18.217 Lecture 12 09/30/2020

Last time: $s_\lambda(\underbrace{1,\dots,1}_{n}) = \#\left\{\begin{array}{l}\text{SSYT's}\\ \text{of shape } \lambda\\ \text{w/ entries} \in\\ \{1,2,\dots,n\}\end{array}\right\}$

$=$ # (integer) Gelfand - Tsetlin patterns with top row $(\lambda_1,\dots,\lambda_n)$

Stanley's hook-content formula

$$= \boxed{\prod_{(i,j)\in\lambda} \frac{n+c_{ij}}{h_{ij}}}$$

h_{ij} hook length

$c_{ij} := j-i$ content of box (i,j)

Weyl's dimension formula

$$= \boxed{\prod_{1\le i<j\le n} \frac{\lambda_i-\lambda_j+j-i}{j-i}}$$

this is the dimension of irreducible representation V_λ of GL_n with highest weight λ

Example. $n=3$

$\lambda=(3,2,0)$

$$s_{\begin{smallmatrix}\square\square\square\\ \square\square\end{smallmatrix}}(1,1,1) =$$

$\begin{array}{|c|c|c|}\hline 4&3&1\\ \hline 2&1\\ \cline{1-2}\end{array}$ hook lengths

$$= \frac{(2+0)(2+1)(2+2)(2-1)(2+0)}{4\cdot3\cdot1\cdot2\cdot1}$$

$\begin{array}{|c|c|c|}\hline 0&1&2\\ \hline -1&0\\ \cline{1-2}\end{array}$ contents

$$= \frac{(3-2+1)}{1}\cdot\frac{(3-0+2)}{2}\cdot\frac{(2-0+1)}{1} =$$

$$= 15.$$

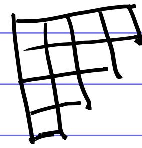
15 SSYT's: $\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2&2\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 2&3\\ \cline{1-2}\end{array}$ $\dots$ $\begin{array}{|c|c|c|}\hline 2&2&3\\ \hline 3&3\\ \cline{1-2}\end{array}$

Which formula is "better"?

- Stanley's: product over boxes of λ
- Weyl's: product of $\binom{n}{2}$ terms

If λ is fixed and n is large then hook-content formula is more efficient.

If n is fixed & $\lambda=(\lambda_1,\dots,\lambda_n)$ an abitrary partition, then Weyl's dim formula is more efficient.

Example $\lambda = (n-1, n-2, \ldots, 0) =$ 

"staircase shape"

$\#$ SSYT's of shape $(n-1, n-2, \ldots, 0)$ with entries $\in \{1, \ldots, n\}$

Weyl's formula

$$\prod_{1 \le i < j \le n} \frac{j-i+j-i}{j-i} = \boxed{2^{\binom{n}{2}}}$$

Ex. $n=2$: $\boxed{1}$ $\boxed{2}$ 2^1

$n=3$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

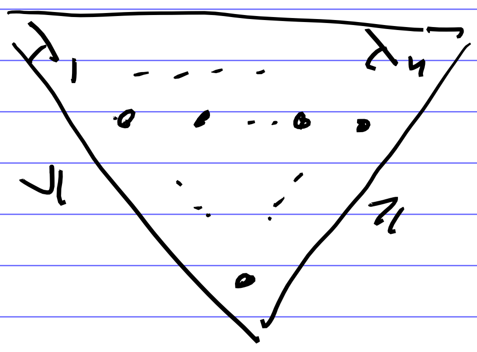
$$\underbrace{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}}_{8 = 2^3}$$

Q: Is there a bijective proof?

