, \dots, n$ with directed edges $\underset{i\ \ j}{\rightarrow}$

$S_n(T) :=$ the set of permutations $\sigma_1, \dots, \sigma_n \in S_n$

such that $\sigma^{-1}(i) < \sigma^{-1}(j)$

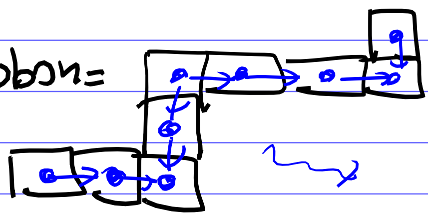
if $\overset{\frown}{i \ \ j}$ is an edge of T

Theorem $\displaystyle\sum_{\sigma \in S_n(T)} \langle \sigma_1 \dots \sigma_n \rangle =$

$$= \prod_{\substack{i \to j \\ \text{edge of} \\ T}} \frac{1}{x_i - x_j}$$

If T is a chain then

this "tree relation" $\Longleftrightarrow$ "ribbon relation"

Ribbon= 

$T =$ 

$S_n(T) \overset{\text{bij.}}{\longleftrightarrow} SYT(\text{Ribbon})$

The above "tree relation" is a special case of a more general formula for PT-factors proved in

[N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Postnikov, J. Trnka] On-shell structures of MHV amplitudes beyond the planar limit.

Geometric Interpretation

of Ribbon Relation (& other rels.) in terms of subdivisions of Root polytopes

The Root polytope introduced in [Gelfand - Graev - Postnikov] "Combinatorics of hypergeometric functions associated with positive roots".

$$R_n := \mathrm{Conv}(0,\ e_i - e_j,\ 1 \le i < j \le n)$$

For a tree T (on vert $1, \dots, n$) with directed edges $i \to j$, $i < j$

$$R_T = R_n \cap \left(\mathbb{R}_{\geq} \text{ span of } e_i - e_j \text{ for } i \to j \text{ edges of } T \right)$$

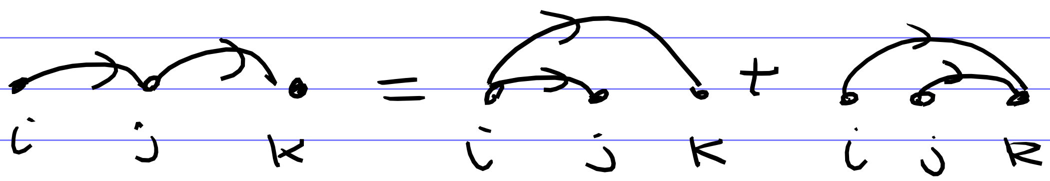
Let's associate

$$\prod_{\substack{i \to j \\ \text{edge of } T}} \frac{1}{x_i - x_j} \quad \text{with} \quad R_T$$

Key Relation:

$$\frac{1}{(x_i - x_j)} \frac{1}{(x_j - x_k)} =$$

$$= \frac{1}{(x_i - x_k)(x_i - x_j)} + \frac{1}{(x_j - x_k)(x_i - x_k)}$$



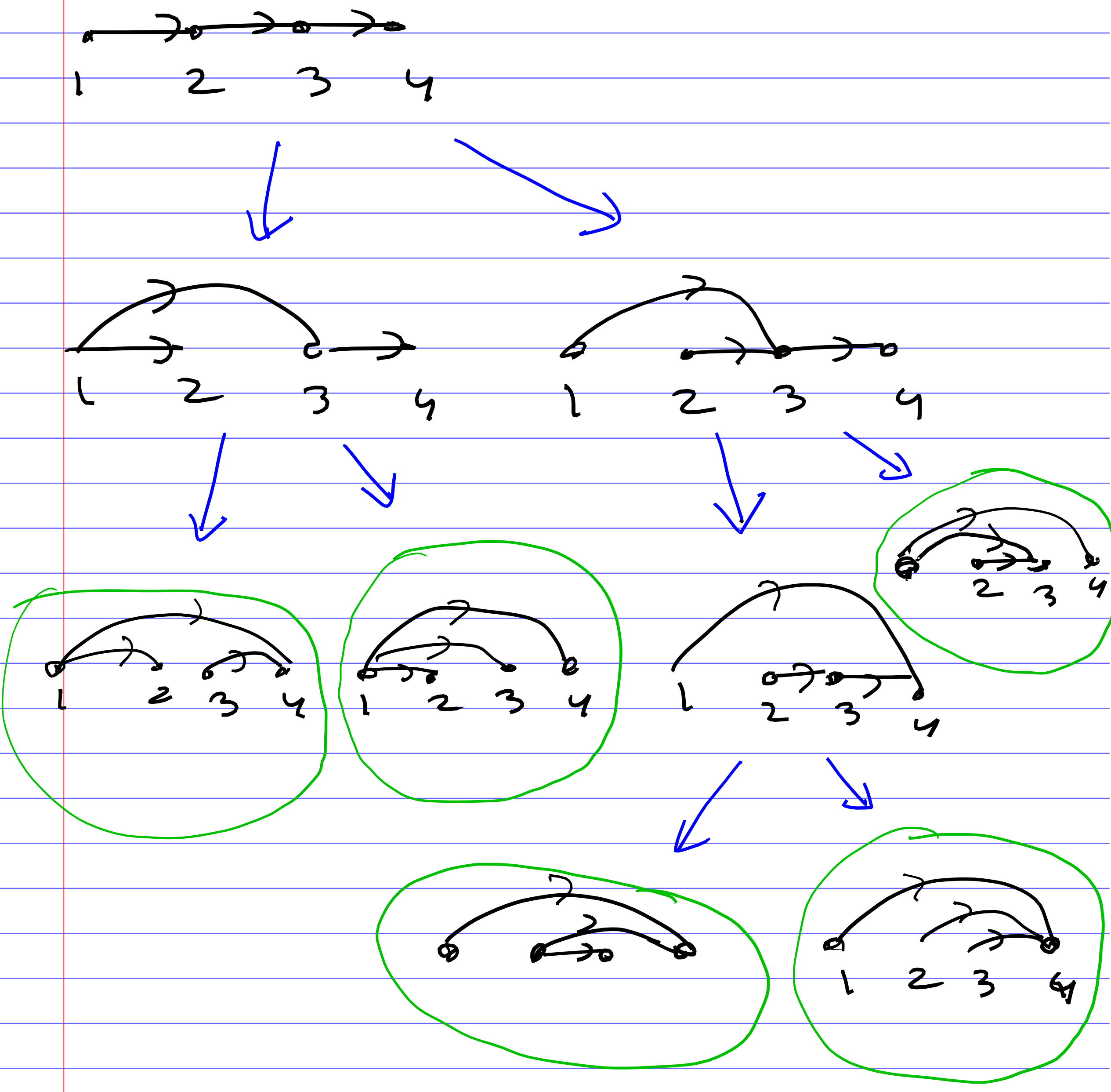
$T \qquad T' \qquad T''$

$$R_T = R_{T'} \cup R_{T''}$$

subdivision of polytope

Example

$R_4 = R_{1 \to 2 \to 3 \to 4}$



Theorem [GGP] This procedure gives produces a subdivison (triangulation of the root polytope R_n into the Catalan number

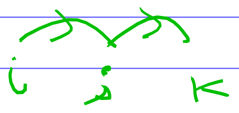
$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

of unit simplices, labelled by non-crossing alternating trees.

