LECTURE 28 Wed 11/13

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{R}^n$

<u>permutohedron</u>: $P(\lambda) = \mathrm{conv}(w(\lambda) \mid w \in S_n)$

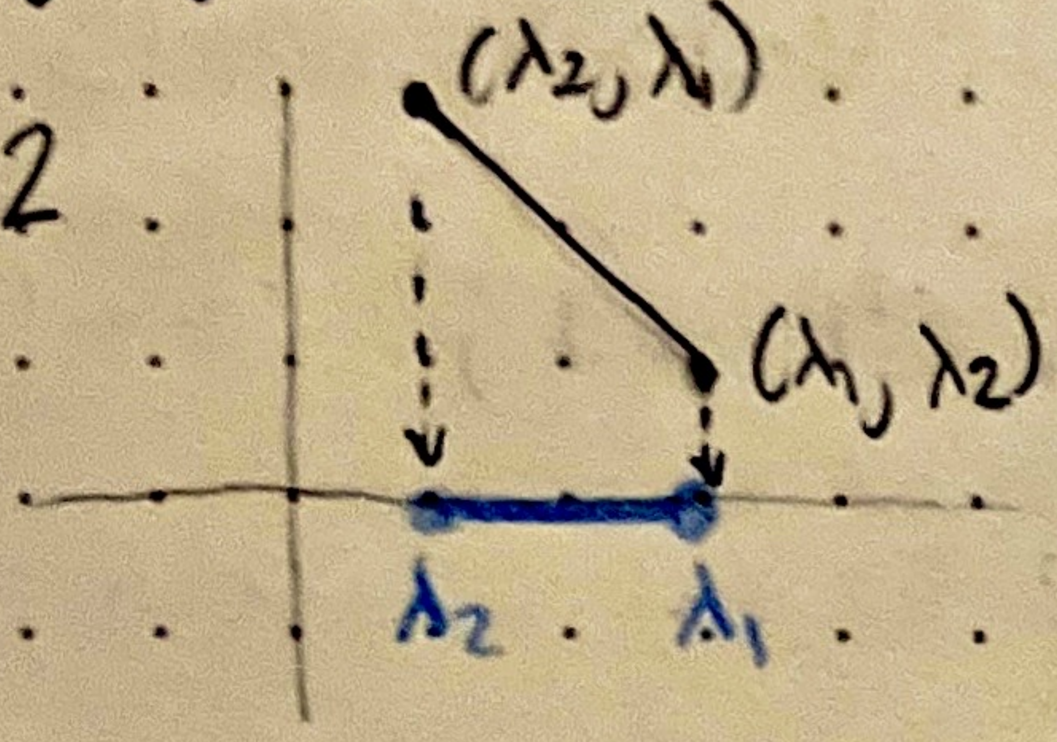
<u>Thrm 1</u>: $\mathrm{Vol}\, P(\lambda) = \frac{1}{(n-1)!} \sum_{w \in S_n} \frac{(\lambda_1 h_{w(1)} + \cdots + \lambda_n h_{w(n)})}{\prod_{i=1}^{n-1} (h_{w(i)} - h_{w(i+1)})}$

(n-1) dim'l volume

permutations act on h_i's but not λ_i's

Here $h_1, h_2, \ldots, h_n$ are some arbitrary distinct numbers. Pick any you like.

<u>Ex.</u> $n=2$



$\mathrm{Vol}(P_\lambda) = \lambda_1 - \lambda_2$ (volume of projection)

$= \frac{1}{1!}\left(\frac{(\lambda_1 h_1 + \lambda_2 h_2)}{h_1 - h_2} + \frac{(\lambda_1 h_2 + \lambda_2 h_1)}{h_2 - h_1} \right)$

$= \frac{\lambda_1 (h_1 - h_2) - \lambda_2 (h_1 - h_2)}{(h_1 - h_2)} = \lambda_1 - \lambda_2$

<u>Note</u>: All the h_i's cancel!

Can we get a more explicit formula?