Gelfand-Tsetlin polytope:

$\lambda = (\lambda_1, \dots, \lambda_n)$

$GT(\lambda) \subset \mathbb{R}^{\binom{n}{2}}$ – the polytope of real-valued GT-patterns with top row λ.



$$\#\left(GT(\lambda) \cap \mathbb{Z}^{\binom{n}{2}}\right) = s_\lambda(\underbrace{1 \dots 1}_{n})$$

integer GT-patterns with top row λ

Ehrhart "polynomial" of a polytope $P \subset \mathbb{R}^N$

$$L_P(t) = \#\left(tP \cap \mathbb{Z}^N\right), \quad t \in \mathbb{Z}_{>0}$$

integer lattice points in the dilated polytope tP.

Theorem (Ehrhart) If P is a lattice polytope (i.e. all vertices $\in \mathbb{Z}^N$), then $L_P(t)$ is a polynomial function in t.

Remark. In general, $L_P(t)$ may not be a polynomial in t when P has non-integer vertices.

If P is a rational polytope then $L_P(t)$ is a quasi-polynomial (i.e. there might be several cases & several different polynomials)

But there are examples of polytopes, which are not lattice polytope, for which $L_P(t)$ is a polynomial in t. (For example, Littlewood-Richardson polytopes aka hive / honeycomb polytopes that we'll discuss later.)

Dilations of GT-polytopes:

$$t \cdot GT(\lambda) = GT(t \cdot \lambda),$$

where $t \cdot \lambda := (t\lambda_1, t\lambda_2, \dots, t\lambda_n)$.

Weyl's character formula $\Longrightarrow$

<u>Corollary</u>. The Ehrhart polynomial of the Gelfand-Tsetlin polytope is

$$L_{GT(\lambda)}(t) = \prod_{1 \le i < j \le n} \frac{t(\lambda_i - \lambda_j) + j - i}{j - i}.$$

In particular, $L_{GT(\lambda)}$ is a polynomial in t with <u>positive</u> coefficients.

Typically, Ehrhart polynomials don't have positive coefficients. But for some special classes of polytopes they do have positive coeffs., e.g. <u>Ehrhart positivity</u> holds for <u>permutohedra</u> and (conjecturally) for <u>generalized permutohedra</u>.

$$\mathrm{Vol}(P) = \text{the top coeff of } L_P(t) = \lim_{t\to\infty} \frac{L_P(t)}{t^N}.$$

Corollary The volume of GT-polytope is

$$\boxed{\mathrm{Vol}\, GT(\lambda) = \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j}{j-i}}$$

Example Staircase Gelfand-Tsetlin polytope $\lambda = (n-1, n-2, \dots, 1, 0)$

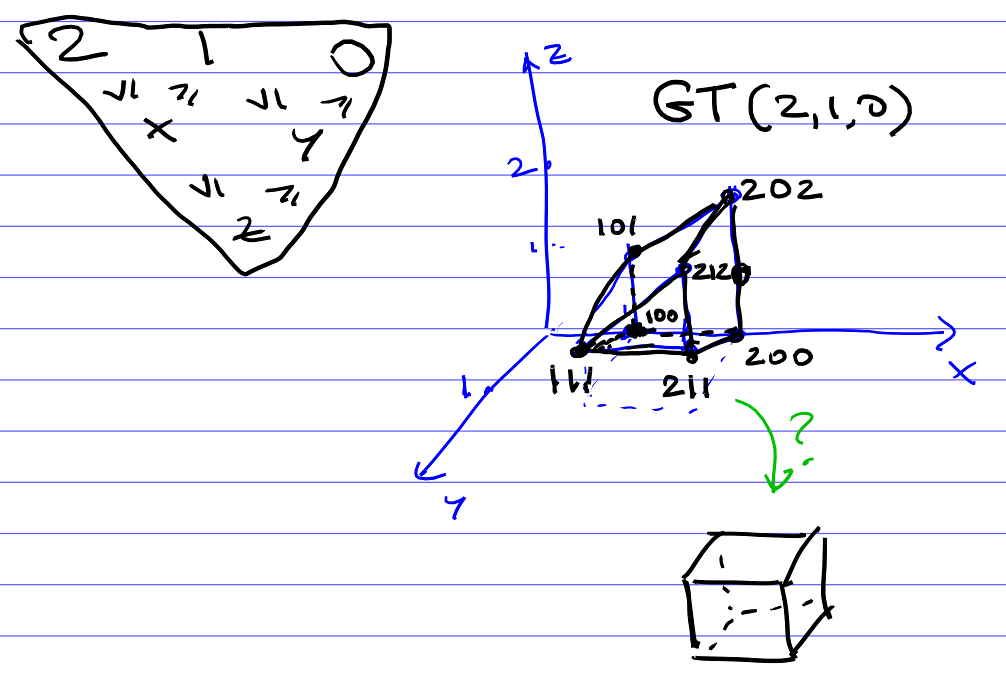
Ehrhart polynomial:

