last time: q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

$$= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)\cdot(1-q)\cdots(1-q^{n-k})}$$

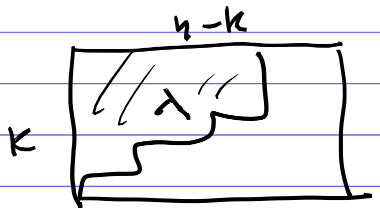
Theorem $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k\times(n-k)} q^{|\lambda|}$.

Not hard to prove by induction using the recurrence relations:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix},$$

It is easy to check this directly from the definition of $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

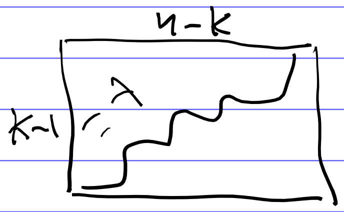
The same relation holds for the RHS: $\sum_{\lambda \subseteq k\times(n-k)} q^{|\lambda|}$:



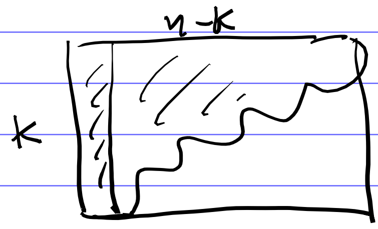
: 2 cases:

(I) λ has $< k$ rows.

then $\lambda \subseteq (k-1)\times(n-k)$



(II) λ has exactly k rows



then $\lambda \setminus 1^{st}$ column fits inside $k\times(n-k-1)$.

The factor q^k accounts for 1^{st} column of λ in this case.

But this proof does not explain where Young diagrams coming from.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q$$

Why is it related to Young diagrams?

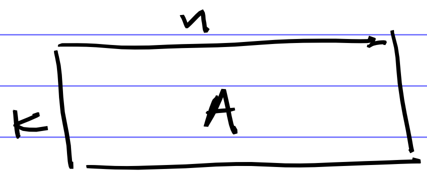
A more conceptual proof:

Fix integers $0 \le k \le n$.
The <u>Grassmannian</u> $Gr_{kn}(\mathbb{F})$ over some field $\mathbb{F}$ is the "space" (<u>set</u>, manifold, variety) of k-dimensional linear subspaces in $\mathbb{F}^n$.

A more "down to earth" way to think about the Grassmannian is

$$Gr_{kn}(\mathbb{F}) = \frac{\left\{ \begin{matrix} k \times n \text{ matrices} \\ \text{of rank } k \end{matrix} \right\}}{\text{row operations}}$$

$$= GL_k(\mathbb{F}) \Big\backslash \left\{ \begin{matrix} k \times n \text{ matrices} \\ \text{of rank } k \end{matrix} \right\}$$



of rank k

$\rightsquigarrow$ row space of A is a k-dim subspace of $\mathbb{F}^n$

row operations on A do not change the row space.

Let us now assume that
$\mathbb{F} = \mathbb{F}_q$ - finite field with
q elements

<u>Fact</u> $\forall\ q = p^r$ (p is a prime number)
there exist a unique (up to isomorphism) finite field, denoted $\mathbb{F}_q$.

Actually, the only thing needed for the following argument is the fact that there are infinitely many finite fields.

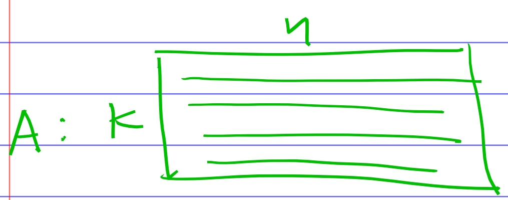
Then $Gr_{kn}(\mathbb{F}_q)$ is a finite set.

Let's count its cardinality
$\# Gr_{kn}(\mathbb{F}_q)$ in 2
different ways:

I. $\#\{k \times n \text{ matrices} / \mathbb{F}_q\}$

$$= (q^n - 1)(q^n - q)(q^n - q^2)\cdots(q^n - q^{k-1})$$

k terms



$q^n - 1$ = # ways to pick the first row of A, which can be any vector in $\mathbb{F}_q^n$, except the 0-vector

$q^n - q$ = # way to pick the second row of A, which can be any vector that is not proportional to the 1^{st} row

$q^n - q^2$: # ways to pick 3^{rd} row (any vector that $\notin$ span of first 2 rows)

etc

In particular, $\# GL_k(\mathbb{F}_q)$

$= \#\{k\times k \text{ matrices of rank } k\}$

$= (q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$

set $n=k$ in previous expression.