- std permutohedron: $\lambda = (n, n-1, \ldots, 1)$ $\mathrm{Vol} = n^{n-2}$
- The hypersymplex $\Delta_{n,k}$ $\lambda = (\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k})$ $\mathrm{Vol} = \frac{1}{(n-1)!} A_{n-1,k-1}$

$A_{n-1,k-1}$ the Eulerian number $= \#\{w \in S_{n-1} \mid \mathrm{des}(w) = k-1\}$

$\mathrm{des}(w) = \#\{i \in w \mid w_i > w_{i+1}\}$

$\mathrm{Vol}\, P(\lambda)$ is a homogeneous polynomial in $\lambda_1, \lambda_2, \ldots, \lambda_n$ of degree $n-1$ (in the region $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$).

How to find the coeff. of some monomial $\lambda_1^{c_1} \lambda_2^{c_2} \cdots \lambda_n^{c_n}$ where $c_i \geq 0$, $c_1 + \cdots + c_n = n-1$?

: Starting with $(c_1, \ldots, c_n)$, replace

$c_i \rightsquigarrow (\underbrace{1, 1, \ldots, 1}_{c_i}, -1)$ $\left(\begin{array}{l} 1 \text{ is upstep} \\ -1 \text{ is downstep} \end{array} \right)$

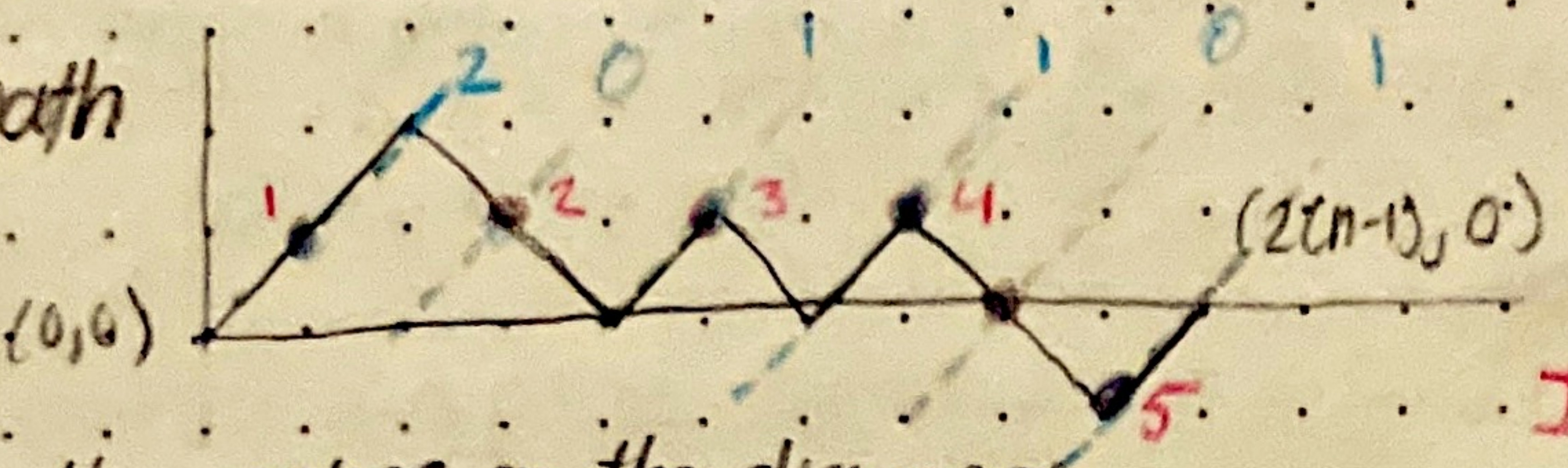
- After doing all the replacements, remove the last -1.

Ex. $n=6$ Want $\lambda_1^2\lambda_3\lambda_4\lambda_6$ coeff.