edges dont cross when a tree is drawn by arcs as above

every vertex is either a source or a sink

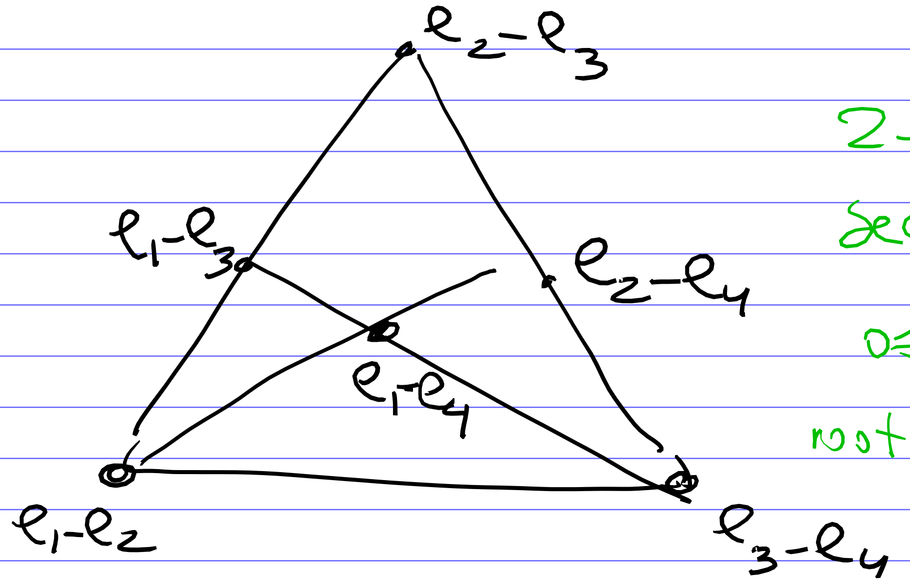
(a tree does not have a pair of edges like



)

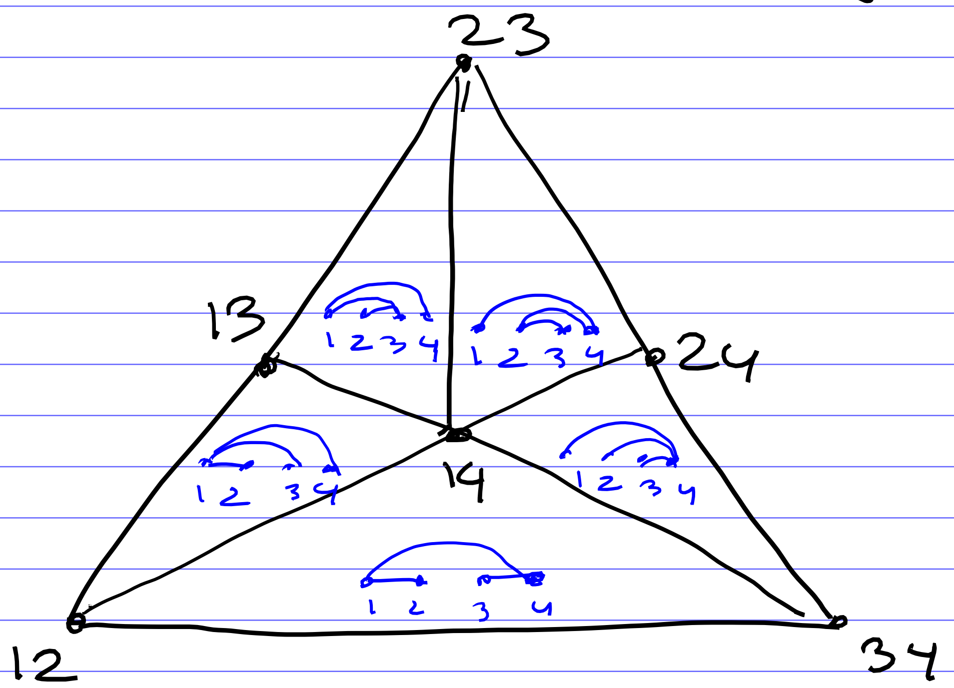
Each tree in the above example correspond to a 3-dimensional simplex in R_4

By we can draw them on the plane, as follows



2-dim section of 3-D root polytope R_4

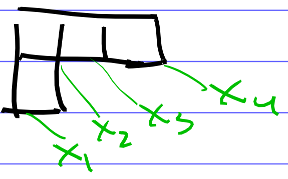
The triangulation of R_4 by non-crossing alternating trees.



5 simplices in the triangulation of R_4 are labelled by $C_3 = 5$ non-crossing alternating trees on n vertices.

We can express the ribbon relations in terms of similar subdivisions of R_n.

