18.217 Problem Set (due Monday, December 03, 2018)

Solve as many problems as you want. Turn in your favorite solutions. You can also solve and turn any other claims that were given in class without proofs, as well as any other problems that will be formulated in class after this problem set is posted.

Is is enough to turn in 3 problems.

Problem 1. Let $f_n(x, y)$, $n \in \mathbb{Z}$, be the sequence of rational functions in two variables x and y given by the initial conditions

$$f_0 = x, \quad f_1 = y$$

and the recurrence relation

$$f_{n+1} = ((f_n)^2 + 1)/f_{n-1}, \quad \text{for } n \in \mathbb{Z}.$$

Prove that $f_n(x, y)$ is a Laurent polynomial in x and y with positive integer coefficients. Find a combinatorial interpretation of these Laurent polynomials.

Problem 2. (Somos sequences) For a positive integer k, the *Somos-k sequence* is the sequence $(a_0, a_1, a_2, \dots)$ satisfying the recurrence relation

$$a_n\, a_{n-k} = a_{n-1}\, a_{n-k+1} + a_{n-2}\, a_{n-k+2} + \cdots + a_{n-\lfloor k/2 \rfloor}\, a_{n-\lceil k/2 \rceil},$$

with the initial condition $a_0 = \cdots = a_{k-1} = 1$.

(a) Prove that the entries of the Somos-4 sequence are integers.

(b) Prove that the entries of the Somos-5 sequence are integers.

Problem 3. (Frieze patterns) A *frieze pattern* of order n is an array of numbers $P = (p_{ij})$, $i, j \in \mathbb{Z}$, $i < j < i + n$, such that

$$\det \begin{pmatrix} p_{ij} & p_{i\,j+1} \\ p_{i+1\,j} & p_{i+1\,j+1} \end{pmatrix} = 1; \;\; p_{i\,i+1} = p_{i\,i+n-1} = 1; \;\; p_{ij} > 0.$$

For a triangulation T of an n-gon (with vertices corresponding to elements of $\mathbb{Z}/n\mathbb{Z}$ clockwise), construct the array $P(T) = (p_{ij})_{i<j<i+n}$ as follows:

For each $i \in \mathbb{Z}$, label the vertex $i \pmod n$ of T by 0. Then label by 1 all vertices connected to $i \pmod n$ by an edge or a diagonal of T. Then extend this labelling to all vertices of T using the recursive rule: For any triangle in T with two vertices labelled by a and b and one unlabelled vertex, assign the label $a + b$ to the third vertex. Then, for $j \in [i + 1, i + n - 1]$, let p_{ij} be the label of the vertex $j \pmod n$ in this labelling.

Prove that the map $T \mapsto P(T)$ is a bijection between triangulations of the n-gon and integer freize patterns of order n.

Problem 4. Let T_0 be the "star triangulation" of the n-gon where the vertex 1 is connected by diagonals with all other vertices. Assign algebraically independent variables x_e to all edges and diagonals e of the triangulation T_0.

(a) Construct a $2\times n$ matrix A (whose entries are rational expressions in the x_e) such that, for any edge or diagonal $e = (i, j)$, $1 \le i < j \le n$, in the triangulation T_0 (i.e., $i = 1$ or $j = i+1$), x_e is equal to the 2×2 minor $\Delta_{ij}(A)$ in the columns i and j.

(b) Use Plücker relations to deduce that, for any other triangulation T of the n-gon, there exists a matrix with the same property. In other words, show that there exists a $2\times n$ matrix A_T such that $\Delta_{ij}(A_T) = \tilde{x}_e$, for any edge or diagonal $e = (i, j)$ in T. Here $\tilde{x}_e$ are algebraically independent variables assigned to all edges and diagonals in T.

(c) Now specialize all $\tilde{x}_e$'s to 1. Prove that, for the integer frieze pattern $P(T) = (p_{ij})$ corresponding to the triangulation T (as in previous problem), we have $p_{ij} = \Delta_{ij}(A_T)$, for any $1 \le i < j \le n$.

