Problem 1. A tree $T$ on vertices labelled $1, 2, 3, \ldots$ is called alternating if it has no pair of edges $(a, b)$ and $(b, c)$ with $a < b < c$. A tree $T$ is called non-crossing if it has no pair of edges $(a, c)$ and $(b, d)$ with $a < b < c < d$. A tree $T$ is called non-nesting if it has no pair of edges $(a, d)$ and $(b, c)$ with $a < b < c < d$.

Prove bijectively that the number of non-crossing alternating trees on $n+1$ vertices equals the number of non-nesting alternating trees on $n+1$ vertices equals the Catalan number $C_n$.

Problem 2. Prove that the number of alternating trees on $n+1$ vertices equals the number of binary trees on $n$ vertices labelled by $1, \ldots, n$ such that the left child of a vertex in always greater than its parent and the right child of a vertex is always less than its parent.

Problem 3. Find a formula for the number of alternating trees on $n$ vertices. (Your formula might involve a single summation.)

Problem 4. Find a formula for the number of non-crossing trees on $n$ vertices.

Problem 5. Calculate the value $\mu_{NC_n}(\hat{0}, \hat{1})$ of the Möbius function for the lattice $NC_n$ of non-crossing partitions.

Problem 6. Prove transitivity of the Hurwitz action for the symmetric group $S_n$. In other words, show that any two factorizations $t_1 t_2 \cdots t_{n-1}$ and $t'_1 t'_2 \cdots t'_{n-1}$ of the long cycle $c = (1, 2, \ldots, n) \in S_n$ into products of $n-1$ transpositions can be obtained from each other by a sequence of Hurwitz moves $\sigma_i$, $i = 1, \ldots, n-2$:

$$\sigma_i : \cdots t_{i-1} t_i t_{i+1} t_{i+2} \cdots \rightarrow \cdots t_{i-1} t_{i+1} (t_{i+1}^{-1} t_i t_{i+1}) t_{i+2} \cdots$$

Problem 7. For the Hurwitz moves $\sigma_i$ acting on factorizations of the long cycle $c \in S_n$ into products of $n-1$ transpositions (as in the previous problem) show that $(\sigma_1 \sigma_2 \cdots \sigma_{n-2})^{n(n-1)}$ is the identity operator.
Problem 8. Prove that the number of $m$-tuples $(t_1, \ldots, t_m)$ of transpositions $t_i \in S_n$ such that

(a) $t_1 t_2 \cdots t_m = 1 \in S_n$,
(b) $t_1, \ldots, t_m$ generate the symmetric group $S_n$, and
(c) $m = 2n - 2$

equals $n^{n-3}(2n - 2)!$.

Problem 9. In class, we showed that the lattice of non-crossing partitions $NC_n$ is isomorphic to the interval $[1, c]_{abs}$ between the identity permutation 1 and the long cycle $c = (1, 2, \ldots, n)$ in the absolute reflection order on the symmetric group $S_n$. Thus the Kreweras complementation map $K : NC_n \to NC_n$ induces a map on saturated chains in the poset $[1, c]_{abs}$. These saturated chains correspond to factorizations $t_1 t_2 \cdots t_{n-1}$ of $c$ in products on $n - 1$ reflections.

Show that the map $t_1 t_2 \cdots t_{n-1} \to t'_1 t'_2 \cdots t'_{n-1}$ acting on factorizations of $c$ obtained from the Kreweras complementation can be described as follows:

$t'_1 = t_{n-1}$,
$t'_2 = t_{n-1}^{-1} t_{n-2} t_{n-1}$,
$t'_3 = t_{n-1}^{-1} t_{n-2}^{-1} t_{n-3} t_{n-2} t_{n-1}$, etc.

Problem 10. Let $\mathcal{A}$ be the affine hyperplane arrangement in the space $\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0 \} \simeq \mathbb{R}^{n-1}$ with $\binom{n}{2}$ affine hyperplanes $H_{ij}$, $1 \leq i < j \leq n$, given by the equations

$x_i - x_j = a_{ij}$,

where $a_{ij}$ are some fixed generic real numbers. Show that

(a) The number of vertices of the arrangement $\mathcal{A}$ (i.e., the number of 0-dimensional intersections of some hyperplanes $H_{ij}$) equals the number $n^{n-2}$ of trees on $n$ labelled vertices.

(b) The number of regions of the arrangement $\mathcal{A}$ equals the number of forests on $n$ labelled vertices.

Problem 11. Find an explicit formula for the number of regions of the hyperplane arrangement in $\mathbb{R}^n$ with $5 \binom{n}{2}$ hyperplanes given by the equations

$x_i - x_j = -2, -1, 0, 1, 2$,

for $1 \leq i < j \leq n$. 
Problem 12. Find an explicit formula for the number of regions of the hyperplane arrangement in $\mathbb{R}^n$ with $4\binom{n}{2}$ hyperplanes given by the equations

$$x_i - x_j = -1, 0, 1, 2,$$

for $1 \leq i < j \leq n$.

