LECTURE 22 Mon 10/28

Orlik-Solomon Algebra

$\mathcal{A} = \{H_1, \ldots, H_N\}$ hyperplane arr.

$$H_i = \{\vec{x} \mid \langle \vec{x}, \vec{v}_i \rangle = h_i\}$$

$OS_{\mathcal{A}}$: assosciative algebra (over $\mathbb{C}$)
anti-commuting generators $e_1, \ldots, e_N$.

relations $\forall$ circuit $\{i_1, \ldots, i_k\}$ (i.e min set w/ lin dep vectors $\vec{v}_i$)
If $H_{i_1} \cap \ldots \cap H_{i_k} \neq \emptyset$ (central circuit)

$$(*) \quad \boxed{\sum_{j=1}^{k} (-1)^j e_{i_1} \ldots \hat{e}_{i_j} \ldots e_{i_k} = 0}$$

If $H_{i_1} \cap \ldots \cap H_{i_k} = \emptyset$, then

$$(**) \quad \boxed{e_{i_1} \cdots e_{i_k} = 0}$$

Note: $(*) \Rightarrow (**)$ (Just multiply $(*)$ by any e_i).

In the case of generic h_i's
$\Rightarrow$ No central circuits
$\Rightarrow$ Only has $(**)$ not $(*)$

$\Rightarrow$ $OS_{\mathcal{A}}$: linear basis $e_I := \prod_{i \in I} e_i$ ← multiplied for i in increasing order

I is any independent subset in $[N]$.

$$e_I \cdot e_J = \begin{cases} \pm e_{I \cup J} & \text{if } I \cap J = \emptyset \text{ \& } I \cup J \text{ is an ind. set} \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\text{Poincaré Poly } P_{\mathcal{A}}(t) = \sum_{I \text{ ind set}} t^{|I|}}$$

In general if h_i not generic, basis of $OS_{\mathcal{A}}$ may be some but not all lin. ind. sets

No-Broken-circuit (NBC) basis of $OS_{\mathcal{A}}$

Fix a lin. order on the set of hyperplanes in $\mathcal{A}$.
$1 < 2 < \dots < N$.

For a circuit $C = \{i_1 < \dots < i_k\}$
<u>broken circuit</u> is $C \setminus \{i_1\}$ $i_1 = \min(C)$

If C is a central circuit, then we call $C \setminus \{i_1\}$ the <u>broken central circuit</u>

<u>Obs:</u>
Exactly one broken central circuit in $(**)$
$\Rightarrow$ Allows us to eliminate it & write as linear combo of the other terms.

<u>Thm</u>: The following set of monomials $e_I = \prod_{i \in I} e_i$ forms a lin. basis of $OS_{\mathcal{A}}$:

(1) I contains no circuits (i.e. I is an indep. set)
(2) I contains no broken central circuits

<u>Part of Proof</u>:
Can write any monomial in terms of basis:
- If (1) fails, get 0
- For broken central circuit, can write using $(*)$

$\Rightarrow$ Our set spans $OS_{\mathcal{A}}$. Showing lin. ind. is harder.

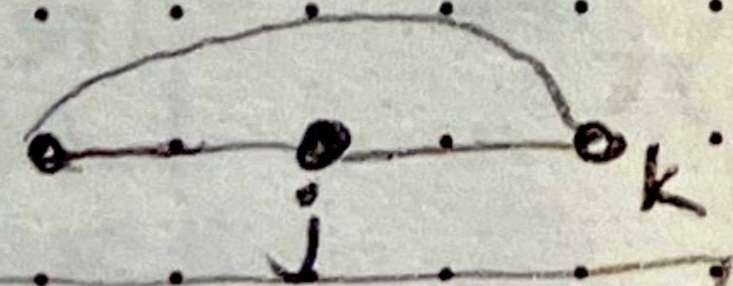
<u>Ex.</u>: The braid arrangement

<u>Arnold-Orlik-Solomon algebra</u>
generators e_{ij} (i,j) is an edge of K_n

(Assume $e_{ij} = e_{ji}$) why not $-e_{ji}$?

b/c $e_{ij} = d\log(x_i - x_j) = \dfrac{d(x_i - x_j)}{x_i - x_j}$ negative signs in numerator & denominator cancel

"circuits" = "cycles in K_n"

3-circuit relations: 

$i<j<k$

$$\boxed{e_{ik} \cdot e_{jk} - e_{ij} \cdot e_{jk} + e_{ij} \cdot e_{ik} = 0}$$

Claim: Only these relations are needed.
i.e. rel. from larger cycles can be written in terms of these.

