last time: q-binom. coeff.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

Plane partitions

Fix integers $k, \ell, m \geq 1$.

$\geq$ (ℓ), k $\geq$: filled w/ $0, 1, \ldots, m$

$k \times \ell$ rectangular array with entries $\in \{0, 1, \ldots, m\}$ weakly decreasing in rows & columns

Example

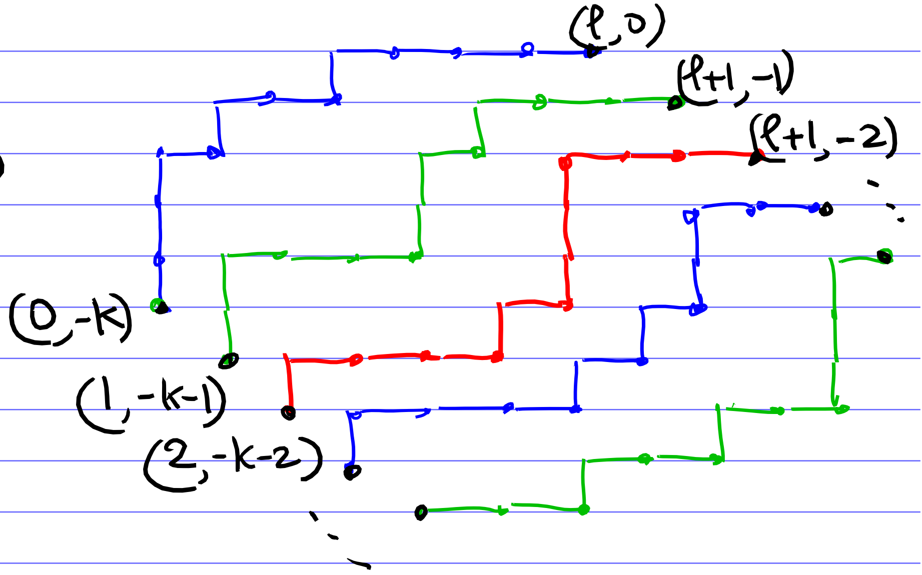
$k = 5$
$\ell = 7$
$m = 5$

$$P = \begin{array}{ccccccc} 5 & 5 & 5 & 4 & 2 & 1 & 0 \\ 5 & 4 & 4 & 3 & 2 & 1 & 0 \\ 4 & 4 & 4 & 3 & 1 & 1 & 0 \\ 3 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

m lattice paths from lower left to upper right corners that separate entries i & $i+1$

- For $i = 1, 2, \ldots, m$, shift the i^{th} path $i-1$ steps in the SE direction (i.e. shift by $(i-1, 1-i)$)

$P \rightsquigarrow$



We obtain a bijection between plane partitions P & collections of m non-crossing lattice paths.

Lindström's Lemma $\Rightarrow$

plane partitions of shape $k \times \ell$ with entries $\in \{0,1,\dots,m\}$

$$PP_{k,\ell,m} = \det\left(\binom{k+\ell}{k+i-j} \right)_{i,j=1}^{m}$$

Example $k=2$, $\ell=4$, $m=3$

$$\begin{vmatrix} \binom{5}{2} & \binom{5}{1} & \binom{5}{0} \\ \binom{5}{3} & \binom{5}{2} & \binom{5}{1} \\ \binom{5}{4} & \binom{5}{3} & \binom{5}{2} \end{vmatrix} = PP_{2,4,3}$$

Notice $\binom{k+l}{k+i-j} = e_{k+i-j}(\underbrace{1 \dots 1}_{k+l})$

The above det is Jacobi-Trudi formula for $S_{k \times m}$ specialized at $(x_1, x_2, \dots) = (\underbrace{1, \dots 1}_{k+l}, 0, 0, \dots)$

So $PP_{k,l,m} = S_{k \times m}(\underbrace{1, \dots 1}_{k+l})$

Also $PP_{k,l,m} = S_{l \times m}(\underbrace{1 \dots 1}_{k+l})$

$PP_{k,l,m} = PP_{l,k,m}$

Is there a better explanation of these equalities?

