18.217 Lecture 22 10/23/2020

<u>last time:</u> Young's orthogonal form

Fix Young diagram λ, $|\lambda| = n$.

Representation of S_n on the space with basis $\{v_T\}$ labelled by SYT's T of shape λ.

For $T \in SYT(\lambda)$, $i = 1, \dots, n$

let $c_i = c_i(T) :=$ the content of box filled with i in T.

Ex. $T = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}$

$$(c_1, \dots, c_5) = (0, -1, 1, 2, 0)$$

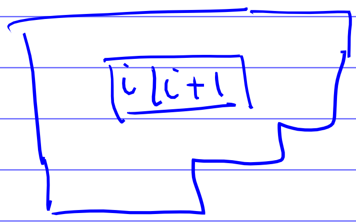
(This is the same as the vector $(\alpha_1, \dots, \alpha_n)$ of eigenvalues of JM-elts)

The action of the generators $s_1, \dots, s_{n-1}$ of S_n on the basis $\{v_T\}$:

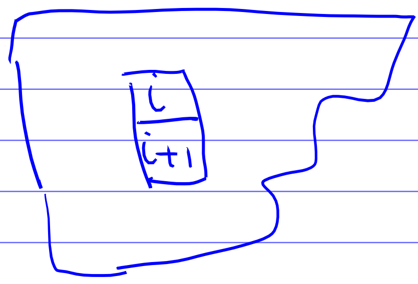
$$s_i : v_T \longmapsto \begin{cases} v_T & \text{if } c_{i+1}(T) - c_i(T) = 1 \\ -v_T & \text{if } c_{i+1}(T) - c_i(T) = -1 \\ \frac{1}{d} v_T + \sqrt{1 - \frac{1}{d^2}}\, v_{s_i(T)}, & \text{otherwise} \end{cases}$$

$d := c_{i+1}(T) - c_i(T)$

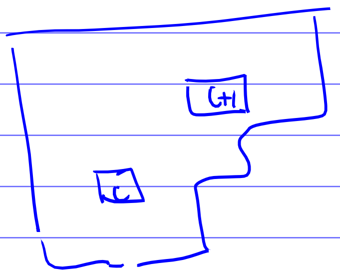
$s_i(T)$ is the SYT obtained from T by switching i & $i+1$

$T =$ 

$s_i : v_T \mapsto v_T$

$T =$ 

$s_i : v_T \mapsto -v_T$

$T =$ 

$s_i : v_T \mapsto \frac{1}{d} v_T + \sqrt{1 - \frac{1}{d^2}}\, v_{s_i(T)}$

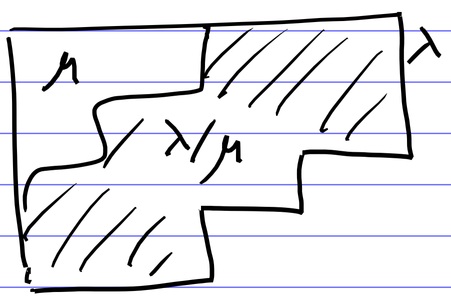
Lemma These operators on $\{\delta_T\}$ satisfy the Coxeter relations for S_n. Thus they define a valid representation of S_n.

Proof Can be checked directly from the definition. $\square$

Theorem This is the irreducible representation V_λ and $\{\delta_T\}$ is the GT-basis of V_λ.

Proof Follows from Vershik-Okounkov's construction. $\square$

Let us now consider a skew Young diagram λ/μ



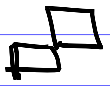
Define the action of S_n on the space $\{\delta_T \mid T\text{-SYT's of shape } \lambda/\mu\}$ by exactly the same formulas.

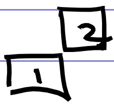
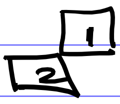
The above lemma still holds (the proof is exactly the same). So we have a valid representation of S_n, which we denote $V_{\lambda/\mu}$.

In general, $V_{\lambda/\mu}$ is not an irreducible representation.

One can also construct $V_{\lambda/\mu}$ using the Young symmetrizer. (Exactly the same construction of Young symmetrizer that we discussed before, except that we use a skew shape λ/μ).

