18.217 PROBLEM SET 2 (due Friday, December 6, 2024)

Problem 1. Fix two positive integers n and k. The *extended Catalan arrangement* is the arrangement of affine hyperplanes in $\mathbb{R}^n$ given by $x_i - x_j = c$, for all $1 \le i < j \le n$ and $c \in \{-k, -k+1, \ldots, k-1, k\}$.

Find an explicit formula for the number of regions of the extended Catalan arrangement.

(You can use the finite field method, or any other method to calculate the number of regions.)

Problem 2. Fix two positive integers n and k. The *extended Shi arrangement* is the arrangement of affine hyperplanes in $\mathbb{R}^n$ given by $x_i - x_j = c$, for all $1 \le i < j \le n$ and $c \in \{-k+1, \ldots, k-1, k\}$.

Find explicit formulas for the number of regions and the number of bounded regions of the extended Shi arrangement.

Problem 3. In class, we defined alternating trees as labelled trees that don't contain an increasing 2-path i–j–k with $i < j < k$. We also defined local-binary-search trees as binary trees with labelled vertices such that a left child is less than its parent and a right child is greater that its parent. Show that the number of alternating trees on $n+1$ vertices equals the number of local binary search trees on n vertices. For example, for $n = 2$, both of these numbers are 2.

Problem 4. Show that the number of alternating trees on $n+1$ vertices (see the previous problem) equals

$$2^{-n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}.$$

Problem 5. Fix n. The *Linial arrangement* is the arrangement of hyperplanes in $\mathbb{R}^n$ given $x_i - x_j = 1$, for all $1 \le i < j \le n$. Show that the number of regions of the Linial arrangement equals the number of alternating trees on $n+1$ vertices. (See the previous two problems.)

Problem 6. The Arnold-Orlik-Solomon algebra A_n is the algebra over $\mathbb{R}$ generated by anti-commutative generators $e_{ij} = e_{ji}$, for $i \neq j \in [n]$, satisfying the relations:

$$e_{ij}e_{jk} = e_{ij}e_{ik} + e_{ik}e_{jk}, \text{ for any } i < j < k.$$

For a graph G on vertex set $[n]$, let e_G be the monomial in e_{ij}'s defined (up to a sign) by

$$e_G := \prod_{(ij) \text{ is an edge of } G} e_{ij}$$

Let us say that a chain C with vertices labelled by integers is *rooted* if its vertex with the minimal label is one of its two ends. (We also view a single vertex as a rooted chain.)

Show that the collection of monomials e_G, where G ranges over all graphs $G \subset K_n$ such that each connected component G is a rooted chain, forms a linear basis of the algebra A_n.

Problem 7. Prove the following identity, which is known in physics as the *Kleiss-Kuijf relation.*

The *Parke-Taylor factor* $PT(x_1, \ldots, x_n)$ is the following rational expression in variables $x_1, \ldots, x_n$:

$$PT(x_1, \ldots, x_n) := \prod_{i=1}^{n} \frac{1}{x_i - x_{i+1}},$$

where we assume that $x_{n+1} = x_1$. Note that this expression depends only on a cyclic ordering of the variables.

Let $x, y, z_1, \ldots, z_k$, $t_1, \ldots t_l$ be some variables. We have

$$PT(x, z_1, \ldots, z_k, y, t_1, \ldots, t_l) = (-1)^l \sum_{\text{Ш}} PT(x, \text{Ш}, y),$$

where the sum is over all shuffles Ш of two lists of variables $z_1, \ldots, z_k$ and $t_l, t_{l-1}, \ldots, t_1$ (with the reversed ordering of t_i's). A *shuffle* Ш of two ordered lists is their permutation that preserves the ordering of each list.

For example, for $k = 1$ and $l = 2$, we have $PT(x, z_1, y, t_1, t_2) =$

$$= PT(x, z_1, t_2, t_1, y) + PT(x, t_2, z_1, t_1, y) + PT(x, t_2, t_1, z_1, y).$$

Problem 8. In class, we discussed the following game on collections of graphs G on the vertex set $[n]$. If we can find a pair of edges e_1 and e_2 in G such that $e_1 = (i, j)$ and $e_2 = (j, k)$, where $i < j < k$, then we can replace G by a pair of graphs G_1 and G_2 such that G_1 is obtained from G by replacing the edge e_1 with the edge $e_3 = (i, k)$ and G_2 is obtained from G by replacing the edge e_2 with the edge e_3. Then we apply similar moves to each G_1 and G_2, etc.

We start with a single initial graph and keep playing this game until we obtain a collection of graphs for each of which no more moves are possible. Note that there might be many different ways to play this game, because, at each step, there might be several possible choices for a pair of edges e_1 and e_2.

(1) Show that this game always terminates after finitely many steps.

(2) Show that the number of graphs in a final collection depends only on the initial graph and does not depend on a way to play the game.