$$L_{GT(n-1,n-2,\dots,0)}(t) = \boxed{(t+1)^{\binom{n}{2}}}$$

Volume $= 1$

same as for $\binom{n}{2}$-dimensional unit cube

For $n=3$:



Problem Construct a piecewise-linear volume preserving bijection

$$GT(n-1, n-2, \dots, 0) \xrightarrow{\sim} [0,1]^{\binom{n}{2}}.$$

Theorem For $\lambda = (\lambda_1 \ge \dots \ge \lambda_n) \in \mathbb{Z}^n$, $GT(\lambda)$ is a lattice polytope.

Proof. $GT(\lambda)$ is given by linear inequalities:

$$\begin{array}{ccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \lambda_4 \\
 & x & & y & & z & \\
 & & t & & u & & \\
 & & & s & & &
\end{array}$$

In order to describe any vertex we need to replace some of these inequalities "$\ge$" by equalities "$=$" s.t.

- A solution exists
- If we replace one more "$\ge$" by "$=$", there will be no solutions.

So each vertex of $GT(\lambda)$ has the form as in this example:

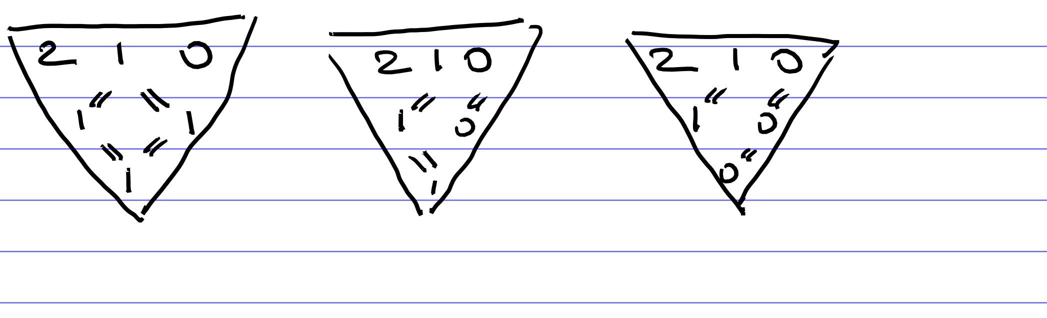
$$\begin{array}{ccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \lambda_4 \\
 & \lambda_1 & & \lambda_3 & & \lambda_3 & \\
 & & \lambda_1 & & \lambda_3 & & \\
 & & & \lambda_3 & & &
\end{array}$$

(each entry equals to one of the two entries above it)

In particular each entry equals to one of λ_i's.