Problem 5. (Diamond Lemma) Let G be a directed graph without infinite directed walks, i.e., any directed walk on G eventually arrives to a sink. Assume that, for any vertex v and two outgoing edges $v \to v'$ and $v \to v''$ in G, there exists a vertex v''' with two incoming edges $v' \to v'''$ and $v'' \to v'''$. Prove that any two directed walks on G that start at the same vertex will eventually arrive to the same sink.

Problem 6. (RSK correspondence and the octahedron recurrence) In class, we constructed the map RSK_n

$$\{\text{nonnegative } n \times n \text{ matrices}\} \to \{n \times n \text{ reverse plane partitions}\}$$

using the tropical octahedron recurrence and split chessboards. For example, for $n = 2$,

$$RSK_2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \min(b,c) & a+b \\ a+c & \max(b,c)+a+d \end{pmatrix}.$$

Show that this map RSK_n is the same as the one obtained via in the classical RSK insertion algorithm. (Here a pair (P, Q) of semistandard Young tableaux in the image of the classical RSK correspondence is identified (via Gelfand-Tsetlin patterns) with a reverse plane partition of square $n \times n$ shape.)

Problem 7. (Geometric RSK correspondence) Let $X = (x_{ij})$ be an $n \times n$ matrix whose entries are variables x_{ij}. For $i, j \in [n]$ and $k \in$

$[\min(i,j)]$, define

$$y_{i,j,k} := \left(\prod_{a\in[i],b\in[j],(a+b\le k \text{ or } a+b\ge i+j-k+2)} x_{ab} \right) \left(\sum_{P_1,\dots,P_k} wt(P_1)\cdots wt(P_k) \right),$$

where the sum is over all families of k non-crossing lattice paths $P_1,\dots,P_k$ on $\mathbb{Z}\times\mathbb{Z}$ (with steps $(1,0)$ and $(0,1)$) connecting the points $(1,k),(2,k-1),\dots,(k,1)$ with the points $(i-k+1,j),(i-k+2,j-1),\dots,(i,j-k+1)$; and the weight of a lattice path P is the product over its vertices $wt(P) := \prod_{(a,b)\in P} x_{ab}$.

Also let $y_{ij} := y_{i,j,n+1-\max(i,j)}$, and let $z_{ij} = y_{ij}/y_{i+1,j+1}$. (Assuming that $y_{ij} = 1$ if $i > n$ or $j > n$.)

Define the *geometric RSK correspondence* as the birational map

$$gRSK_n : (x_{ij})_{i,j\in n} \mapsto (z_{ij})_{i,j\in[n]}.$$

(a) The *tropicalization* $\mathrm{Trop}(gRSK_n)$ of this birational map is the piecewise-linear map $\{n\times n \text{ matrices}\} \to \{n\times n \text{ matrices}\}$ obtained from $gRSK_n$ by replacing the arithmetic operations: "$\times$" $\to$ "$+$", "$/$" $\to$ "$-$", "$+$" $\to$ "max". Show that $\mathrm{Trop}(gRSK_n)$ is exactly the classical RSK correspondence, i.e., the map RSK_n from the previous problem.

(b) Let $\tilde{y}_{i,j,k} := y_{i,j,k}/\prod_{a\le i+k-1,b\le j+k-1} x_{ab}$. Show that the array $(\tilde{y}_{i,j,k})$ satisfies the *octahedron recurrence:*

$$\tilde{y}_{i,j,k+1}\,\tilde{y}_{i,j,k} = \tilde{y}_{i,j-1,k+1}\,\tilde{y}_{i,j+1,k} + \tilde{y}_{i-1,j,k+1}\,\tilde{y}_{i+1,j,k}.$$

Problem 8. (Totally positve and totally nonnegative matrices) Let $Mat(m,n) \simeq \mathbb{R}^{mn}$ be the set of all real $m\times n$ matrices. Prove that the closure (in the usual topology on $\mathbb{R}^{mn}$) of the subset of totally positive matrices in $Mat(n,n)$ is exactly the set of totally nonnegative matrices. In other words, any totally nonnegative matrix can be approximated by a totally positive matrix.