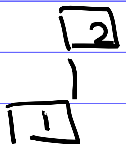
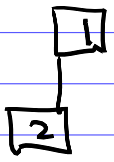
Proposition The construction of $V_{\lambda/\mu}$ based on Young's orthogonal form gives an isomorphic rep. as the representation given by Young's symmetrizer.

Ex. $\lambda/\mu =$  $\lambda = (2,1)$, $\mu = (1)$

$T_1 =$  , $T_2 =$ 

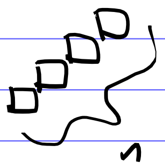
$$s_1 : \begin{cases} \delta_{T_1} \mapsto \frac{1}{2}\delta_{T_1} + \frac{\sqrt{3}}{2}\delta_{T_2} \\ \delta_{T_2} \mapsto \frac{\sqrt{3}}{2}\delta_{T_1} - \frac{1}{2}\delta_{T_1} \end{cases}$$

$\lambda/\mu =$  $\lambda = (2,1,1)$ $\mu = (1,1)$

$T_1' =$  $T_2' =$ 

$$s_1 : \begin{cases} \delta_{T_1'} \mapsto \frac{1}{3}\delta_{T_1'} + \frac{\sqrt{8}}{3}\delta_{T_2'} \\ \delta_{T_2'} = \frac{\sqrt{8}}{3}\delta_{T_1'} - \frac{1}{3}\delta_{T_2'} \end{cases}$$

different formulas, but the rep. is isomorphic to the previous one.

Actually V  is isomorphic to the regular representation of S_n, i.e., the action of S_n on the group algebra

$w : f \in \mathbb{C}[S_n] \longmapsto w.f$

$$\simeq \bigoplus_{\lambda \vdash n} (\underbrace{V_\lambda \oplus \dots \oplus V_\lambda}_{\dim V_\lambda})$$

Characters of representations

$R: G \to GL(V)$ a rep. of a finite group

The character of R

$\chi = \chi_R = \chi_V$

$$\chi_R : \begin{array}{ccc} G & \to & \mathbb{C} \\ \cup & & \\ g & \mapsto & tr(R_g) \end{array}$$

the trace of matrix R_g.

$$tr(A) := a_{11} + a_{22} + \dots + a_{nn}$$

$$A = \begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \\ & & & a_{nn} \end{bmatrix}$$

We know that

$$\boxed{tr(A) = tr(CAC^{-1})}$$

Thus we have

__Lemma__. (1) The character χ_R does not depend on choice of basis in V.

(2) If $R_1 \sim R_2$ are two isomorphic representations, then $\chi_{R_1} = \chi_{R_2}$

(3) The character χ_R is <u>class function</u> of G, i.e. χ_R is constant on conjugacy classes of G

$$\chi_R(g) = \chi_R(hgh^{-1}) \quad \forall g, h \in G.$$

__Lemma__ (1) $\chi_V(id) = \dim V$.

(2) $\chi_{V \oplus W} = \chi_V + \chi_W$

(3) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

Moreover,

<u>Theorem</u> (1) The characters $\chi_1, \dots, \chi_N$ of all irreps. of G form a linear basis of the space $\mathbb{C}_{class}(G)$ of of class functions on G.

(N = # conj. classes in G
= # (isom. classes of) irreps. of G.)

(2) The basis $\chi_1, \dots, \chi_N$ is orthogonal w.r.t. the inner product on $\mathbb{C}_{class}(G)$

$$\langle \chi, \xi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\xi(g)}.$$

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(1) $\Rightarrow$ Two representations R_1 & R_2 of G are isomorphic iff $\chi_{R_1} = \chi_{R_2}$.

Example $G = \mathbb{Z}/n\mathbb{Z}$
$= \{1, g, g^2, \dots, g^{n-1}\}$

G is an abelian group $\Rightarrow$ all irreps R_k of G are 1-dim'l.