$\Rightarrow$ all vertices of $GT(\lambda)$ are integer. $\square$

Example: 7 vertices of $GT(2,1,0)$:



A <u>more refined</u> <u>Gelfand-Tsetlin</u> <u>polytope</u>. Fix $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$

$GT(\lambda, \mu)$ = the polytope of real-valued GT-patterns with top row λ and weight μ, i.e. with row sums

$\mu_1, \ \mu_1 + \mu_2, \ \mu_1 + \mu_2 + \mu_3, \dots$
(from the bottom)

<u>Example</u> $n = 3$

$$\begin{array}{ccccccl}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \\
 & x & & y & & & \mu_1 + \mu_2 \\
 & & z & & & & \mu_1
\end{array}$$

$GT(\lambda, \mu) =$

$= \{ (x, y, z) \in \mathbb{R}^3 :$

$\lambda_1 \geq x \geq \lambda_2,$

$\lambda_2 \geq y \geq \lambda_3$

$x \geq z \geq y$

$z = \mu_1$

$x + y = \mu_1 + \mu_2 \}$

Assume λ, μ are integer vectors.

integer lattice points in $GT(\lambda, \mu)$

is $\# GT(\lambda, \mu) \cap \mathbb{Z}^{\binom{n}{2}}$

$= K_{\lambda \mu}$ (Kostka number)

SSYT's of shape λ & weight μ.

Some questions to think about:

- Is $GT(\lambda,\mu)$ always a lattice polytope?

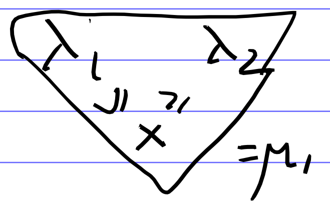
- Is its Ehrhart "polynomial"

$$L_{GT(\lambda,\mu)}(t) = K_{t\lambda, t\mu}$$

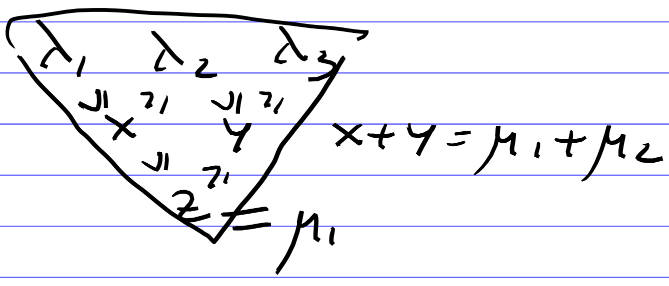
a polynomial in t?

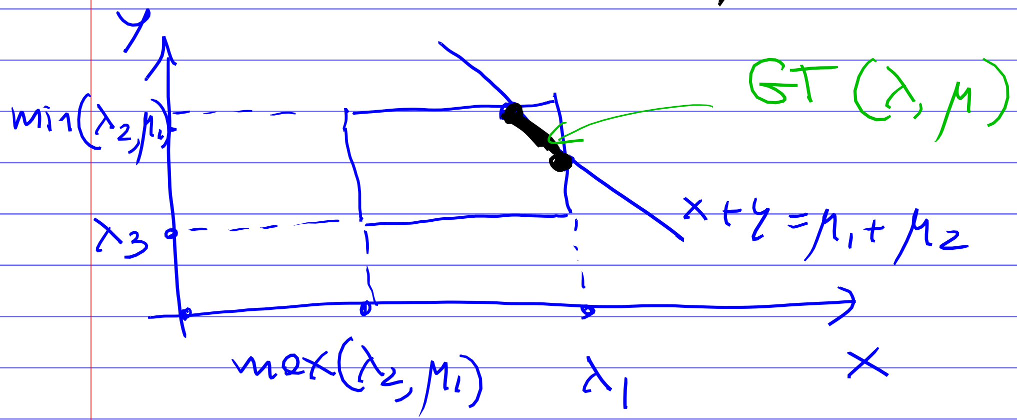
- Does it have positive coefficients?

Examples: $n=2$, if $K_{\lambda\mu} \neq 0$.
(i.e. if $\lambda_1 \geq \mu_1 \geq \lambda_2$)
then $K_{t\lambda, t\mu} = 1$, $\forall\, t \in \mathbb{Z}_{\geq 0}$.



$n=3$





If $K_{\lambda\mu} = \ell + 1$ then

$$K_{t\lambda, t\mu} = \underbrace{\ell \cdot t + 1}$$

\# lattice points in t-dilation of a line segment of length ℓ.

this is a polynomial in t with positive coeffs.