Pick the lexicographical order of edges of K_n

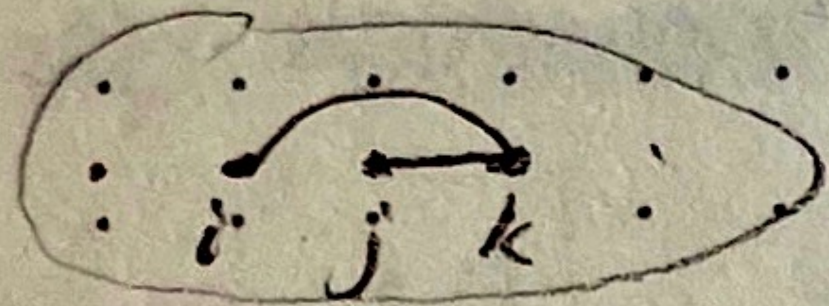
$i<j<k \Rightarrow ij < ik < jk$

broken 3-circuit $\{ik, jk\}$ for $i<j<k$

Prop: The corresponding NBC basis consists of monomials

$$e_F = \prod_{\substack{(i,j)\text{ edges} \\ \text{of } F}} e_{ij}$$

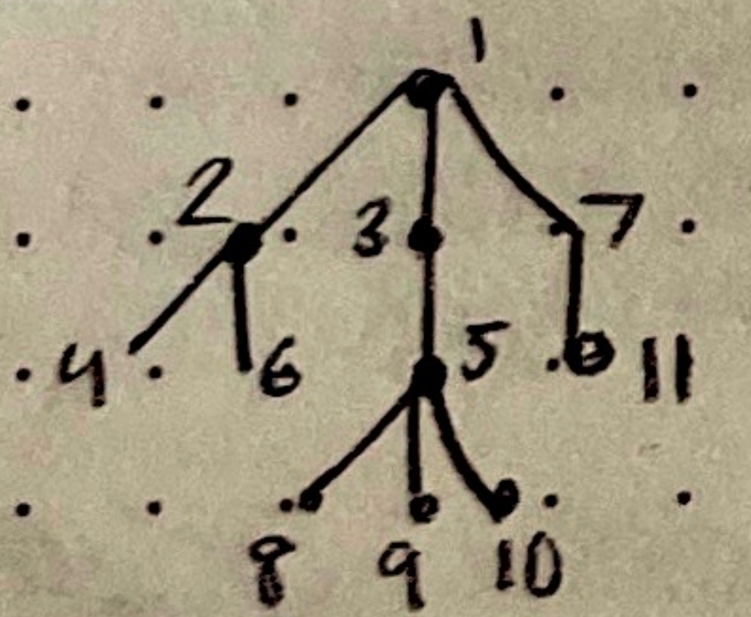
s.t. F is a forest in K_n w/ n edges and F contains no pair of edges ik & jk for $i<j<k$

— Forbidden pattern 

Def: An increasing tree T is a labelled tree s.t. if we direct all edges away from min. vertex of T, then $\forall$ directed edges $i \rightarrow j$ we have $i<j$

$\rightarrow$ This pattern is equivalent to having incr. tree!

Ex: An incr. tree



n	1	2	3	4
incr. trees				
# incr. trees	1		2	6

Lemma: # incr. trees on n verts $= (n-1)!$

Proof: Any time you add new vertex, can add to any of the n previous verts in any way.

Def: An increasing forest is a forest F s.t. every conn. comp. of F is an incr. tree.

