LECTURE 21 Fri 10/25

Orlik-Solomon Algebras

History:
- Arnold: braid arrang.
- Briestkorn: Coxeter arrangements.
- Orlik-Solomon: any arrang.

$\mathcal{A}$ a hyperplane arr. in $\mathbb{F}^n$

$\chi_{\mathcal{A}}$ its char poly

$C_{\mathcal{A}} = \mathbb{F}^n \setminus \bigcup_{H \in \mathcal{A}} H$ the compliment

$\mathbb{F} = \mathbb{R}$ then $|\chi(-1)| = \#$ connected components in $C_{\mathcal{A}}$

$\mathbb{F} = \mathbb{F}_q$ $\chi(q) = \# C_{\mathcal{A}}$ ($q = p^r$. True for all but fin. many primes p)

$\mathbb{F} = \mathbb{C}$: $\chi(q) \xleftrightarrow{?} C_{\mathcal{A}}$ How are these related?

Assume that $\mathbb{F} = \mathbb{C}$.

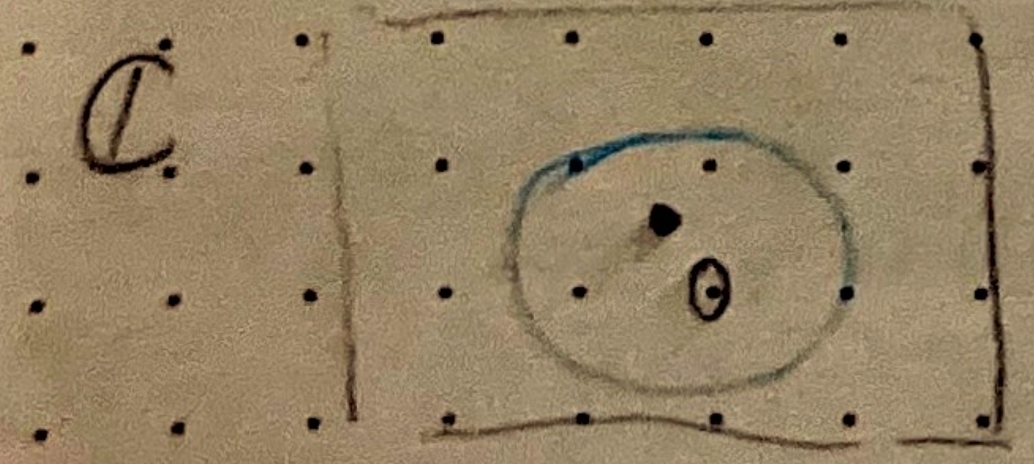
$H^*(C_{\mathcal{A}}) := H^*(C_{\mathcal{A}}; \mathbb{C})$ the cohomology ring of the compliment to $\mathcal{A}$.

Many ways to construct cohomology
(e.g. de Rham cohomology relevant to today).

"I don't want to go into definitions, if you know then you know, if not it's okay".

$P_{\mathcal{A}}(t) := \sum_{k \geq 0} \dim H^k(C_{\mathcal{A}}) t^k$ Poincaré polynomial

Ex. $n=1$ $\mathcal{A} = \{\{0\}\}$ $\mathcal{A} = \{\{0\}, \{1\}\}$



$P_{\mathcal{A}} = 1 + 1t$ $P_{\mathcal{A}} = 1 + 2t$

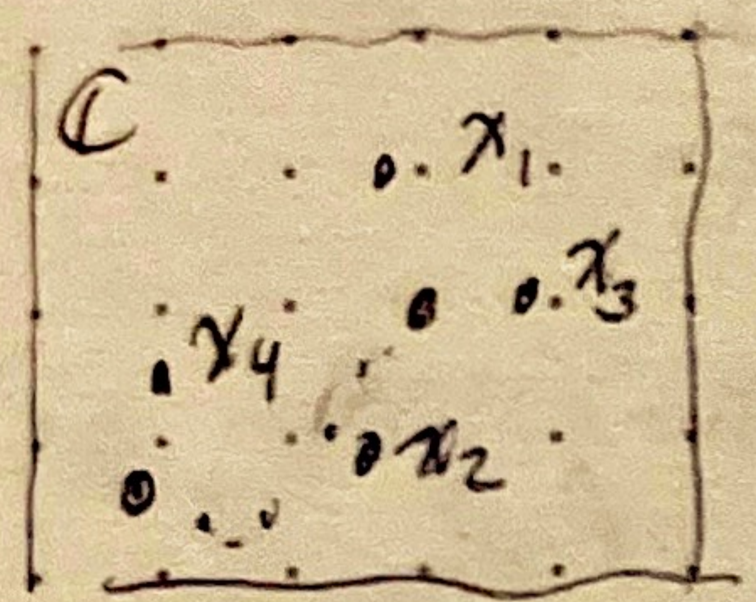
Betti numbers := dim of cohomology of various k.