For a plane partition P,

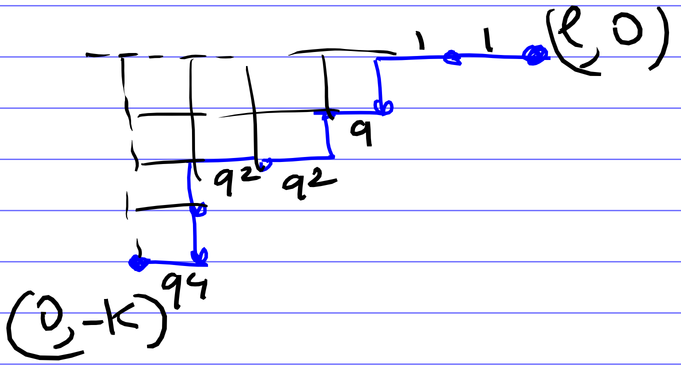
let $s(P) = \sum$ entries of P.

How about the generating function

$$PP_{k,l,m}(q) = \sum_{\substack{\text{plane} \\ \text{part.} \\ P}} q^{s(P)}$$

Let us assign weights 1 to vert. edges and weight q^j to horizontal edges $(i,-j) — (i+1,-j)$

Ex.



Then $\displaystyle\sum_{\substack{\text{path from}\\ (0,-k) \text{ to } (\ell,0)}} wt(P) = \sum_{\text{paths}} q^{\text{area above the path}}$

$$= \sum_{\lambda \subseteq k \times \ell} q^{|\lambda|} = \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q$$

---

plane partition

$P \rightsquigarrow$ non-crossing paths $P_1, P_2, \dots, P_m$

$$\prod_{i=1}^{m} \text{weight}(p_i) = q^{s(P)} \cdot q^{\ell \cdot \frac{m(m-1)}{2}}$$

extra factor because we shift the paths

Lindström's Lemma $\Longrightarrow$

$$P_{k,\ell,m}(q) =$$

$$= q^{-\ell \cdot m(m-1)/2} \det\left(q^{(i-1)(\ell-j+i)} \begin{bmatrix} k+\ell \\ k+i-j \end{bmatrix}_q\right)_{i,j=1}^{m}$$

extra factors coming from shifting the paths

We can get more explicit formulas.

<u>Theorem</u> (MacMahon, 1896)

$$\boxed{PP_{k,\ell,m}(q) = \prod_{a=1}^{k} \prod_{b=1}^{\ell} \prod_{c=1}^{m} \frac{[a+b+c-1]_q}{[a+b+c-2]_q}}$$

gen. function for plane partitions of shape $k \times \ell$ with entries in $\{0,1,\ldots,m\}$

In particular, # such plane partitions is

$$\boxed{\prod_{a=1}^{k} \prod_{b=1}^{\ell} \prod_{c=1}^{m} \frac{a+b+c-1}{a+b+c-2}}$$

Example $\quad k = \ell = m = 2$

$$\begin{array}{ccccc}
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\
1 & q & q^2 & q^2 & q^2
\end{array}$$

$$\begin{array}{ccccc}
\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \\
q^3 & q^3 & q^3 & q^4 & q^4
\end{array}$$

$$\begin{array}{ccccc}
\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\
q^4 & q^4 & q^5 & q^5 & q^5
\end{array}$$

$$\begin{array}{ccccc}
\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\
q^6 & q^6 & q^6 & q^7 & q^8
\end{array}$$

$$PP_{3,2,2}(q) = 1 + q + 3q^2 + 3q^3 +$$

$$+ 4q^4 + 3q^5 + 3q^6 + q^7 + q^8 =$$

MacMahon

$$= \frac{[2]_q}{[1]_q} \frac{[3]_q}{[2]_q} \frac{[3]_q}{[2]_q} \frac{[3]_q}{[2]_q} \frac{[4]_q}{[3]_q} \frac{[4]_q}{[3]_q} \frac{[4]_q}{[3]_q} \frac{[5]_q}{[4]_q}$$

$$= \frac{(1+q+q^2+q^3)^2}{(1+q)^2} (1+q+q^2+q^3+q^4) =$$

$$= (1+q^2)^2 (1+q+q^2+q^3+q^4)$$

such plane partitions $= 20$.