Problem 9. (Minors of upper-triangular matrices) Prove that a generic upper-triangular unipotent $n\times n$ matrix U (i.e., U has 1's on the main diagonal) has exactly the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$ of different non-zero minors. For example, the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ has $C_2 = 2$ nonzero minors $1, x$; and the matrix $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ has $C_3 = 5$ nonzero minors $1, x, y, z, xz - y$. Find a bijection between these minors and some of the known combinatorial interpretations of C_n.

Problem 10. (LDU decomposition for totally positive matrices) Let A be an $n \times n$ matrix A with the LDU decomposition $A = LDU$, i.e., L is lower-triangular unipotent, D is diagonal, and U is upper-triangular unipotent. Show that A is a totally positive matrix if and only if each of the factors L, D, and U is a totally positive lower-triangular/diagonal/upper-triangular matrix. Here we say that L and U are totally positive unipotent lower/upper-triangular matrices if all of their C_n minors that are not identically zero are strictly positive. Likewise, we say that D is a totally positive diagonal matrix if all its diagonal entries are strictly positive.

Problem 11. (Moves of double wiring diagrams) Prove the relations of the form $EE' = AC+BD$ for the minors $\Delta_{I,J}$ of an $n\times n$ matrix that correspond to all moves of Fomin-Zelevinsky's *double wiring diagrams.* More explicitly, prove the following relations for minors.

Let $I, J \subset [n]$ such that $|I| = |J|$, let $i', i'', i''' \in [n] \setminus I$, and let $j', j'', j''' \in [n] \setminus J$.

(a) $$\Delta_{\{i''\}\cup I,\{j'\}\cup J}\,\Delta_{\{i',i'''\}\cup I,\{j',j''\}\cup J} = \Delta_{\{i'\}\cup I,\{j'\}\cup J}\,\Delta_{\{i'',i'''\}\cup I,\{j',j''\}\cup J} + \Delta_{\{i',i''\}\cup I,\{j',j''\}\cup J}\,\Delta_{\{i'''\}\cup I,\{j'\}\cup J}.$$

(This relation corresponds a Coxeter move of 3 strands of the same color in a double wiring diagram.)

(b) *Lewis Carrol identity* aka *Dodgson condensation:*

$$\Delta_{\{i''\}\cup I,\{j'\}\cup J}\,\Delta_{\{i'\}\cup I,\{j''\}\cup J} = \Delta_{I,J}\,\Delta_{\{i',i''\}\cup I,\{j',j''\}\cup J} + \Delta_{\{i'\}\cup I,\{j'\}\cup J}\,\Delta_{\{i''\}\cup I,\{j''\}\cup J}\,.$$

(This relation corresponds to commuting a "black crossing" through a "red crossing" in a double wiring diagram.)

Problem 12. (Positivity test via solid minors) A minor $\Delta_{I,J}(A)$ of an $n \times n$ matrix A is called *solid* if I and J are consecutive intervals in $[n]$ and $1 \in I \cup J$. Show that the collection of solid minors $\Delta_{I,J}$ is a *positivity test,* that is, if all solid minors of A are strictly positive then all minors of A are strictly positive.

Problem 13. Find an example of a positivity test for $n \times n$ matrices that does not come from a double wiring diagram. What is the smallest n for which such an example exists?

Problem 14. (Bruhat decomposition) Let $G = GL_n$ (over some field $\mathbb{F}$), let $B \subset G$ be the Borel subgroup of invertible upper-triangular matrices, and let $W = S_n \subset G$ be subgroup of permutation matrices.

Prove that G has the following disjoint decomposition

$$G = \bigcup_{w \in W} BwB.$$

Problem 15. (Lusztig's transformations) (a) Find the "birational subtraction-free bijection" $(x, y, z) \mapsto (x', y', z')$ from $\mathbb{R}^3_{>0}$ to itself such that

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Find the bijective map $(x, y) \mapsto (\tilde{x}, \tilde{y})$ from $\mathbb{R}^2_{>0}$ to $\mathbb{R}^2_{>0}$ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tilde{y} & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

for some $t_1, t_2 \in \mathbb{R}_{>0}$.

