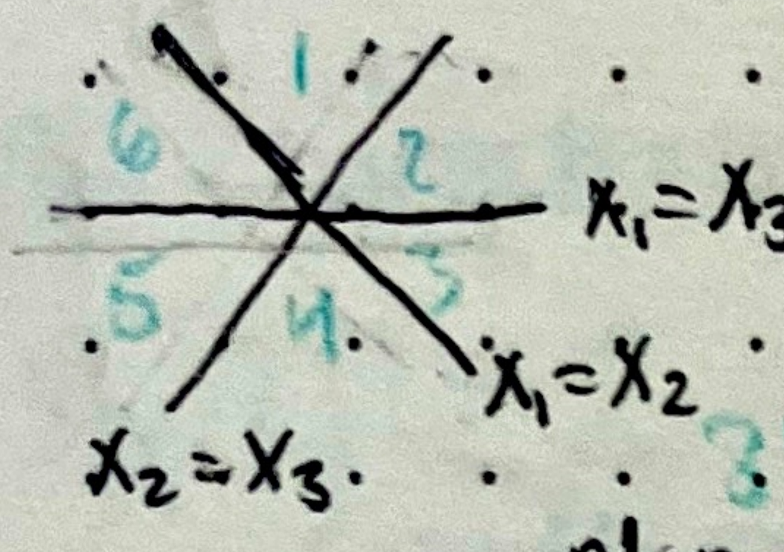
LECTURE 14 Mon 10/7

Counting regions of hyperplane arrangements

Motivating Example: Arrange. in $\mathbb{R}^n$

(1) **Braid arrangement**

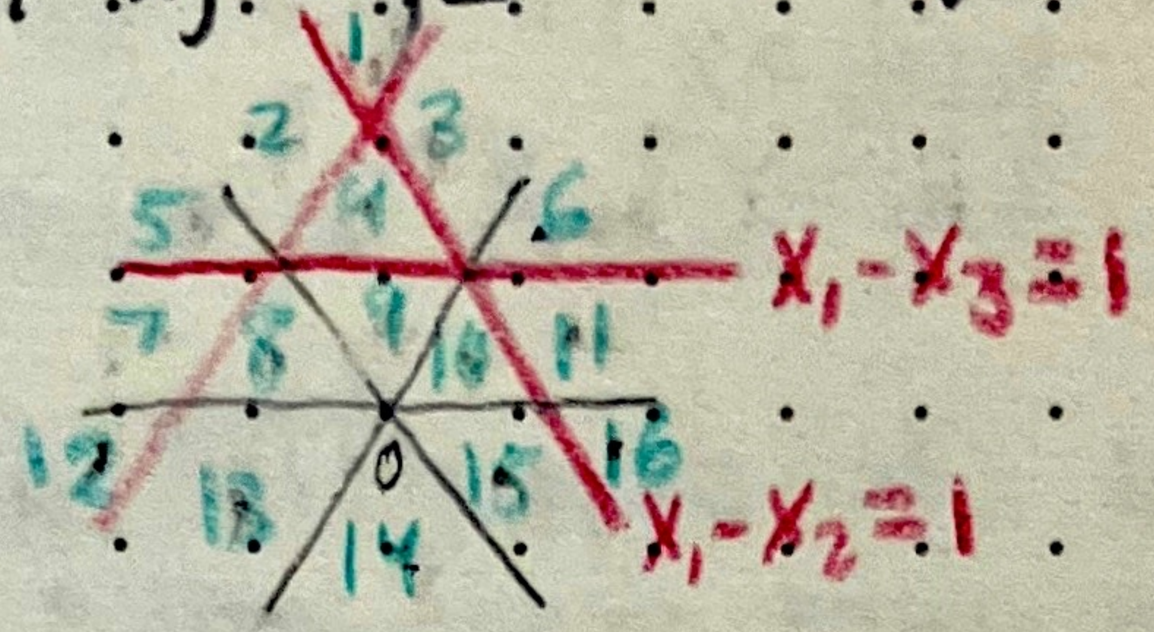
$x_i - x_j = 0 \quad 1 \le i < j \le n$



$n!$ regions

(2) **Shi arrangement**

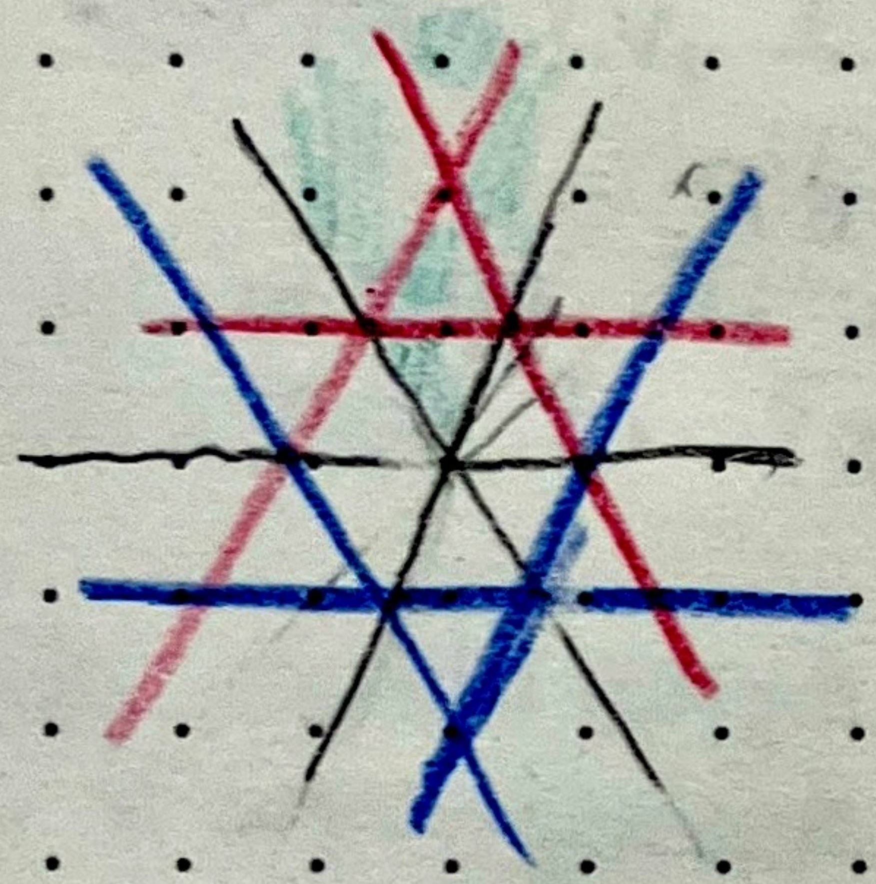
$x_i - x_j = 0, 1 \quad 1 \le i < j \le n$



16 regions

in general $(n+1)^{n-1}$ regions = # spanning trees on n verts.

(3) **Catalan arrang.** $x_i - x_j = -1, 0, 1 \quad 1 \le i < j \le n$



regions = $n! \cdot$ (# regions in shaded portion)