Observation. The answer is symmetric in k, ℓ, m.

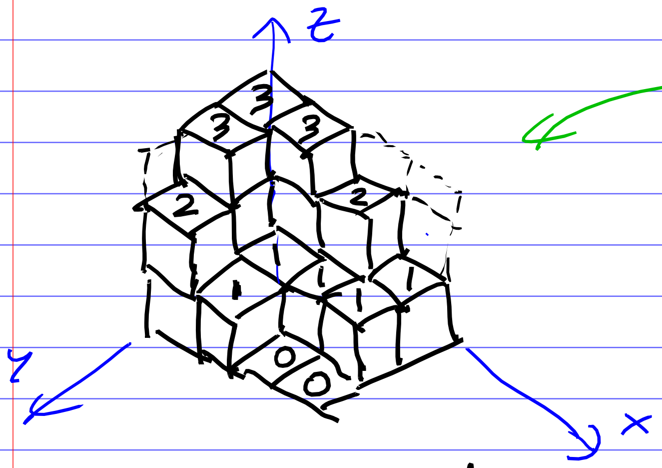
Why? Symmetry $k \leftrightarrow \ell$ is clear. (transposition of p.p.)

Plane partitions can be viewed as "3-dimensional Young diagrams" that fit inside the $k \times \ell \times m$ box

Example

$$P = \begin{array}{|cccc|} \hline 3 & 3 & 2 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ \hline \end{array}$$

For each box of P filled with "a" put a little cubes above it.



3 dim Young diagram

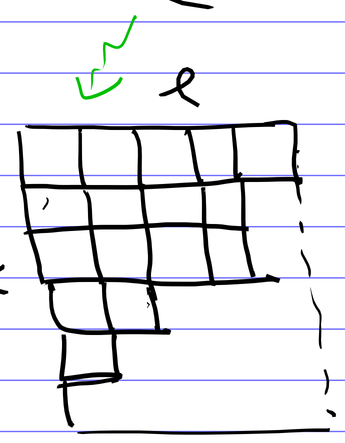
3D view of a plane partition

$s(P) = \#$ little cubes in this "Young diagram"

This explains the symmetry between k, ℓ, m.

This is a 3D version of how usual partitions can be viewed as usual Young diagrams

Ex. $\lambda = (5, 4, 4, 2, 1, 0)$



partitions with $\le k$ parts & entries $\le \ell$

$\updownarrow$ bij.

Young diagrams $\subseteq k \times \ell$ rectangle

1-dim case:

$$1 + q + \dots + q^{n-1} = [n]_q$$

2-dim case

$$\sum_{\lambda \subseteq k \times \ell} q^{|\lambda|} = \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q$$

3-dim case:

$$\sum_{P \subseteq k \times \ell \times n} q^{s(P)} = \text{MacMahon formula}$$

Q: Is there a nice product formula for higher dimensional Young diagrams?

A: No. for dim > 3.

How to prove MacMahon's formula?

- Deduce from the determinantal formula
- Deduce from Stanley's hook-content formula

$$\left\{\begin{array}{l}\text{Plane partitions}\\ \text{of shape } k\times \ell\\ \text{w/ entries } \leq m\end{array}\right\} \overset{\text{bij}}{\longleftrightarrow} \left\{\begin{array}{l}\text{Reverse}\\ \text{plane part.}\\ \text{of shape } k\times \ell\\ \text{w/ entries} \leq m\end{array}\right\}$$

rotate by 180°

$$\begin{array}{cc} & \geq \\ \geq & \begin{bmatrix} 4 & 4 & 3 & 1 & 1\\ 4 & 3 & 3 & 1 & 0\\ 3 & 3 & 2 & 1 & 0\\ 2 & 1 & 1 & 1 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}\end{array} \longleftrightarrow \begin{array}{cc} & \leq \\ \leq & \begin{bmatrix} 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 1 & 1 & 2\\ 0 & 1 & 2 & 3 & 3\\ 0 & 1 & 3 & 3 & 4\\ 1 & 1 & 3 & 4 & 4\end{bmatrix}\end{array}$$