$R_k(g) = \left(e^{\frac{2\pi i}{n}k}\right)$ ← 1×1 matrix

$i = \sqrt{-1}$, $k = 0, 1, \dots, n-1$

$\chi_k : g^\ell \mapsto e^{\frac{2\pi i}{n}\cdot k\cdot \ell}$

$k = 0, \dots, n-1$; $\ell = 0, \dots, k-1$

In general, characters are $\mathbb{C}$-valued functions (as in this example)

Example $G = S_3$

The character table of S_3

irreps of S_3 \ conj. classes (cyclic types)	$(1,1,1)$ id	$(2,1)$ s_1	(3) $s_1 s_2$
trivial rep. $V_{(3)}$	1	1	1
sign rep. $V_{(1,1,1)}$	1	-1	1
$V_{(2,1)}$	2	0	-1

2 dim subrep. of the defining rep. of S_3

dim $V_{(2,1)}$? ?

The defining rep. of S_3 is
$V_{(2,1)} + V_{(3)}$ ($w \mapsto$ perm matrix of w)

$\chi_{(2,1)} + \chi_{(3)} : w \mapsto \mathrm{tr}(\text{perm matrix of } w)$
$= \#$ fixed points of w.

	id	s_1	$s_1 s_2$
$\chi_{(2,1)} + \chi_{(3)}$	3	1	0

One can also calculate the last row of the character table of S_3 using orthogonality.

The inner product

	id	s_1	$s_1 s_2$
χ	a	b	c
ξ	d	e	f

← 2 class functions on S_3

sizes of conj. classes

$$\langle \chi, \xi \rangle := \frac{1}{6}\left(a\cdot\bar d + 3\cdot b\cdot \bar e + 2\, c\cdot \bar f\right)$$

The rows of the char. table are orthonormal w.r.t. this inner product

Also the columns of the char. table are orthogonal w.r.t. the "usual" inner product

	a	d
	b	e
	c	f

$a\bar d + b\bar e + c\bar f$

We can also figure out the last row using char of reg. representation:

$\mathbb{C}[S_3] = V_{(3)} \oplus V_{(1,1,1)} \oplus V_{(2,1)} \oplus V_{(2,1)}$

Lemma. $\forall$ group G

$$\chi_{\substack{\text{reg. rep.}\\ \text{on } \mathbb{C}[G]}}(g) = \begin{cases} |G| & \text{if } g = id \\ 0 & \text{otherwise} \end{cases}$$

So in the above char. table
1st row + 2nd row + 2·3rd row
$= (6, 0, 0)$

<u>Example</u> The character table of S_4

irreps. \ conj classes	(1^4) id	(21^2) s_1	(31) s_1s_2	(4) $s_1s_2s_3$	$(2\,2)$ s_1s_3
trivial $V_{\ydiagram{4}}$	1	1	1	1	1
sign $V_{\ydiagram{1,1,1,1}}$	1	-1	1	-1	1
subrep of def. rep. $V_{\ydiagram{3,1}}$	3	1	0	-1	-1
$V_{\ydiagram{2,1,1}} = V_{\ydiagram{3,1}} \otimes V_{\ydiagram{1,1,1,1}}$	3	-1	0	1	-1
$V_{\ydiagram{2,2}}$	2	0	-1	0	2

dim = # SYT's of shape $\ydiagram{2,2}$

can calculate these values using orthogonality, or using the regular rep. of S_4

Observation Apriori, χ_V is a $\mathbb{C}$-valued function.

But in these 2 examples for S_3 and S_4, we see that the characters of reps. of symmetric groups are $\mathbb{Z}$-valued.

Why?

How to calculate the character table of S_n?