So $\# Gr_{kn}(\mathbb{F}_q) =$

$$= \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}$$

$$= \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

<u>Theorem</u> $\# Gr_{kn}(\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$

the value of q-binom. coeff for $q = p^r$.

Now let's count $\# Gr_{kn}(\mathbb{F}_q)$ using Gaussian elimination

Gauss: A $k \times n$ matrix modulo row operations can be transformed into a unique reduced row echelon form.

Ex. $k=5$, $n=11$

$$k \begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$
n

A has k pivots because $\operatorname{rank}(A) = k$.

"$*$" - any element of $\mathbb{F}_q$.

Remove the pivot columns:

we get
$$\begin{array}{cc} & n-k \\ k & \begin{bmatrix} 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \end{array}$$

shape λ corresponds to positions of pivots

$\lambda = (5, 4, 2, 2, 1)$

The shape formed by "$*$"'s is a (reflected) Young diagram $\lambda \subseteq k \times (n-k)$.

$\#$ ways to pick "$*$" $= q^{|\lambda|}$

Theorem $\# Gr_{kn}(\mathbb{F}_q) =$

$$= \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

Remark. This decomposition of $Gr_{kn}(\mathbb{F})$ into "cells" given by $\lambda \subseteq k \times (n-k)$ is called the Schubert decomposition

Here $\mathbb{F}$ can be any field.

So we proved that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}$$

for any $q = p^r$ (power of prime)

Since there are infinitely many numbers of this form

$$\Rightarrow \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}$$

for any q.

$\square$

<u>Remark.</u> This proof might be a little longer than the proof by induction.
But it explains where Young diagrams coming from.

"Young diagrams" =

= "shapes formed by $*$'s in a reduced echelon form"

Now the know that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial with nonnegative coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = a_0 + a_1 q + \dots + a_N q^N$$

a_i's are called the <u>Gaussian coefficients</u>.

$$a_i = \#\left\{ \begin{array}{l} \text{Young diagrams } \lambda \\ \text{s.t. } \lambda \subseteq k \times (n-k) \\ |\lambda| = i \end{array} \right\}$$

$N = k(n-k)$ ← largest possible size of such Young diagram

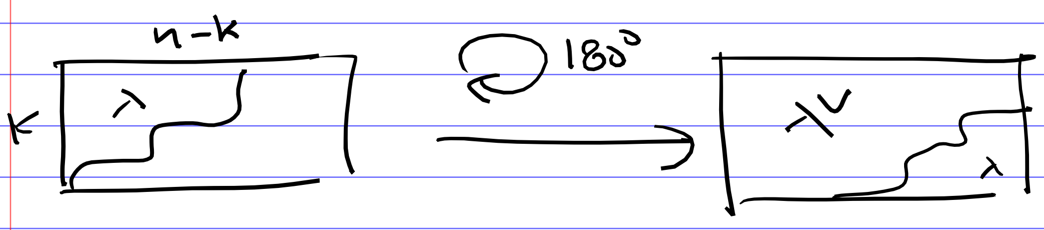
<u>Theorem</u> (Symmetry of the Gaussian coefficients):

$\boxed{a_i = a_{N-i}}$ for $i = 0, 1, \dots, N$.

<u>Proof</u> (easy)

Involution of $\lambda \subseteq k \times (n-k)$

$\lambda \longleftrightarrow \lambda^\vee$



$\lambda^\vee$ is the skew diagram $k \times (n-k)/\lambda$ rotated by $180°$.

Equiv, $\lambda^\vee = (n-k-\lambda_k, \dots, n-k-\lambda_1)$

Clearly $|\lambda^\vee| = k\cdot(n-k) - |\lambda|$, which proves the symmetry. $\square$

<u>Example</u> $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q =$

$= 1 + q + 2q^2 + q^3 + q^4$

a <u>palindromic</u> polynomial.

Theorem (Unimodality of the Gaussian coefficients)

$$a_0 \le a_1 \le \dots \le a_{\lfloor \frac{N}{2} \rfloor} \ge \dots \ge a_N.$$

harder to prove

We will give a proof due to Sylvester 1878.