Poincaré Poly: $P_A(t) = \sum_{\substack{F \subseteq K_n \\ \text{incr. forest}}} t^{\#\text{edges in } F} = (n-1)!\,t^{n-1} + \dots + 1$

$= \sum_{k=1}^{n-1} c(n,k)\, t^{n-k}$ The signless Stirling #'s of the first kind $= \#$ perms $\in S_n$ w/ exactly k cycles

These #'s equal b/c # ways to specify cycle is $(n-1)!$ = # trees on that many verts

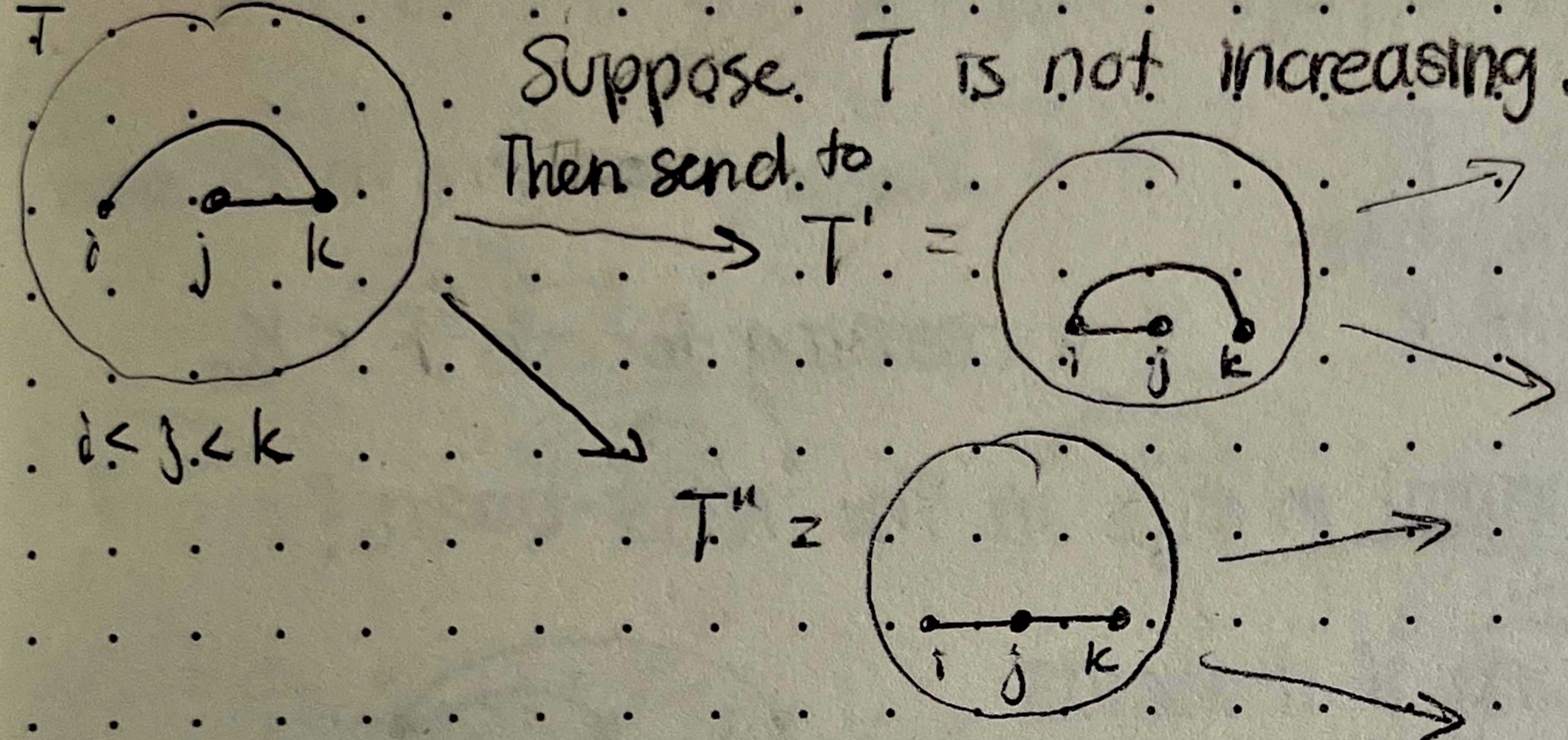
$\rightarrow = (t+1)(2t+1)\dots((n-1)t+1)$ Generating function for $c(n,k)$

How do we write things in terms of basis?

Play a game on forests
(Really a game on trees):

Start w/ tree T.

Suppose T is not increasing.



Keep going on each tree until it is an increasing tree, at which point stop.

A priori, playing game in different orders could give different results.

BUT Thrm from 2 pages ago $\Rightarrow$

<u>Thrm</u>: The resulting collection of increasing trees will always be the same.