Example

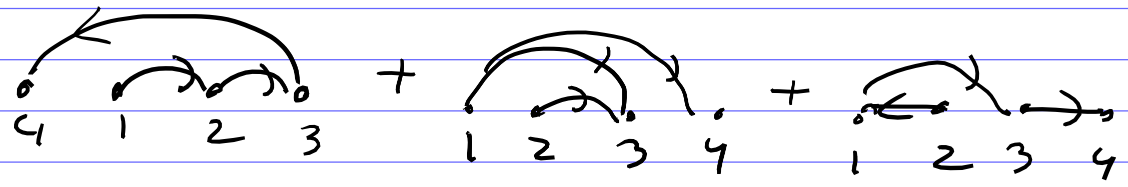


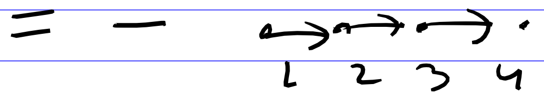
$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&&\\\cline{1-1}\end{array}$ $\quad$ $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&&\\\cline{1-1}\end{array}$ $\quad$ $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&&\\\cline{1-1}\end{array}$

$$\langle 2,3,4,1\rangle + \langle 2,3,1,4\rangle + \langle 2,1,3,4\rangle$$

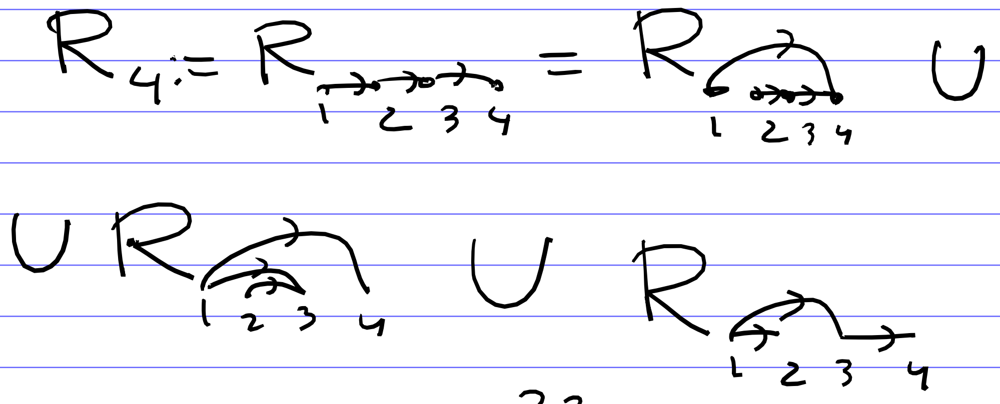
$$= -\langle 1,2,3,4\rangle$$

$\Updownarrow$

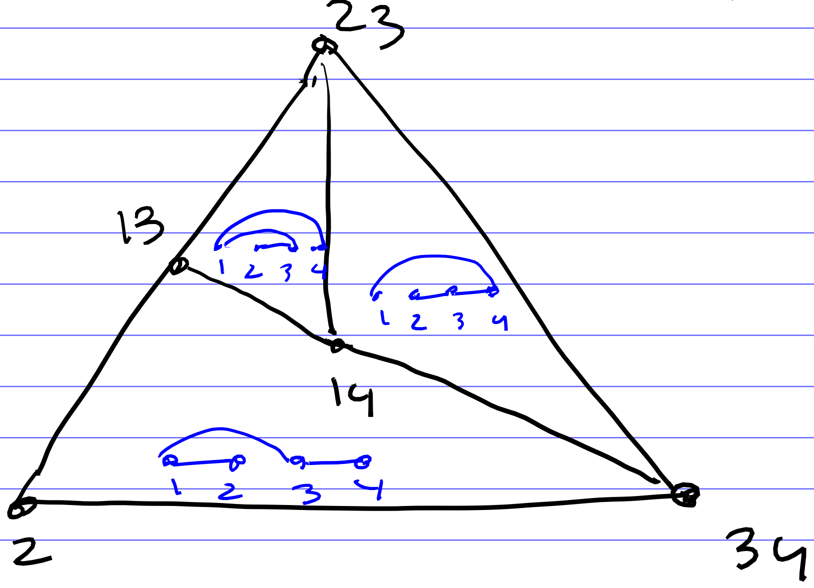




Geometrically,

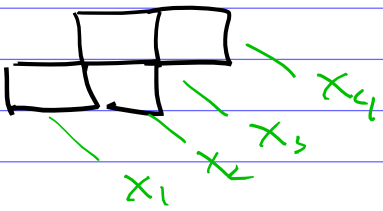


$R_4 =$

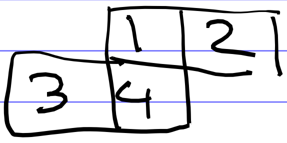


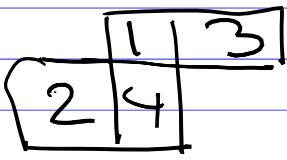