We can answer some of the above questions using Kostant's partition function...

Kostant's partition function $p(\beta)$
(of type A)

$\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ any vector, (not a partition)
s.t. $\beta_1 + \dots + \beta_n = 0$

$p(\beta) :=$ # ways to express β as a <u>nonnegative integer linear</u> combination of vectors $e_i - e_j$, $1 \le i < j \le n$ (called <u>positive roots</u>)

(Here $e_1, \dots, e_n$ are the coord. vectors in $\mathbb{R}^n$.)

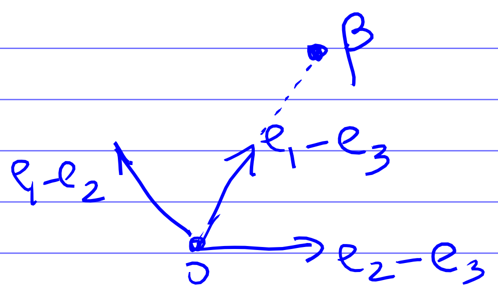
<u>Ex.</u> $n = 3$

$\beta = (2, 0, -2)$

$$\beta = 2(e_1 - e_3) = (e_1 - e_3) + (e_1 - e_2) + (e_2 - e_3)$$

$$= 2(e_1 - e_2) + 2(e_2 - e_3)$$

So $p((2,0,-2)) = 3$.



Kostant multiplicity formula

(for type A)

For $\lambda = (\lambda_1, \ldots, \lambda_n)$ $\mu = (\mu_1, \ldots, \mu_n)$

$$\boxed{K_{\lambda\mu} = \sum_{w \in S_n} (-1)^{\ell(w)} p(w(\lambda+\rho) - (\mu+\rho)),}$$

Kostka numbers are known as weight multiplicities in Lie theory

where $\rho = (n-1, n-2, \ldots, 1, 0)$

the same thing that we earlier denoted δ.

Example. $n=3$

In lecture 4, we calculated

$$S_{(4,2,0)}(x_1, x_2, x_3) = \ldots + 3x_1^2 x_2^2 x_3^2 + \ldots$$

$$K_{(420),(222)} = 3$$

Kostant's formula: $K_{(420),(222)} =$

$$= \underset{\overset{\|}{3}}{p(2,0,-2)} - \underset{\overset{\|}{0}}{p(-1,3,-2)} - \underset{\overset{\|}{0}}{p(-1,3,-2} + \ldots$$

$= 3.$

Proof. classical def. of $S_\lambda(x_1 \ldots x_n)$

$$S_\lambda := \frac{a_{\lambda+\rho}}{a_\rho} = \frac{\sum_{w \in S_n} (-1)^{\ell(w)} x^{w(\lambda+\rho)}}{x^\rho \prod_{1 \le i < j \le n} \left(1 - \frac{x_j}{x_i}\right)}$$

sign(w)

$$= \sum_{w \in S_n} (-1)^{\ell(w)} x^{w(\lambda+\rho)-\rho} \cdot \prod_{i<j} \frac{1}{1 - \frac{x_j}{x_i}}$$

We have $\displaystyle \prod_{i<j} \frac{1}{1 - \frac{x_j}{x_i}} =$

$$= \prod_{i<j} \frac{1}{1 - x^{-(e_i - e_j)}} = \prod_{i<j} \left(\sum_{k_{ij} \ge 0} x^{-k_{ij}(e_i - e_j)} \right)$$

$$= \sum_{\beta \in \mathbb{Z}^n} P(\beta)\, x^{-\beta}$$

this is basically the def. of Kostant's partition function

$$S_\lambda = \sum_{\substack{w \in S_n \\ \beta \in \mathbb{Z}^n}} (-1)^{\ell(w)} x^{w(\lambda+\rho)-\rho-\beta} P(\beta)$$

$$= \sum_{\substack{w \in S_n \\ \mu \in \mathbb{Z}^n}} (-1)^{\ell(w)} P(w(\lambda+\rho) - \mu - \rho)\, x^\mu$$