$(c_1, c_2, \dots, c_6) = (2, 0, 1, 1, 0, 1)$

$\rightsquigarrow (1, 1, -1, -1, 1, -1, 1, -1, -1, 1, \cancel{-1})$

Lattice path



$I = \{5\}$

c_i's are the upsteps on the diagonal

Delete $I = I(c_1, \dots, c_n) = \{i \mid p(2i+1) < 0\} \subset [n-1]$

red pts. falling below 0-line

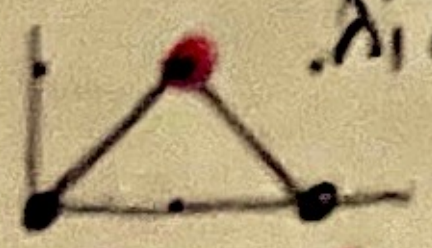
Denote $D_n(I) := \#\{w \in S_n \mid w_i > w_{i+1} \text{ iff } i \in I\}$

where $I \subseteq [n-1]$.

Thrm 2: $\operatorname{Vol} P(\lambda_1, \dots, \lambda_n)$

$$= \sum_{\substack{c_1, \dots, c_n \geq 0 \\ c_1 + \dots + c_n = n-1}} (-1)^{|I(c_1, \dots, c_n)|} D_n(I(c_1, \dots, c_n)) \frac{\lambda_1^{c_1}}{c_1!} \cdots \frac{\lambda_n^{c_n}}{c_n!}$$

Ex. $n=2$



$\lambda_1^1\lambda_2^0$ $\boxed{\lambda_1}$

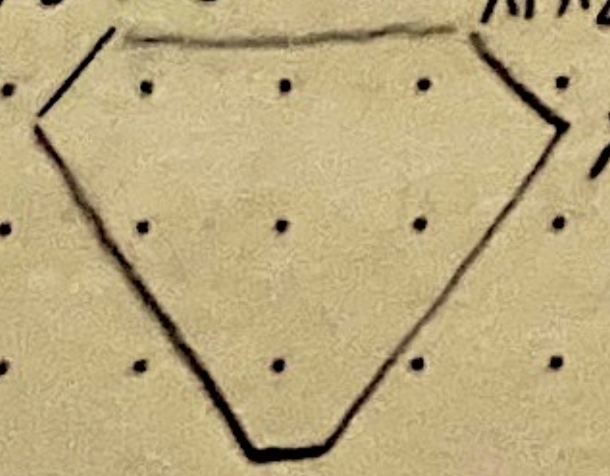
$I = \emptyset$

$D_2(I) = \#\{12\}$.

$\lambda_1^0\lambda_2^1$ $\boxed{-\lambda_2}$

$I = \{1\}$ $D_2(I) = \#\{21\}$

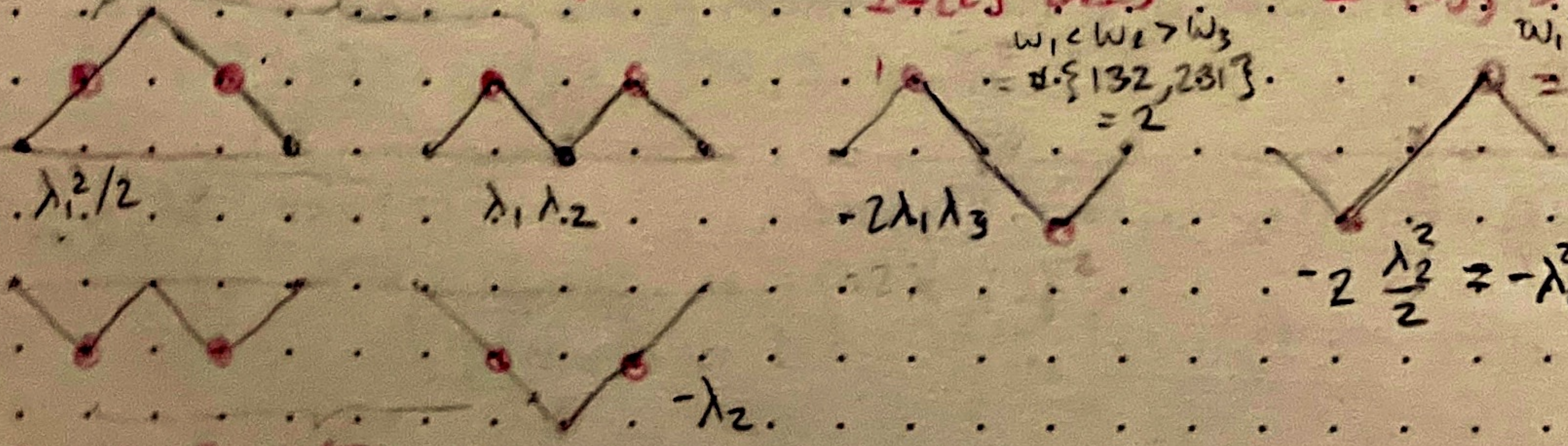
$n=3$ $P(\lambda_1, \lambda_2, \lambda_3) =$