Def For 2 partitions λ, μ of n

$$\chi_\lambda(\mu) := \chi_{V_\lambda}\left(\begin{matrix}\text{conj. class} \\ \text{with cyclic} \\ \text{type } \mu\end{matrix}\right)$$

$$\chi_\lambda(\mu) = ?$$

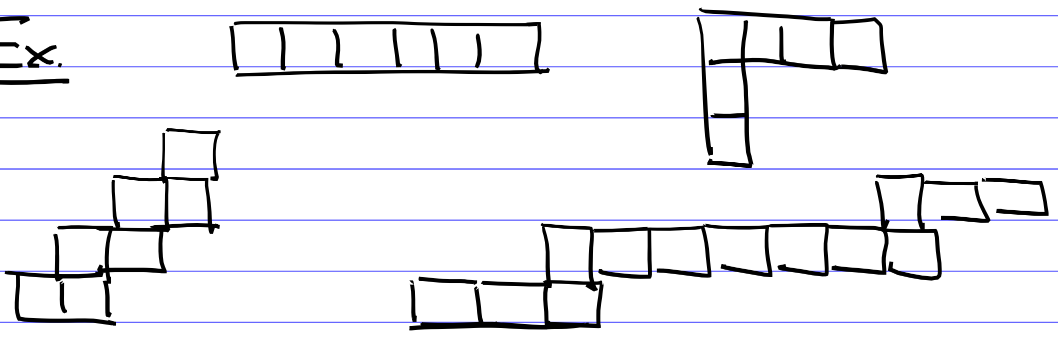
Murnaghan – Nakayama Rule

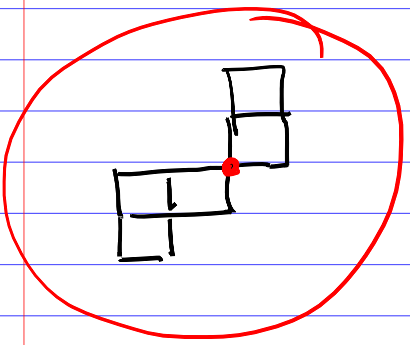
Theorem.
$$\boxed{\chi_\lambda(\mu) = \sum_{\substack{T \text{ ribbon tableaux} \\ \text{of shape } \lambda \text{ and} \\ \text{type } \mu}} (-1)^{ht(T)}}$$

Defs. A ribbon (aka rim-hook or border strip) is a skew Young diagram λ/μ.

- it is connected
- all diagonal contain ≤ 1 boxes

Ex.



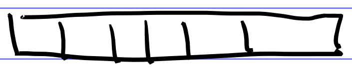


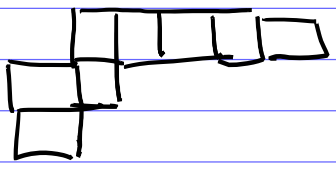
not a ribbon because it is not connected

A ribbon tableau T of shape λ and type $\mu = (\mu_1, \dots, \mu_k)$ is a filling of Young diagram λ with $1, 2, \dots, k$, weakly increasing in rows and columns, s.t., $\forall i = 1, \dots, k$ all i's in T form a ribbon with μ_i boxes

The height of a ribbon

$$ht(\text{a ribbon}) := \#\text{ of its rows} - 1$$

 $ht = 0$

 $ht = 2$

$$ht\begin{pmatrix}\text{a ribbon} \\ \text{tableau } T\end{pmatrix} = \sum_{i=1}^{k} ht\begin{pmatrix}\text{its } i\text{-th} \\ \text{ribbon}\end{pmatrix}$$

Example A ribbon tableau

$$\leq$$

$$T = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 3 & 3 & 4 & 6 \\ 3 & 3 & 4 & 4 & \\ 5 & 5 & & & \\ 5 & & & & \end{array} \quad \wedge$$

shape: $\lambda = (5, 5, 5, 4, 2, 1)$

type: $\mu = (6, 5, 4, 3, 3, 1)$

height: $ht(T) = 2 + 1 + 1 + 1 + 1 + 0$
$= 6$

Examples. From char. table of S_4

$$\chi_{(2,1,1)}((3,1)) = (-1)^1 + (-1)^2 = 0.$$

$$\begin{array}{cc} 1 & 1 \\ 1 & \\ 2 & \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 1 & \\ 1 & \end{array}$$

$ht = 1 + 0$ $\qquad$ $ht = 2 + 0$

$$\chi_{(2,2)}((2,2)) = 1 + (-1)^2 = 2$$

$$\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array}$$

$ht = 0$ $\qquad$ $ht = 2$

A more general version of Murnaghan-Nakayama rule

Theorem λ/μ a skew shape $\nu = (\nu_1, \dots, \nu_k)$ any seq. of positive integers (not necessarily decreasing)