So $K_{\lambda\mu} :=$ the coeff of x^μ in S_λ $=$

$$= \sum_{w \in S_n} (-1)^{\ell(w)} P(w(\lambda+\rho) - \mu - \rho).$$

$\square$

... back to polytopes

$P(\beta)$ is # of integer lattice points of <u>flow polytope</u>

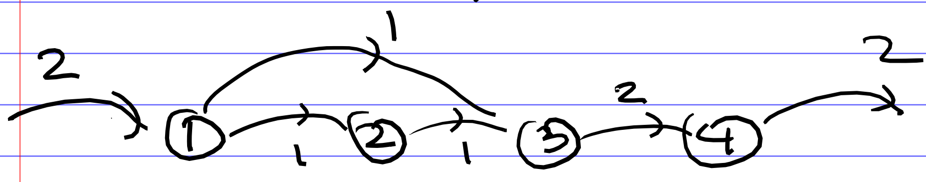
$$\mathrm{Flow}(\beta) := \left\{ (x_{ij})_{1\le i<j\le n} \in \mathbb{R}^{\binom{n}{2}} \;\middle|\; \begin{array}{l} x_{ij} \ge 0 \\ \sum x_{ij}(e_i - e_j) = \beta \end{array} \right\}$$

We can think of (x_{ij}) as flows along edges (i,j) of K_n (directed $i \to j$ for $i<j$) s.t. $\forall$ vertex $k \in [n]$

$$\underbrace{\sum_{i\to k} x_{ik}}_{\text{in-flow to vertex } k} + \underbrace{\beta_k}_{\text{excess in-flow}} = \underbrace{\sum_{j:\, k\to j} x_{kj}}_{\text{out-flow from vertex } k}$$

$(\beta_1, \ldots \beta_n)$ is <u>excess flow vector</u>.

<u>Example</u> $\beta = (2, 0, 0, -2)$



a flow on K_4

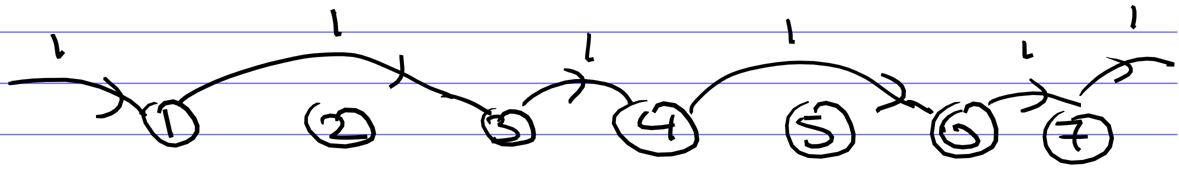
<u>Lemma</u> $\mathrm{Flow}(\beta)$ is an lattice polytope in $\mathbb{R}^{\binom{n}{2}}$ (all vertices are integer vectors)

<u>Exercise</u>, Prove this by describing the vertices of $\mathrm{Flow}(\beta)$.

Example.

Let's describle vertices of Flow(β) for $\beta = (1,0,0,\ldots,0,-1)$

Flow$(1,0,\ldots,0,-1)$ has 2^{n-2} vertices that correspond to all directed paths in K_n from 1 to n (with edges directed as $i \to j$ for $i<j$)



a flow in K_7 with excess in-flow 1 for 1^{st} vertex & excess out-flow 1 for the last vertex.

It is not hard to see that any other flow in K_n for $\beta = (1,0,\ldots,0,-1)$ is a non-negative linear combination of such "path flows".

Theorem Fix $\beta, \lambda, \mu \in \mathbb{Z}^n$.

(1) Kostant's partition function $p(t \cdot \beta)$ is a polynomial in t.