Problem 13. For a finite matroid $M$ without loops, let $L(M)$ be the lattice of flats of $M$. Prove that

(a) $L(M)$ is a geometric lattice, and
(b) any geometric lattice $L$ is isomorphic to $L(M)$ for some $M$.

Problem 14. Let $M = (M, \mathcal{B})$ be a matroid, where $E$ is the ground set of $M$, and $\mathcal{B}$ is the set of bases of $M$. Let $M^* = (E, \mathcal{B}^*)$, where $\mathcal{B}^* := \{ E \setminus I \in \mathcal{B} \} \subset 2^E$. Prove that $M^*$ is a matroid. In other words, show that the Exchange Condition for the set of bases $\mathcal{B}$ is equivalent to the Exchange Condition for the set of bases $\mathcal{B}^*$.

Problem 15. Let $A_{m,n}$ be the number of acyclic orientations of the complete bipartite graph $K_{m,n}$. Prove that

(a) $A_{m,n}$ equals the number of $m \times n$ matrices filled with 0’s and 1’s such that no $2 \times 2$ submatrix equals \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) or \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

(b) $A_{m,n}$ equals the number of $m \times n$ matrices filled with 0’s and 1’s such that no $2 \times 2$ submatrix equals \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) or \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \).

(A “$2 \times 2$ submatrix” means a submatrix located in any 2 rows and any 2 columns of a matrix, not necessarily consecutive rows/columns.)

Problem 16. Find an explicit formula for the number $A_{m,n}$ of acyclic orientations of $K_{m,n}$. (Your answer that might involve a single summation.)

Problem 17. An increasing forest is a forest with vertices labelled by $1, 2, 3, \ldots$ that contains no pair of edges $(a, c)$ and $(b, c)$ with $a < b < c$.

Prove bijectively that the number of increasing forests on $n$ labelled vertices with $k$ edges equals the Stirling number of the first kind $s(n, n-k)$, i.e., the number of permutations in $S_n$ with $n-k$ cycles.
Problem 18. Construct a linear basis of the Orlik-Solomon algebra for the Catalan hyperplane arrangement. Describe a bijection between elements of your basis and some set of combinatorial objects of cardinality $n! C_n$.

Problem 19. In class, we discussed the following map $\phi$ from the set of Young diagrams $\lambda$ that fit inside the staircase shape $(n - 1, n - 2, \ldots, 1)$ and certain posets $P$ on $n$ labelled vertices $1, \ldots, n$. (Clearly, such Young diagrams $\lambda$ correspond to Dyck paths with $2n$ steps.) The poset $P = \phi(\lambda)$ is given by $i < P j$ if and only if the box $(i, n + 1 - j)$ belongs to $\lambda$.

Prove that the map $\phi$ induces a bijection between Dyck paths with $2n$ steps and all unlabelled semiorders on $n$ vertices.

Problem 20. Let $v_1, \ldots, v_N$ be a collection of vectors that span a vector space $V \simeq \mathbb{R}^d$, and let $\Lambda \subset V$, $\Lambda \simeq \mathbb{Z}^d$, be the $\mathbb{Z}$-span of these vectors $v_i$ (i.e., the set of their integer linear combinations). We say that a collection of vectors $v_1, \ldots, v_N$ is unimodular if, whenever a subset of these vectors forms a linear basis of $V$, the $\mathbb{Z}$-span of this subset equals $\Lambda$. For example, the collection of vectors $(1, 0), (0, 1), (1, 1) \in \mathbb{Z}^2 \subset \mathbb{R}^2$ is unimodular, but the collection of vectors $(1, 0), (0, 1), (1, 2)$ is not unimodular because the $\mathbb{Z}$-span of $(1, 0)$ and $(1, 2)$ is not $\mathbb{Z}^2$.

In class, we discussed the graphical arrangement $A_G$ and the co-graphical arrangement $A_G^*$ associated with a graph $G$. Show that, for each of these arrangements, one can pick normal vectors to the hyper-planes so that they form a unimodular collection of vectors.

Problem 21. For $n \geq 4$, the wheel graph $W_n$ is the simple graph on $n$ vertices obtained from the $(n - 1)$-cycle graph $C_{n-1}$ by adding one extra vertex connected to all vertices of $C_{n-1}$. Show that the number of acyclic orientations of the wheel graph $W_n$ equals the number of totally cyclic orientations of $W_n$.

Problem 22. Show that the evaluation $T_{K_{n+1}}(1, -1)$ of the Tutte polynomial for the complete graph $K_{n+1}$ equals the number of alternating permutations in $S_n$. (Recall that a permutation $w \in S_n$ is alternating if $w_1 < w_2 > w_3 < w_4 > \cdots$.)