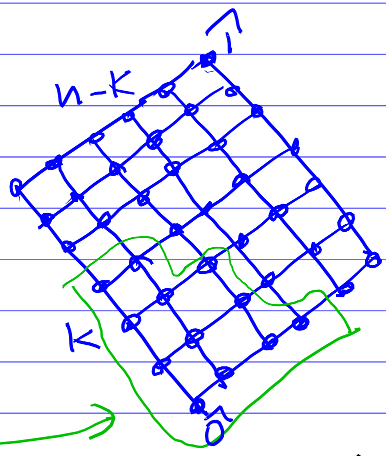
Let express this theorem in the language of posets

P – a poset

an order ideal in P is a subset λ of elements of P s.t. $x \in \lambda$, $y < x \Rightarrow y \in \lambda$.

Ex.

$P =$



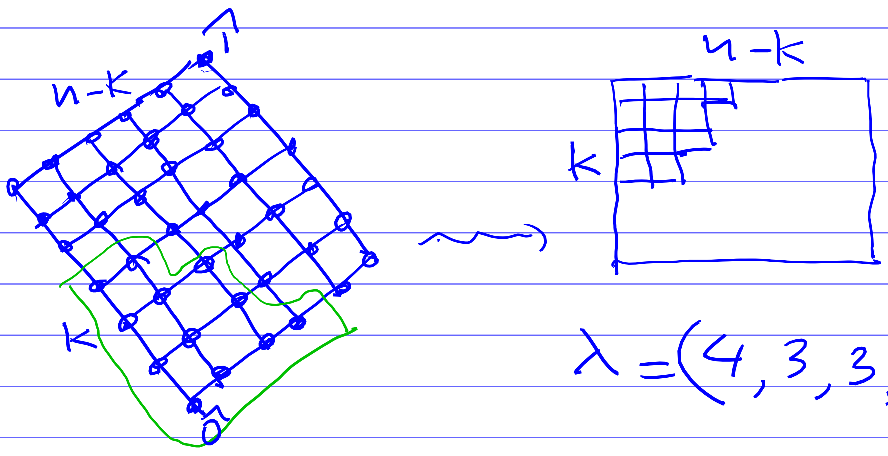
an order ideal

$P = [k] \times [n-k]$ product of 2 chains

= the poset whose Hasse diagram is the $k \times (n-k)$ grid (rotated by $45°$)

Order ideals λ in $P = [k] \times [n-k]$ correspond to Young diagrams $\lambda \subseteq k \times (n-k)$

Vertices of the Hasse diagram of $P = [k] \times [n-k]$ correspond to boxes of λ.



$\lambda = (4,3,3,2)$

$J(P)$ - the lattice of order ideals of P
(all order ideals ordered by inclusion)

$J(P)_r$ = the set of all order ideals with r elements
i.e. elts. of $J(P)$ of rank r.

Example For $P = [k] \times [n-k]$

$$J(P)_r \longleftrightarrow \left\{ \begin{array}{l} \text{Young diagrams } \lambda \subseteq k \times (n-k) \\ \text{s.t. } |\lambda| = r \end{array} \right\}$$

For $\lambda \in J(P)$, let

$$\mathrm{Add}(\lambda) = \{ x \in P \mid \lambda \cup \{x\} \in J(P) \}$$

$$\mathrm{Remove}(\lambda) = \{ y \in P \mid \lambda \setminus \{y\} \in J(P) \}$$

In our example

$$\mathrm{Add}(\lambda) = \left\{ \begin{array}{l} \text{all "addable" boxes} \\ \text{to } \lambda \end{array} \right\}$$

i.e. outer corners of λ

$$\mathrm{Remove}(\lambda) = \left\{ \begin{array}{l} \text{all removable} \\ \text{boxes from } \lambda \end{array} \right\}$$

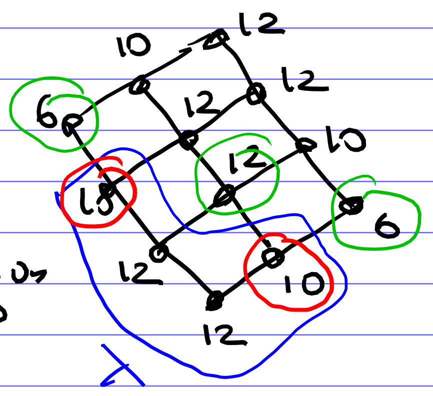
i.e. inner corners of λ.