a visualization of the ribbon relation for $\begin{array}{|c|c|c|}\hline &&\\\hline &&\\\cline{1-1}\end{array}$ in terms of subdivision of the root polytope

Example

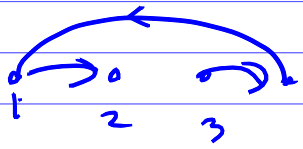


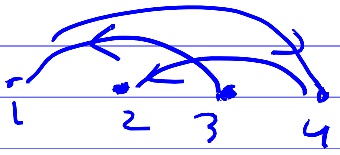
5 SYT's:

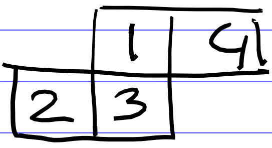


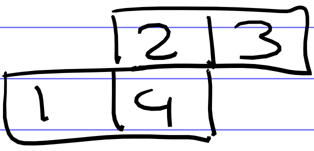


$\langle 3,4,1,2\rangle + \langle 3,1,4,2\rangle$

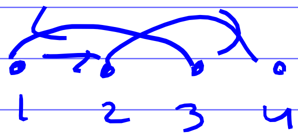


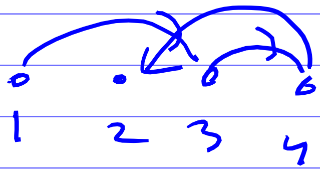


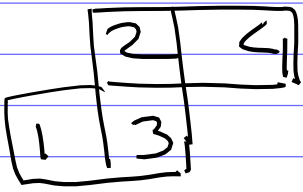


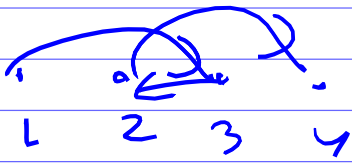


$+ \langle 3,1,2,4\rangle \quad + \langle 1,3,4,2\rangle$



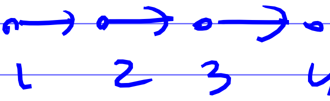


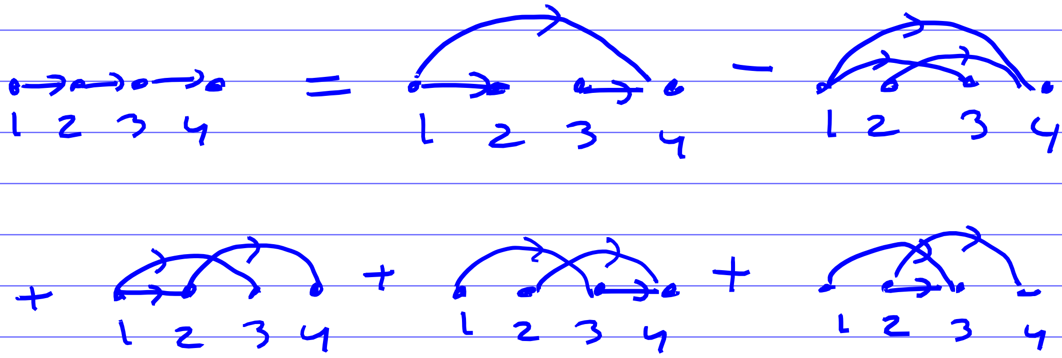
$+$ 



$\langle 1,3,2,4\rangle$

$\overset{\text{ribbon rel.}}{=} \; - \langle 1,2,3,4\rangle$





Geometrically

$R_4 =$