$$\left\{\begin{array}{l}\text{RPP's of}\\ \text{shape } k\times \ell\\ \text{w/ entries} \leq m\end{array}\right\} \overset{\text{bij.}}{\longleftrightarrow} \left\{\begin{array}{l}\text{SSYT's}\\ \text{of shape}\\ k\times \ell\\ \text{with entries}\\ \in \{1,2,\dots,k+m\}\end{array}\right\}$$

add i to all entries in i^{th} row for $i=1,\dots,k$

$$\begin{array}{cc} & \leq \\ \leq & \begin{bmatrix} 1 & 1 & 1 & 1 & 2\\ 2 & 3 & 3 & 3 & 4\\ 3 & 4 & 5 & 6 & 6\\ 4 & 5 & 7 & 7 & 8\\ 6 & 6 & 8 & 9 & 9\end{bmatrix}\end{array}$$

SSYT filled with $1,2,\dots,k+m$

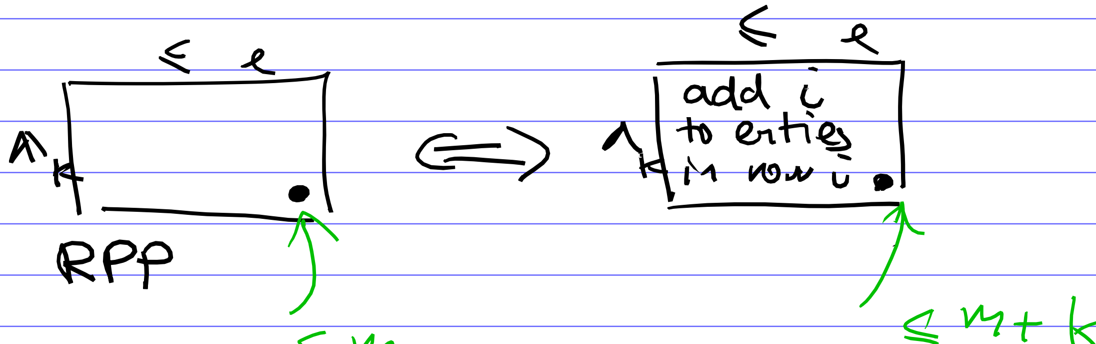
<u>Note</u>: This correspondence between RPP's with bounded entries and SSYT's with bounded entries works <u>only</u> for <u>rectangular</u> shapes

Indeed, for RPP's & SSYT's of rectangular shape, we have

<u>all</u> entries are bounded by some number

$$\Updownarrow$$

the one entry in the <u>bottom left corner</u> is bounded by this number



Now apply Stanley's hook-content formula

$$s_\lambda(1,q,\dots,q^{n-1}) = q^{m(\lambda)} \prod_{x\in\lambda} \frac{\overbrace{[n+c(x)]_q}^{\text{content}}}{\underbrace{[h(x)]_q}_{\text{hook length}}}$$

For $\lambda = k\times l$, $n = m+k$

We obtain

$$PP_{k,l,m}(q) = q^{-m(\lambda)} s_{k\times l}(1,q,\dots,q^{m+k-1})$$

$$= \prod_{x\in k\times l} \frac{[m+k+c(x)]_q}{[h(x)]_q},$$

which is equivalent to MacMahon's formula. $\square$

<u>Remark</u> We have a nice product formula for # SSYTs with bounded entries for <u>any</u> shape λ.