(2) Kostka number (or weight multiplicities)

$$K_{t\lambda, t\mu} \text{ is a polynomial}$$

in t.

In both cases, we assume that t is a positive integer)

Proof (1) Follows from Ehrhart Thm
(2) Follows from (1) & Kostant's formula. $\square$

Remark This argument proves polynomiality, but does not prove positivity...

Remark. Kostant's partition function $p(\beta)$ and Kostka numbers $K_{\lambda,\mu}$ considered as multivariate functions of $\beta_1, \dots, \beta_n$ and $\lambda_1, \dots, \lambda_n, \mu_1 \dots \mu_n$ are piecewise polynomial functions.

<u>Conjecture</u> $p(t\beta)$ is a polynomial in with <u>positive</u> coefficients.

<u>Theorem</u>. This conjecture holds for $(\beta_1, \dots, \beta_n)$ s.t. $\beta_1, \beta_2, \dots, \beta_{n-1} \geq 0$.

(In this case, there is an explicit polynomial formula for $p(\beta)$

<u>Lidskii, P.-Stanley, Baldoni-Vergne</u>)

A more general conjecture

<u>Conjecture</u> $K_{t\lambda, t\mu}$ is a polynomial in t with positive coefficients.

Why "more general"?

<u>Lemma</u> $p(\beta) = K_{\beta + N\rho, N\rho}$ for sufficiently large N.

<u>Proof</u>. Kostant's formula for $K_{\beta+N\rho, N\rho}$ has only one term $p(\beta)$ if N is large. Indeed,

$$K_{\beta+N\rho, N\rho} = \sum_{w \in S_n} (-1)^{\ell(w)} p(w(\beta + N\rho + \rho) - N\rho - \rho)$$

for $w = id$, we get $p(\beta)$

for all other w's we get

$$p\left(\begin{array}{l}\text{some vector that} \\ \text{cannot be expressed} \\ \text{as a non-negative} \\ \text{combination of positive} \\ \text{roots } e_i - e_j,\ i<j.\end{array}\right) = 0$$

$$\approx w(N\rho) - N\rho$$

even more general conjecture...

Later in this course we'll talk about the <u>Littlewood–Richardson coefficients</u> $c_{\lambda\mu}^{\nu}$:

(λ, μ, ν partitions) defined by

$$s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$$

The LR-coeffs $c_{\lambda\mu}^{\nu}$ include the Kostka numbers $K_{\lambda\mu}$ as special cases:

"special cases of"

$$\underbrace{\{P(\beta)\}}_{\text{Kostant's part. funt.}} \subset \underbrace{\{K_{\lambda\mu}\}}_{\text{Kostka number}} \subset \underbrace{\{c_{\lambda\mu}^{\nu}\}}_{\text{Littlewood-Rich. coeffs.}}$$

$c_{\lambda\mu}^{\nu}$ can also be expressed as linear combinations of Kostant's partition functions

<u>Conjecture</u> (King–Tolly–Toumazet)

For any partitions λ, μ, ν

$c_{t\lambda, t\mu}^{t\nu}$, $t \in \mathbb{Z}_{>0}$ is a polynomial $f(t)$ with <u>positive integer</u> coeffs. , or $f(t) \equiv 0$.

The previous conjecture can be viewed as a strong form of the saturation conjecture.

Saturation Conjecture

(proved by Knutson-Tao based on works of Horn, Klyachko, Berenstein-Zelevinsky)

If $c_{t\cdot\lambda, t\cdot\mu}^{t\cdot\nu} \neq 0$ for some t, then

$c_{\lambda\mu}^{\nu} \neq 0$.

Clearly, if $c_{t\lambda, t\mu}^{t\nu}$ is a polynomial $f(t)$ in t with positive coeff, then saturation holds: if one value of $f(t)$ (for $t>0$) is non-zero then all values of $f(t)$ (for $t>0$) are non-zero.

We'll talk more about this stuff later in this course...