Thrm: (Orlik-Solomon)

$P_{\mathcal{A}}(t) = t^n \chi_{\mathcal{A}}(-t^{-1})$

Ex. The braid arrangement in $\mathbb{C}^n$. $x_i - x_j = 0$

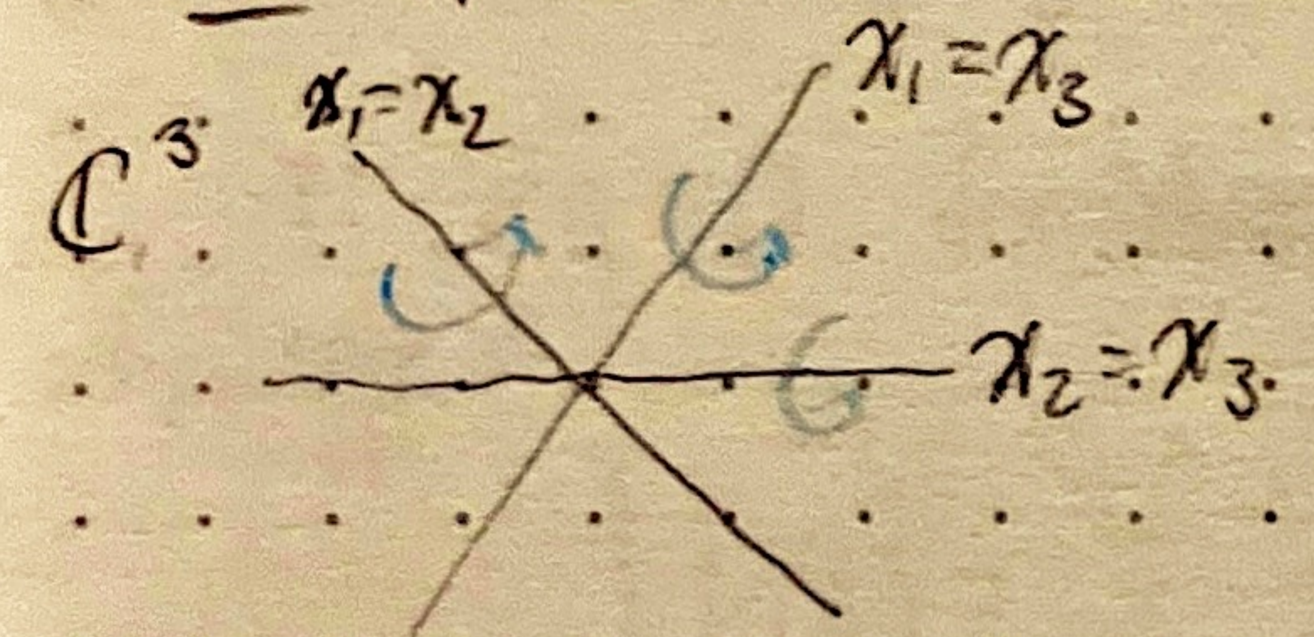
$C_{\mathcal{A}}$ = the space of n-tuples of distinct marked pts in $\mathbb{C}$



$\chi_{\mathcal{A}}(t) = t(t-1)(t-2)\ldots(t-n+1)$

$P_{\mathcal{A}}(t) = (t+1)(2t+1)(3t+1)\ldots((n-1)t+1)$

Ex. $n=3$.



$P_{\mathcal{A}} = (1+t)(1+2t)$
$= 1 + 3t + 2t^2$

Above Ex.'s not compact
↳ No Poincare duality
↳ Not a symmetric poly

$\mathcal{A} = \{H_1, \ldots, H_n\}$ any hyperplane arrangement

Def: Orlik-Solomon alg of $\mathcal{A}$.