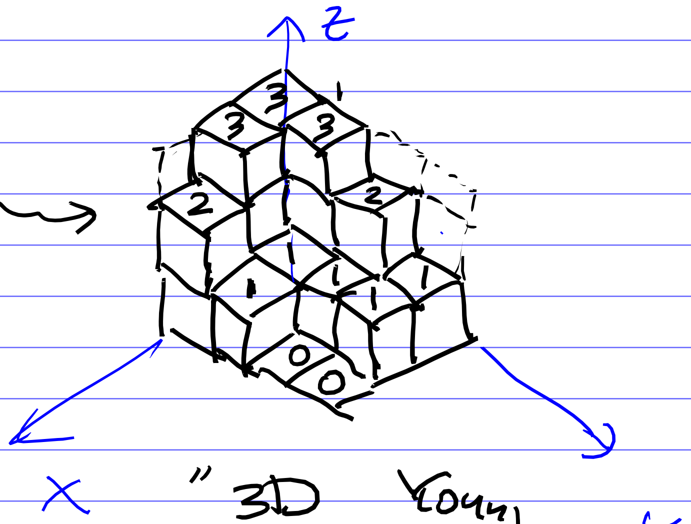
However, for (reverse) plane partitions with bounded entries there is a nice product formula only for <u>rectangular</u> shape.

Other interpretations of plane partitions:

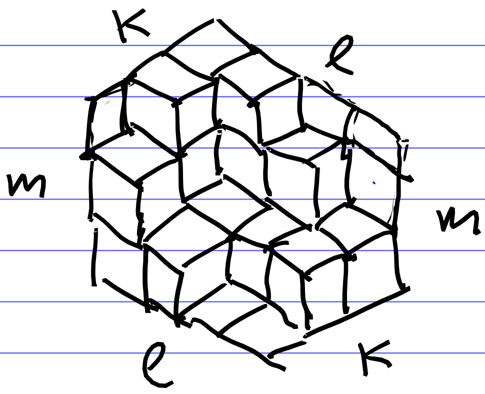
- Rhombus tilings
- Perfect matchings
- Pseudo-line arrangements

$$\begin{array}{cc} & \le m \quad \ell \\ k & \begin{bmatrix} 3 & 3 & 2 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow$$

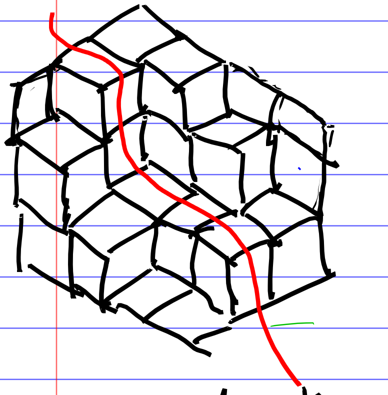
a plane partition



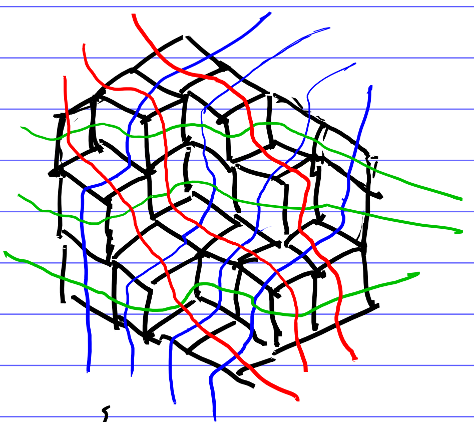
"3D Young diagram"

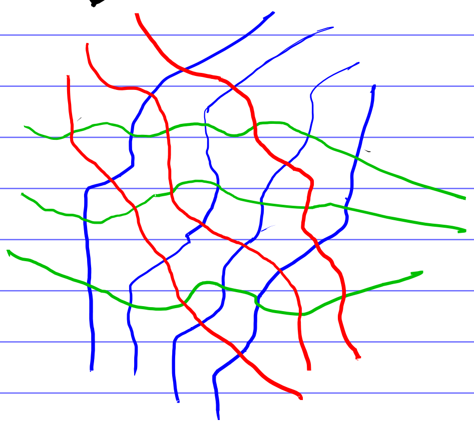


a rhombus tiling



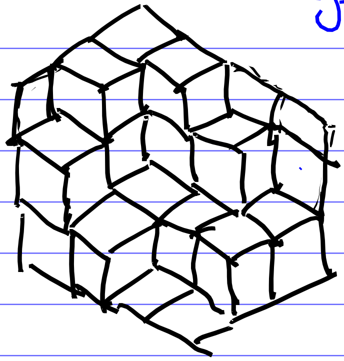
a pseudo-line (a "line that can bend")