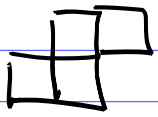
$$|\lambda/\mu| = |\nu| = n$$

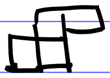
Then $\chi_{\lambda/\mu}(\nu) := \chi_{V_{\lambda/\mu}}\left(\begin{array}{c}\text{conj.}\\ \text{class of}\\ \text{cyclic}\\ \text{type } \nu\end{array}\right)$

$$= \sum_{\substack{T \text{ ribbon tableau} \\ \text{of shape } \lambda/\mu \text{ and} \\ \text{type } \nu}} (-1)^{ht(T)}$$

Remark This number does not change if we permute the entries of ν.

Example

$\lambda/\mu =$ 

	1^4	21^2	31	4	22
χ 	5	1	-1	-1	1

\# SYT's of shape λ/μ $= \dim V_{\lambda/\mu}$

Under 21^2:

type$(2,1,1)$
$\begin{array}{ccc} & 1 & 1 \\ 2 & 3 & \end{array}$

type$(1,2,1)$

$\begin{array}{ccc} & 1 & 3 \\ 2 & 2 & \end{array}$ ht $=0$

$\begin{array}{ccc} & 2 & 3 \\ 1 & 2 & \end{array}$ ht $=1$

$\begin{array}{ccc} & 2 & 2 \\ 1 & 3 & \end{array}$ ht $=0$

$1 = 1 + (-1) + 1$

Under 31:
$\begin{array}{ccc} & 1 & 2 \\ 1 & 1 & \end{array}$

Under 4:
$\begin{array}{ccc} & 1 & 1 \\ 1 & 1 & \end{array}$

Under 22:
$\begin{array}{ccc} & 1 & 1 \\ 2 & 2 & \end{array}$

How to prove the M-N rule?

We can use Young's orthogonal form. We have explicit matrices for the action of s_i's. So we should be able to calculate the trace.

$$R_{\lambda/\mu} : S_n \to GL(V_{\lambda/\mu})$$

$V_{\lambda/\mu}$ = the linear space with basis $\{v_T \mid T\text{-}SYT(\lambda/\mu)\}$

$\chi_{\lambda/\mu}(\nu) := \operatorname{tr} R_{\lambda/\mu}(w)$, where

w is <u>any</u> permutation in S_n with cyclic type ν.

Let us pick w smartly in order to simplify the calculation

$$w = (1\,2 \ldots \nu_1)(\nu_1+1, \ldots, \nu_1+\nu_2) \ldots$$
$$\ldots (\nu_1+\ldots+\nu_{k-1}+1, \ldots, \nu_1+\ldots+\nu_k) =$$
$$= \underbrace{s_1 \ldots s_{\nu_1-1}}_{\nu_1-1 \text{ terms}} \underbrace{s_{\nu_1+1} \ldots s_{\nu_1+\nu_2-1}}_{\nu_2-1 \text{ terms}} \ldots$$
$$\ldots \underbrace{s_{\nu_1+\ldots+\nu_{k-1}+1} \ldots s_{\nu_1+\nu_2+\ldots+\nu_k-1}}_{\nu_k-1 \text{ terms.}}$$

<u>Ex.</u> $n=12$, $\nu=(3,3,2,2,1,1)$

$$w = \underbrace{s_1 s_2}, \underbrace{s_4 s_5}, \underbrace{s_7}, \underbrace{s_9}$$
$$= (1,3,3)(4,5,6)(7,8)(9,10)$$

$$R_w := (R_{s_1} R_{s_2} \dots R_{s_{\nu_1 - 1}})(R_{s_{\nu_1+\nu_2+1}} \dots)$$

$\dots$

Each R_{s_i} acts on the basis

$$R_{s_i} : v_T \mapsto \frac{1}{c_{i+1} - c_i} v_T + \underbrace{\sqrt{1 - \dots}\, v_{s_i(T)}}_{\text{or } 0 \text{ if } s_i(T) \text{ is not define}}$$