$OS_{\mathcal{A}}$: generators $e_1, e_2, \ldots, e_N$
relations (1) $e_i e_j = -e_j e_i$, $e_i^2 = 0$ $\forall\, i,j$

$H_i = \{\vec{x} \mid \langle \vec{v}_i, x \rangle = h_i\}$ (2) $e_{i_1} e_{i_2} \cdots e_{i_k} = 0$ if $H_{i_1} \cap \cdots \cap H_{i_k} = \emptyset$ (*)

(3) $\sum_{j=1}^{k} (-1)^j e_{i_1} \cdots \hat{e}_{i_j} \cdots e_{i_k} = 0$

for any central circuit $\{i_1, \ldots, i_k\} \subset [N]$

Def: $\{i_1, \ldots, i_k\}$ is a central circuit if
(1) $\exists$ a unique (up to rescaling) lin. dependance $c_1 \vec{v}_{i_1} + \ldots + c_k \vec{v}_{i_k} = 0$, and
(2) $c_1 h_1 + \ldots + c_k h_k = 0$ $\quad (\Leftrightarrow H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_k} \neq 0)$

If $\mathcal{A}$ is a central arrang. then OSA relations

(1) anti-commutes.

(3) $\sum (-1)^j e_{i_1} \cdots \hat{e}_{i_j} \cdots e_{i_k} = 0 \ \forall$ circuit $\{i_1, \dots, i_k\}$

$$OS_{\mathcal{A}} = H^*_{DR}(\mathcal{C}_{\mathcal{A}})$$

$$e_i \mapsto d\log(\langle \vec{v}_i, x\rangle - h_i) = \frac{d\langle \vec{v}_i, x\rangle}{\langle \vec{v}_i, \vec{x}\rangle - h_i}$$

Rmk: $OS_{\mathcal{A}}$ depends only on intersection semi-lattice of $\mathcal{A}$.
(matroidal data)

Ex. 1:
$\mathbb{C}^2$.

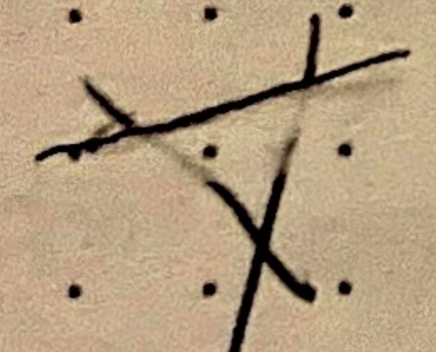
$OS_{\mathcal{A}}$. e_1, e_2, e_3 commute

Also $e_1 e_2 - e_2 - e_3 + e_2 e_3 = 0$

A linear basis in $OS_{\mathcal{A}}$

k	0	1	2
OS^k	1	e_1, e_2, e_3	$e_1 e_2$
dim	1	3	2

Ex.
$\mathcal{A} =$



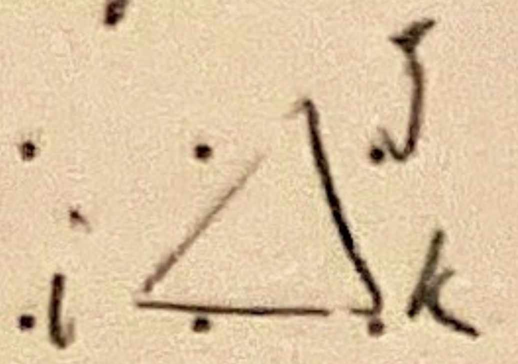
anti-com generators e_1, e_2, e_3
relations $e_1 e_2 e_3$.

k	0	1	2
	1	e_1, e_2, e_3	$e_1 e_2, e_2 e_3, e_{13}$
dim	1	3	3

Ex. The braid arrang: $x_i - x_j = \mathcal{A}$

$OS_{\mathcal{A}}$: generators e_{ij} $\quad 1 \le i \le j \le n$
relations (1) anti-commutative
(3) $e_{ij} e_{ik} - e_{ij} e_{jk} = 0$

circuits $\longleftrightarrow$ cycles in k



In this case having rule for 3-cycles implies it for all cycles.

<u>E.x.</u> arrang. of hyperplanes $x_i - x_j = h_{ij}$ for generic h_{ij}'s.

$OS_{\mathcal{A}}$: Anti-commutative generators e_{ij} $1 \leq i \leq j \leq n$

Having cycle in graph corresponds to statement (*) from second page

$\prod_{(i,j) \text{ edge of } H} e_{ij} = 0$ for any subgraph H that has a cycle

$$\boxed{P(t) = \sum_{F \text{ forest}} t^{\# \text{ edges in } F}}$$

Next lecture: Will talk about how to find one nice linear basis in Orlik-Solomon alg. for any arrangement