(c) For a reduced double wiring diagram D, the corresponding *double Bruhat cell* in $GL_n^{TNN}(\mathbb{R})$ is the subset of matrices that can be factorized as

$$X_{i_1}(t_1) \cdots X_{i_l}(t_l)\, \mathrm{diag}(z_1, \dots, z_n),$$

where $t_1, \dots, t_l, z_1, \dots, z_n \in \mathbb{R}_{>0}$ and each factor $X_i(t)$ has the form $I + t\, E_{i,i+1}$ or $I + t\, E_{i+1,i}$. (The sequence of these factors is given by black and red crossings in the double wiring diagram D.) Show that this double Bruhat cell depends only on the pair of permutations associated with D (and not on a choice of a double wiring diagram).

Problem 16. ("Inverse Lindström Lemma") Show that, for any totally nonnegative $m \times n$ matrix A (not necessarily square), one can find a planar directed acyclic graph G with positive weights on the edges, that can be drawn on the plane so that all its sources $A_1, \dots, A_m$ are on the left and all its sinks $B_1, \dots, B_n$ are on the right (both sources and sinks are ordered from bottom to top), such that

$$a_{ij} = \sum_{P : A_i \to B_j} weight(P), \qquad \text{for any } i, j,$$

where the sum is over directed paths P in G from the source A_i to the sink B_j, and the weight $weight(P)$ of a path P is the product of weights of all its edges.

Problem 17. Prove that a point $(p_I)_{I \in \binom{[n]}{k}} \in \mathbb{R}_{\geq n}^{\binom{[n]}{k}}$ represents a point of the positive Grassmannian $Gr^{>0}(k, n)$ (i.e., the p_I are the Pücker coordinates of a point in $Gr^{>0}(k, n)$) if and only if the p_I satisfy the *3-term* Plücker relations.

Problem 18. Prove that a matroid M is a positroid if and only if any minor of M of rank 2 on 4 elements is a positroid.

Problem 19. (a) Calculate the number of d-dimensional cells in the totally nonnegative Grassmannian $Gr_{\geq 0}(2, n)$.

(b) The same question for $Gr_{\geq 0}(3, 6)$.

Problem 20. (Strong Bruhat order) Prove the equivalence of the 3 definitions of the strong Bruhat order on S_n:

(1) Covering relations: $u < w$ iff $w = u \cdot (i, j)$ and $\ell(w) = \ell(u) + 1$.

(2) $u \leq w$ if *any* reduced decomposition of w has a subword which is a reduced decomposition of u.

(3) $u \leq w$ if *some* reduced decomposition of w has a subword which is a reduced decomposition of u.

Problem 21. (Stembridge's formula) Assign the weight $(j - i)$ to a covering relation $w < w \cdot (i, j)$, where $i < j$, in the strong Bruhat order on S_n. The weight of a saturated chain from the minimal element id to the maximal element w_0 is the product of weights of all covering relations in the chain. Prove that the weighted sum over all saturated chains from id to w_0 equals $\binom{n}{2}!$.

Problem 22. (Circular Bruhat order) In class, we defined the circular Bruhat order on decorated permutations. Show that it is a ranked poset with the corank function given by the number of alignments of a decorated permutation.

Problem 23. (Skew-symmetrizable matrices) Prove the equivalence of the following 2 definitions:

(1) An $n \times n$ matrix B is *skew-symmetrizable* if there exists a diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ with $d_i > 0$ such that DB is skew-symmetric.

(2) An $n \times n$ matrix B is *skew-symmetrizable* if there exists a diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ with $d_i > 0$ such that BD is skew-symmetric.