$I = \emptyset$ $D_3(I) = \#\{123\}$

$I = \{2\}$ $D(I)$ $w_1 < w_2 > w_3$ $= \#\{132, 231\} = 2$

$I = \{1\}$, $D(I) = 2$ $w_1 > w_2 < w_3$ $= \#\{213, 312\}$



$\lambda_1^2/2$ $\lambda_1\lambda_2$ $-2\lambda_1\lambda_3$ $-2\frac{\lambda_2^2}{2} = -\lambda_2^2$

λ_1 $-\lambda_2$

$I = \{1, 2\}$ $D(I) = \#\{321\} = 1$

$f(h_1, \ldots, h_n)$ a polynomial

divided symmetrization $\langle f \rangle := \sum_{w \in S_n} w\left(\frac{f(h_1, \ldots, h_n)}{\prod_{i=1}^{n-1} (h_i - h_{i+1})} \right)$

__Lemma 1__: $\langle f \rangle$ is a symmetric poly in $h_1, \ldots, h_n$ of degree $= \deg(f) - (n-1)$

In particular, if $\deg f = n-1$, then $\langle f \rangle$ is a constant

__Proof__: We know that it's symmetric b/c it's a symmetrization

degree also follows from dividing degree of top by degree of bottom.

Not obvious that it is a polynomial! Here's why!

Expand to get $\langle f \rangle = \frac{g(h_1, h_2, \ldots, h_n)}{\prod_{1 \le i \le j \le n} (h_i - h_j)}$ $\leftarrow$ for some polynomial g in $h_1, h_2, \ldots, h_n$

Since $\langle f \rangle$ is symmetric, and denominator anti-symmetric, g must be anti-symmetric

$\Rightarrow g(h_1, \ldots, h_n) = 0$ if we plug in some $h_i = h_j$ for some $i < j$

$\Rightarrow (h_i - h_j)$ is a root of g so g divisible by $(h_i - h_j)$ $\forall$ $h_i - h_j$

$\Rightarrow g$ divisible by $\prod_{i<j} (h_i - h_j)$ $\rightsquigarrow$ dividing gives that $\langle f \rangle$ a polynomial

__Lemma 2__: $c_1, \ldots, c_n \geq 0$ $\sum c_i = n-1$

$\langle h_1^{c_1} h_2^{c_2} \cdots h_n^{c_n} \rangle = (-1)^{|I|} D_n(I)$ where $I = I(c_1, \ldots, c_n)$

__Proof idea__: $\frac{1}{h_i - h_j} = h_i^{-1} \frac{1}{1 - h_j/h_i} = h_i^{-1} \sum_{k \geq 0} \left(\frac{h_j}{h_i} \right)^k$ $\leftarrow$ conveges if $\{h_1 > h_2 > \cdots > h_n > 0\}$

$i < j$.

Somehow write formula explicitly for permutation,

expand using this lemma,

pick a constant term in this expansion

and it gives the answer.