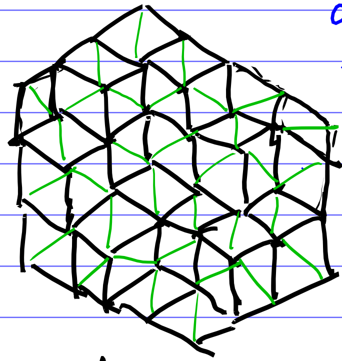




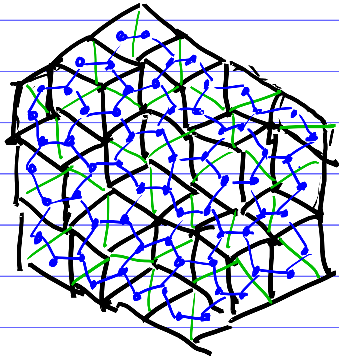
a pseudo line arrangement

a rhombus tilings

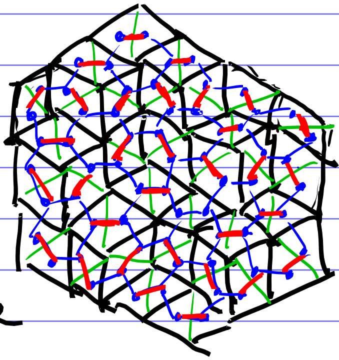




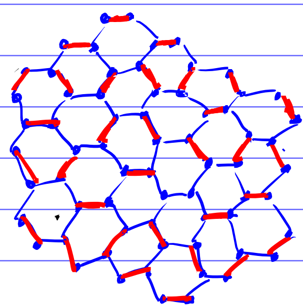
divide each rhombus by chord into 2 triangles



take the plane dual graph



red edges are the edges that cross the dividing chords



a perfect matching of the honeycomb graph

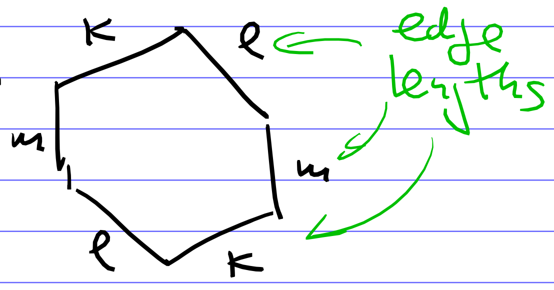
aka dimers

Theorem The number $pp_{k,\ell,m}$
(# plane partitions of shape $k \times \ell$ with entries in $\{0,1,\dots,m\}$)

equals:

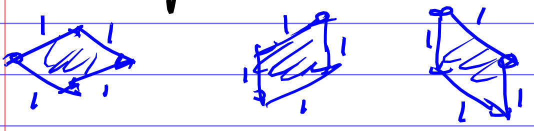
- (rhombus tilings)

  # ways to subdivide the hexagon

  by rhombuses:

  3 tile pieces:

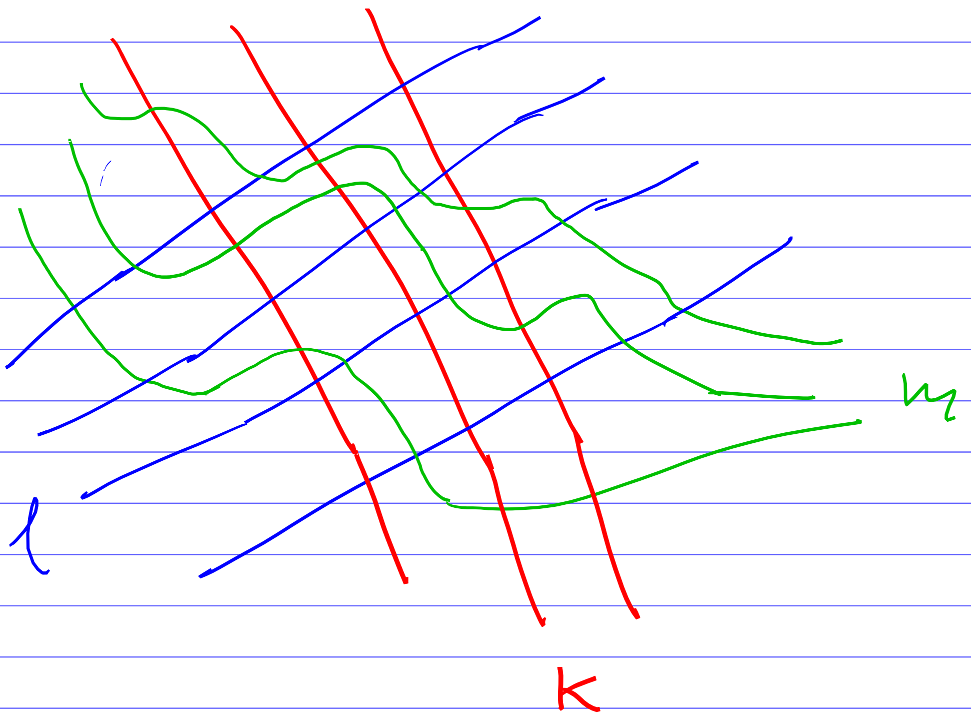
  

- (pseudoline arrangements)

  # combinatorial types of arrangements of pseudolines of 3 colors

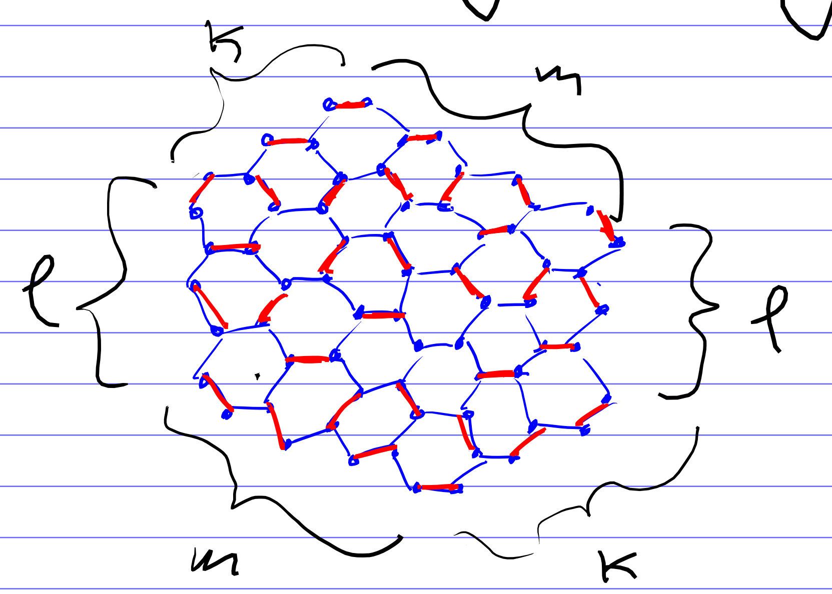
  (k red pseudolines, ℓ blue pseudoline, m green pseudolines)

  s.t. pseudolines of the same color don't intersect & 2 pseudolines of different colors intersect exactly once

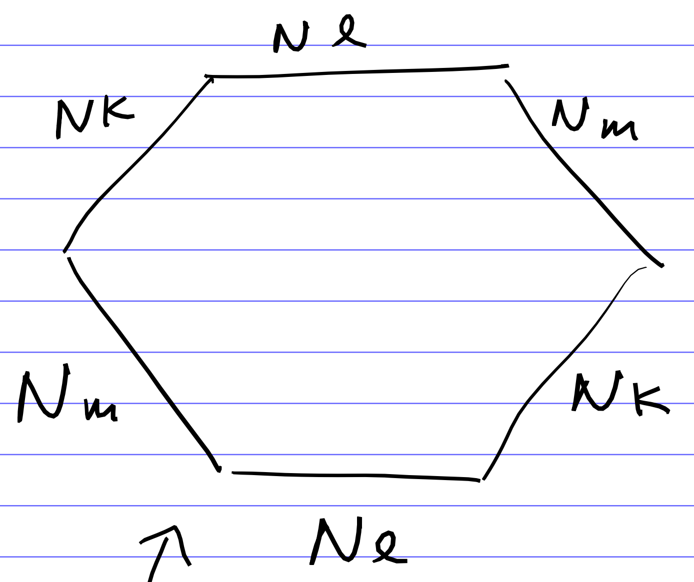


- perfect matchings:

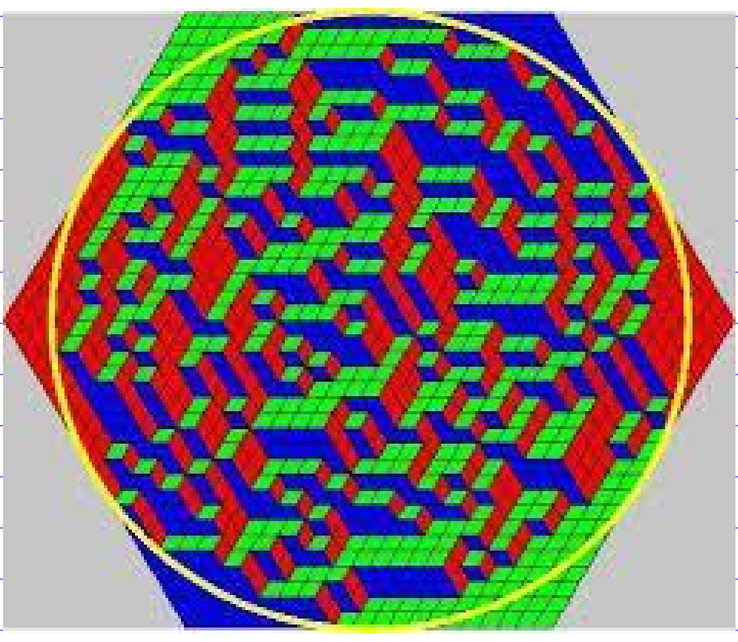
perfect matching of the honeycomb graph:

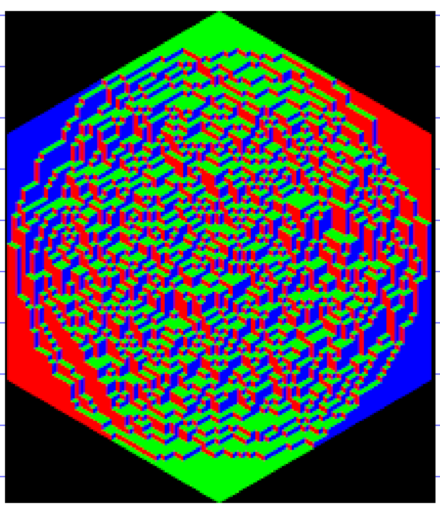


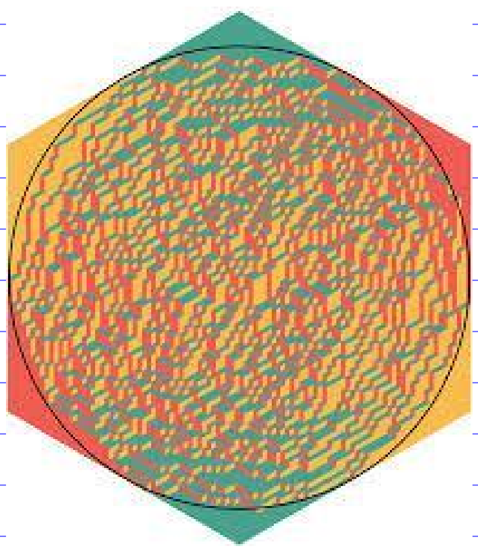
Arctic circle phenomenon:



Consider random rhombus tilings of hexagon for large N.
(Let's say $k=l=m$)







The area where tiling is "random" approaches a circle as $N \to \infty$.
The tiling is "frozen" around the vertices of the hexagon