Problem 24. (Cluster algebras of rank 2) Show that a cluster algebra of rank 2 with exchange matrix $B = \begin{pmatrix} 0 & b \\ -c & c \end{pmatrix}$ is of finite type if and only if $|b \cdot c| \leq 3$.

Problem 25. The diagram $D(B)$ of an exchange matrix B is the directed graph with edges $i \to j$ for $b_{ij} > 0$ weighted by positive integers $|b_{ij} b_{ji}|$. Show that

(a) If $D(B)$ is the cyclically oriented n-cycle with weights $1, \dots, 1$, then B mutation equivalent to an exchange matrix whose Cartan companion is the type D_n Cartan matrix.

(b) If $D(B)$ is the cylically oriented 3-cycle with edge weights 2, 2, 1, then B is mutation equivalent to an exchange matrix whose Cartan companion is the type B_3 Cartan matrix.

(b) If $D(B)$ is the cylically oriented 4-cycle with edge weights 2, 1, 2, 1, then B is mutation equivalent to an exchange matrix whose Cartan companion is the type F_4 Cartan matrix.

Problem 26. (a) Show that the quiver given by any orientation of the n-cycle, except the 2 cylic orientations, is not 2-finite. Thus it is of infinite type.

(b) More generally, if B is any exchange matrix whose diagram is an n-cycle (with some edge orientations and some weights), then B is of infinite type in all cases except the 3 cases from the previous problem.

Problem 27. (Associahedron) Let $R = \{e_i - e_j \mid i, j \in [n], i \neq j\} \subset V = \{x_1 + \cdots + x_n = 0\} \simeq \mathbb{R}^{n-1}$ be a type A_{n-1} root system. Let $R_{\geq -1} = \{e_i - e_j \mid i < j \text{ or } j = i - 1\} \subset R$ be almost positive roots.

Describe all functions $F : \{\text{almost positive roots}\} \to \mathbb{R}$ such that the polytope

$$P_F := \{x \in V \mid (x, \alpha) \leq F(\alpha) \text{ for any } \alpha \in R_{\geq -1}\}$$

is combinatorially equivalent to the associahedron (i.e., P_F has the same normal fan as the associahedron).

Problem 28. ("Generalized" generalized associahedra) The same problem for any root system and the Chapoton-Fomin-Zelevinsky's generalized associahedron.

Problem 29. (Cyclohedron) Prove the equivalence of the following 3 descriptions of combinatorial structure of the cyclohedron:

(1) (symmetric triangulations) The description in terms of centrally-symmetric triangualations of the $(2r + 2)$-gon

(2) (graph-associahedron) The description as the nested set complex $\mathcal{N}_G$ for the graph G equal to the $(r + 1)$-cycle.

(3) (CFZ-associahedron) The description in terms of pairwise compatible collections of almost positive roots for a type B_r root system.

Problem 30. (Type C cluster algebras) In class, we mentioned that type B cluster algebra can be constructed by folding from a type A cluster algebra. Do a similar construction for a type C cluster algebra.

Problem 31. (G-Catalan numbers) For a graph G, we defined the G-Catalan number as the number of vertices of the corresponding graph-associahedron P_G, i.e., the number of maximal nested sets in the nested set complex $\mathcal{N}_G$. Calculate the G-Catalan number when G is the Dynkin diagram of type D_r.

Problem 32. (Haiman's model for Coxeter-Catalan numbers in type A) Let $Q = \{(a_1, \ldots, a_{r+1}) \in \mathbb{Z}^{r+1} \mid a_1 + \cdots + a_{r+1} = 0\}$ (the root lattice), and $h = r+1$ (the Coxeter number). The Weyl group $W = S_{r+1}$ acts on Q by permutations of the coordinates. Show that the number of W-orbits in $Q/(h+1)Q$ equals the Catalan number $\frac{1}{r+2}\binom{2r+2}{r+1}$.

Problem 33. Show that, for type B_r, the number of W-orbits in $Q/(h+1)Q$ equals the central binomial coefficient $\binom{2r}{r}$.

Problem 34. Calculate γ-vectors for CFZ generalized associahedra.