(3) Show that, if the initial graph G in the n-chain $G = 1–2–\cdots–n$, then the number of graphs in any final collection equals the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Problem 9. For a partition $\lambda = (\lambda_1 \geq \cdots \lambda_n \geq 0)$, define two symmetric polynomials in n variables. The *Schur polynomial* is defined as

$$s_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n} \left(\frac{x_1^{\lambda_1} \cdots x_n^{\lambda_n}}{\prod_{1 \leq i < j \leq n} (1 - x_j/x_i)} \right).$$

Also define

$$b_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n} \left(\frac{x_1^{\lambda_1} \cdots x_n^{\lambda_n}}{\prod_{i=1}^{n-1} (1 - x_{i+1}/x_i)} \right).$$

The Schur polynomial $s_\lambda(x_1, \dots, x_n)$ is the sum of $K(\lambda, \beta) x_1^{\beta_1} \cdots x_n^{\beta_n}$ over all lattice points $(\beta_1, \dots, \beta_n)$ of the permutohedron $P(\lambda_1, \dots, \lambda_n)$, and $K(\lambda, \beta)$ is a certain integer called the *Kostka number*. On the other hand, according to Brion's formula, $b_\lambda(x_1, \dots, x_n)$ is the sum of monomials $x_1^{\beta_1} \cdots x_n^{\beta_n}$ over the same set of lattice points $(\beta_1, \dots, \beta_n)$.

Prove that $\{s_\lambda\}$ and $\{b_\lambda\}$ are two linear bases of the same linear space.

Problem 10. Show that the Eulerian number $A_{n,k}$, i.e., the number of permutations of size n with exactly k descents, is given by the following formula:

$$A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n.$$

Problem 11. Fix positive integers $k \le n$. Let M be a nonempty subset of $\binom{[n]}{k}$. Show that the following two conditions are equivalent:

(Exch) For any $I, J \in M$ and any $i \in I$, there exists $j \in J$ such that $(I \setminus \{i\}) \cup \{j\} \in M$.

(StrngExch) For any $I, J \in M$ and any $i \in I$, there exists $j \in J$ such that $(I \setminus \{i\}) \cup \{j\} \in M$ and $(J \setminus \{j\}) \cup \{i\} \in M$.

Problem 12. In this problem we consider simple graphs $G = (V, E)$ on the vertex set $V = [n]$ with vertices arranged on a circle in the clockwise order and edges represented by chords of the circle. We say that two edges of G *intersect* if they have a common vertex or the chords intersect inside the circle. For example, the edges $(1, 3)$ and $(2, 4)$ intersect; but the edges $(1, 4)$ and $(2, 3)$ don't intersect.

Such a graph G is called a *thrackle*[1] if any two edges of G intersect.

(1) Show that the maximal possible number of edges in a thrackle on n vertices is n.

(2) Prove that the number of thrackles on n vertices with n edges equals the Eulerian number $A_{n-1,1}$, i.e., the number of permutations of size n with exactly 1 descent.

(3) Give an explicit closed-form formula for the number of thrackles on n vertices with n edges.

Problem 13. For a thrackle G on n vertices with n edges (as in the previous problem), let S_G be the convex hull of n points $e_i + e_j$ for every edge (i, j) of G. (Here $e_1, \dots, e_n$ are the standard coordinate vectors of $\mathbb{R}^n$.) Show that

(1) All polytopes S_G are $(n-1)$-dimensional simplices.

(2) For any two thrackles G and G' with n vertices and n edges, the interiors of polytopes S_G and $S_{G'}$ don't intersect.

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[1] A side note: This is a special case of a more general class of Conway's thrackles.

(3) The union of polytopes S_G for all thrackles on n vertices with n edges is the second hypersimplex $\Delta_{n,2}$.

Problem 14. For a simple graph G on the vertex set $[n]$, define the *root polytope* $R_G \subset \{x_1 + \cdots + x_n = 0\} \subset \mathbb{R}^n$ as the convex hull of the origin and points $e_i - e_j$ for all edges (i, j), $i < j$, of G.

Show that the collection of root polytopes R_T, where T ranges over all non-crossing alternating trees on $[n]$, forms a triangulation of the root polytope $R_n := R_{K_n}$.

A non-crossing alternating tree is an alternating tree T (defined in Problem 3) such that any two edges of T with all distinct vertices do not intersect (defined in Problem 12).

Problem 15. Let $G = (V, E)$ be a simple connected graph on the vertex set $V = [n]$. We say that a non-empty subset $I \subseteq [n]$ is G-connected if the induced graph $G|_I$ is connected. The *graph-associahedron* A_G is defined as the Minkowski sum of simplices:

$$A_G := \sum_{I \text{ is } G\text{-connected}} \operatorname{conv}\{e_i \mid i \in I\}.$$

(1) Show that the number of vertices $N(G)$ of the graph-associahedron A_G satisfies the following recurrence relation:

$$N(G) = \sum_{i \in [n]} N(G_1) \cdots N(G_{m(i)}),$$

where $G_1, \ldots, G_{m(i)}$ are the connected components of the graph obtained from G be removing vertex i, i.e., the connected components of the induced graph $G|_{[n]\setminus\{i\}}$.

(2) Find explicit formulas for numbers $N(G)$ of vertices of graph-associahedra for the following graphs on n vertices: the complete graph K_n, the n-chain 1–2–$\cdots$–n, the n-cycle, the $(n-1)$-star. (Here the $(n-1)$-*star* is the graph on the vertex set $[n]$ with edges $(1, i)$ for all $i = 2, \ldots, n$.)