Theorem Fix P & r. Suppose $\exists$ weight function $wt : P \to \mathbb{R}_{>0}$ s.t. $\forall \lambda \in J(P)_r$, we have
$$\sum_{x \in Add(\lambda)} wt(x) > \sum_{y \in Remov(\lambda)} wt(y)$$

Then $\# J(P)_r \leq J(P)_{r+1}$

Example $P = [4] \times [3]$
$r = 4$

a weight function on P



$$6 + 12 + 12 > 10 + 10$$

As Young diagrams:

12	10	6
12	12	10
10	12	12
6	10	12

Remark. Remember that for $\lambda \in \mathbb{Y}$, $\#$ outer corners $= \#$ inner corners $+1$

Why we cannot just take $wt(x) = 1$?

A: Because our Young diagr, λ should belong to $k \times (n-k)$ rectangle.

If we restrict to part of $\mathbb{Y}$ formed by $\lambda \subseteq k \times (n-k)$, then the identity
$$\#\{\text{outer corners}\} = \#\{\text{inner corners}\} + 1$$
inside the $k \times (n-k)$ rectangle

might no longer be true.

But for some other choice of weight we can get an interesting identity.

Proof Define the (weighted) up & down operators on $\mathbb{R}[J(P)]$ — space of formal lin. combinations of elts of $J(P)$

$$U : \lambda \mapsto \sum_{x \in Add(\lambda)} \sqrt{wt(x)}\ \lambda \cup \{x\}$$

$$D : \lambda \longmapsto \sum_{y \in Remove(\lambda)} \sqrt{wt(y)}\ \lambda \setminus \{y\}$$

Notice that $D = U^T$.

D is given by the conjugate matrix U^T to U.

Lemma The commutator $H = DU - UD$ has diagonal form

$$H : \lambda \mapsto \left(\sum_{x \in Add(\lambda)} \sqrt{wt(x)} \cdot \sqrt{wt(x)} - \sum_{y \in Remove(\lambda)} \sqrt{wt(y)}\sqrt{wt(y)} \right) \lambda$$

Proof The fact that H is a diagonal operator is proved in basically the same way as for the (unweighted) up & down operators that we discussed earlier.

If we want to add an element x and then remove a <u>different</u> elt. $y \neq x$, we can do these operations in different order (first remove y and then add x) and get we same result (with the same weight $\sqrt{wt(x)} \cdot \sqrt{wt(y)}$)

It remains to consider the case when we add and then remove the same elt. x, or remove and then add the same elt. y, which gives the needed expression $\square$

Now we have

$$DU = UD + H$$

$$= D^T D + H$$

Symmetric matrix with eigenvalues ≥ 0

i.e. positive <u>semi</u>definite matrix

diagonal

Let $V_r = \mathbb{R}[J(P)_r]$

formal lin. comb. of order ideals with r elements

$$\begin{tikzcd}
& V_{r+1} \arrow[dr, "D_{r+1}"] & \\
V_r \arrow[ur, "U_r"] \arrow[rr, "H_r"] \arrow[dr, "D_r"'] & & V_r \\
& V_{r-1} \arrow[ur, "U_{r-1}"'] &
\end{tikzcd}$$

Let D_r, U_r, H_r be the components of operators D, U, H restricted to V_r.

i.e. $D_r : V_r \to V_{r-1}$

$U_r : V_r \to V_{r+1}$

$H_r : V_r \to V_r$

(They are given by matrices which are blocks of bigger matrices D, U, H)

We have

$$D_{r+1} U_r : V_r \to V_{r+1} \to V_r$$

$$D_{r+1} U_r = U_{r-1} D_r + H_r$$

$$= D_r^T D_r + H_r$$

positive semi.def

positive
<u>definite</u>
(diag. matrix
with positive
entries on diag.)