<u>Observation</u>

Only diagonal terms make a contribution to trace $(R_w) :=$

$$:= \sum_{T : SYT(\lambda/\mu)} \text{coeff of } v_T \text{ in } R_w(v_T)$$

If we use the second term at any step, we <u>will not</u> be able to get back to v_T

So we get $\chi_{\lambda/\mu}(\nu) =$

$$\sum_{T : SYT(\lambda/\mu)} \prod_{\substack{i \in \{1,\dots,\nu_1 - 1\} \\ \cup \{\nu_1+1,\dots,\nu_1+\nu_2-1\} \\ \dots \\ \cup \{\nu_1+\dots+\nu_{k-1}+1,\dots,\nu_1+\dots+\nu_k-1\}}} \frac{1}{c_{i+1}-c_i}$$

corresp. to the factors s_i in decomp. for w

$c_i = c_i(T)$:= the content of box i in T

Basically we just used the def of trace

How do we get ribbon tableaux from this?

The M-N rule reduces to the following claim:

Proposition For λ/μ with n boxes

$$\sum_{T : SYT(\lambda/\mu)} \frac{1}{c_2 - c_1} \cdot \frac{1}{c_3 - c_2} \cdots \frac{1}{c_n - c_{n-1}} = \begin{cases} (-1)^{ht(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a ribbon} \\ 0 & \text{otherwise.} \end{cases}$$

This Proposition is a special case of M-N for $w = s_1 \dots s_{n-1}$.

But actually, it is enough to prove the general case of M-N rule.

For example, suppose w consists of 2 cycles

$$w = (1,2,3,4,5)(6,7,8,9) =$$
$$= (s_1 s_2 s_3 s_4)(s_6 s_7 s_8) \in S_9$$

$$\chi_{\lambda/\mu}(w) =$$

$$\sum_{T \in SYT(\lambda/\mu)} \underbrace{\frac{1}{c_2-c_1}\,\frac{1}{c_3-c_2}\,\frac{1}{c_4-c_3}\,\frac{1}{c_5-c_4}} \cdot \underbrace{\frac{1}{c_7-c_6}\,\frac{1}{c_8-c_7}\,\frac{1}{c_9-c_8}}$$

$$= \sum_{\tilde{\lambda}\,:\, \mu \subset \tilde{\lambda} \subset \lambda} \Big(\qquad \Big)\Big(\qquad \Big)$$

$T_1 : SYT(\tilde{\lambda}/\mu)$ with 5 boxes

$T_2 : SYT(\lambda/\tilde{\lambda})$ with 4 boxes

$$= \sum_{\tilde{\lambda}} \Big(\sum_{T_1\ SYT(\tilde{\lambda}/\mu)} \cdots \Big)\Big(\sum_{T_2 : SYT(\lambda/\tilde{\lambda})} \cdots \Big)$$

given by Proposition

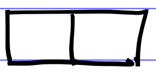
$$= \sum_{\substack{\tilde{\lambda}\,:\ \tilde{\lambda}/\mu \text{ ribbon w/ 5 boxes} \\ \lambda/\tilde{\lambda} \text{ ribbon w/ 4 boxes}}} (-1)^{ht(\tilde{\lambda}/\mu)} \cdot (-1)^{ht(\lambda/\tilde{\lambda})}$$

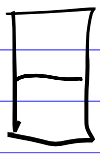
$$= \sum_{\text{ribbon tableaux}} (-1)^{ht(\text{ribbon tableau})}$$

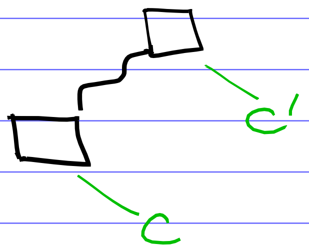
(with 2 ribbons in this example)

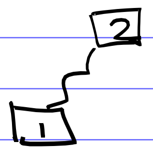
Let us look at some examples of Proposition.

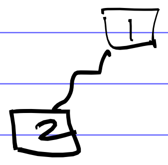
Examples $\lambda/\mu =$


: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$ $\quad \frac{1}{c_2 - c_1} = \boxed{1}$

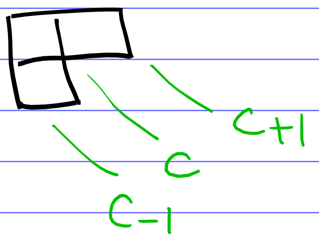

: $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ $\quad \frac{1}{c_2 - c_1} = \boxed{-1}$




$\frac{1}{c'-c}$


$\frac{1}{c-c'}$

$= \boxed{0}$

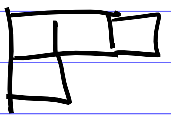


$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ $\quad \frac{1}{((c+1)-c)((c-1)-(c+1))}$

$+$

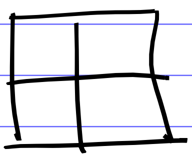
$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$ $\quad \frac{1}{((c-1)-c)((c+1)-(c-1))}$

$$= \frac{1}{1\cdot(-2)} + \frac{1}{(-1)\,2} = \boxed{-1}$$


: $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}$ $\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$ $\quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$

$$\frac{1}{1\cdot 1\cdot(-3)} + \frac{1}{1\,(-2)\cdot 3} + \frac{1}{(-1)\cdot 2\cdot 1}$$

$$= -\frac{1}{3} - \frac{1}{6} - \frac{1}{2} = \boxed{-1}$$


: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ $\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$

$$\frac{1}{1\cdot(-2)\,1} + \frac{1}{(-1)\,2\,(-1)}$$

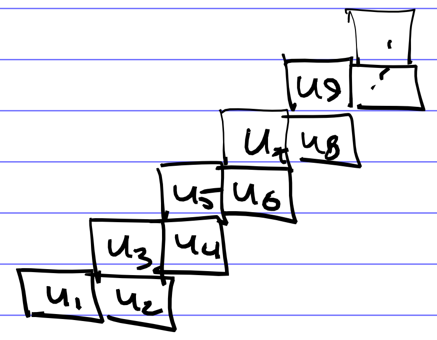
$$= \boxed{0}$$

Claim We will always get 0 if λ/μ is disconnected shape or has ≥ 2 boxes in 1 diagonal.

Otherwise (if λ/μ is a ribbon), we get $(-1)^{ht(\lambda/\mu)}$

Special case

The staircase ribbon



SYT's of this shape are alternating permutations

$$u_1 < u_2 > u_3 < u_4 > u_5 < \ldots$$

the content vector $(c_1, c_2, \ldots, c_n)$ = the inverse permutation to u

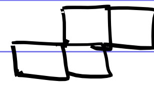
(if we assume that the content of the leftmost box is 1)

Corollary

$$\sum_{\substack{\text{alternating permutations}\\ u_1<u_2>u_3<\ldots \\ \text{in } S}} \prod_{i=1}^{n-1} \frac{1}{u^{-1}(i+1) - u^{-1}(i)} = (-1)^{\lfloor \frac{n-1}{2} \rfloor}$$

Moreover, a similar claim is true not only for alternating perms, but for any set of perm. with specified positions of ascents and descents.

Ex. $n = 4$



1	2	3	4	
$1 <$	$3 >$	$2 <$	4	$\frac{1}{(3-1)}\frac{1}{(2-3)}\frac{1}{(4-2)}$
$2 <$	$3 >$	$1 <$	4	$\frac{1}{(1-3)}\frac{1}{(2-1)}\frac{1}{(4-2)}$
$1 <$	$4 >$	$2 <$	3	$\frac{1}{(3-1)}\frac{1}{(4-3)}\frac{1}{(2-4)}$
$2 <$	$4 >$	$1 <$	3	$\frac{1}{(1-3)}\frac{1}{(4-1)}\frac{1}{(2-4)}$
$3 <$	$4 >$	$1 <$	2	$\frac{1}{(4-3)}\frac{1}{(1-4)}\frac{1}{(2-1)}$

$$-\frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{3\cdot 4} - \frac{1}{3}$$

$$= \boxed{-1}$$

<u>Exercise</u> Prove the Proposition.