$\Rightarrow$ $D_{r+1} U_r$ is a

positive <u>definite</u> matrix

- D_{r+1} is $|J(P)_r| \times |J(P)_{r+1}|$
rectangular matrix
- U_r is its transpose
- $D_{r+1} U_r$ is $|J(P)_r| \times |J(P)_r|$
square matrix
with $\det \neq 0$

$$\Rightarrow \operatorname{rank}(D_{r+1} U_r) =$$
$$= |J(P)_r|$$
$$\leq |J(P)_{r+1}|.$$

\# columns in U_r

We obtained the needed inequality. $\square$

Now to prove the Unimodality of Gaussian coefficient

$$a_r = |J([k] \times [n-k])_r|$$

it remains to construct a weight function

$$wt : [k] \times [n-k] \to \mathbb{R}_{>0}$$

with needed properties

<u>Claim.</u> The following weight function works:

$$wt : [k] \times [n-k] \to \mathbb{R}_{>0}$$

given by

$$wt(x) = (n-k-c(x)) \cdot (k+c(x))$$

where $c(x)$ is the <u>content</u> of box $x \in k \times (n-k)$ rectangle.

Example $k=4, \quad n-k=3$

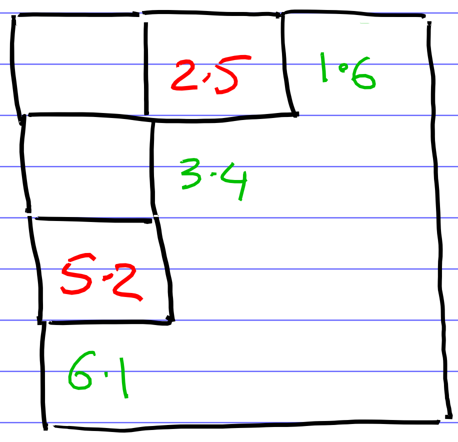
$$\begin{array}{c|ccc} & & n-k & \\ \hline & 3\cdot 4 & 2\cdot 5 & 1\cdot 6 \\ k & 4\cdot 3 & 3\cdot 4 & 2\cdot 5 \\ & 5\cdot 2 & 4\cdot 3 & 3\cdot 4 \\ & 6\cdot 1 & 5\cdot 2 & 4\cdot 3 \end{array}$$

Lemma, For this weight function, for any $\lambda \subseteq k \times (n-k)$

$$\sum_{x \in Add(\lambda)} wt(x) - \sum_{y \in Remove(\lambda)} wt(y)$$

$$= k\cdot(n-k) - 2|\lambda|.$$

Example. $\lambda = (2, 1, 1)$



$$1\cdot 6 + 3\cdot 4 + 6\cdot 1 - 2\cdot 5 - 5\cdot 2 =$$

$$= 4\cdot 3 - 2\cdot 4$$

$k\cdot(n-k)$ $\qquad$ $2|\lambda|$

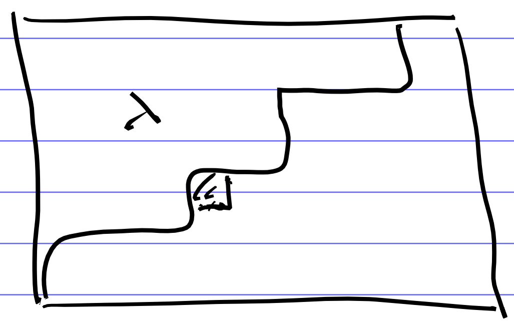
Proof of Lemma

- True for $\lambda = \emptyset$

  $LHS = k(n-k) = RHS$.

- When we add a box to λ, the RHS decreases by 2.

  We can check that the LHS also decreases by 2:



there are several cases how the sets $Add(\lambda)$ & $Remov(\lambda)$ will change.

It is easy to check that in all cases $\sum_{Add} wt(x) - \sum_{Remove} wt(y)$

decreased by 2. $\square$

So for $r = |\lambda| < \frac{k(n-k)}{2}$

$$\sum_{x \in Add(\lambda)} wt(x) - \sum_{y \in Remove(\lambda)} wt(y) > 0$$

$\Rightarrow$ we can apply the argument with positive definite matrices and prove that $a_r \leq a_{r+1}$.

This is exactly what we need for the unimodality of Gaussian coeffs.

$$a_0 \leq a_1 \leq \dots \leq a_{\lfloor N/2 \rfloor} \geq \dots \geq a_N$$

$$N = k(n-k).$$

$\square$

Remark Sylvester's proof is non-constructive

First constructive proof of unimodality of the Gaussian coeff., i.e.

a bijection between Young diagrams $\lambda \subseteq k \times (n-k)$ with r boxes and some subset of Young diagrams with $r+1$ boxes, for $r < \frac{k(n-k)}{2}$,

was found by K. O'Hara in 1990.