$= n!\, C_n$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number

Q: How to prove # of regions?

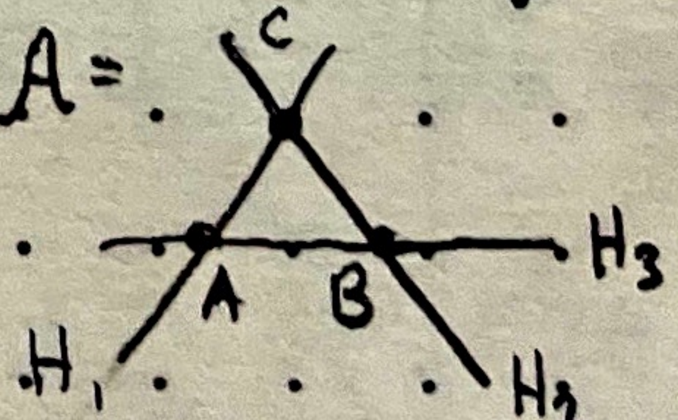
A: ↳ Zaslavsky's formula (most general)
↳ Finite field method
↳ bijective method (works only on specific arrangements that have some nice comb. object to be in bijection with)

Today: Zaslavsky's Formula

$\mathcal{A} = \{H_1, \ldots, H_N\}$ arrange. in $\mathbb{R}^n$.

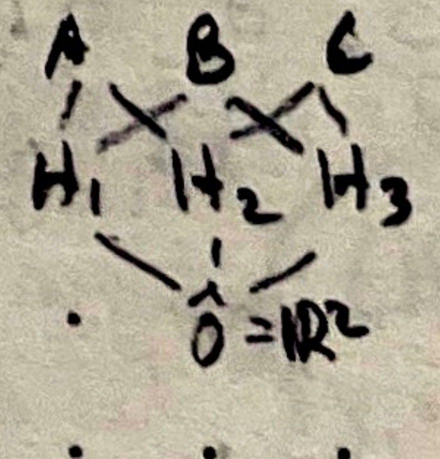
$L_{\mathcal{A}}$ the intersection poset of $\mathcal{A}$
elts. $H_{i_1} \cap \ldots \cap H_{i_k} \neq \emptyset$ (including $\bigcap_{\emptyset} = \mathbb{R}^n$)
ordered by reverse inclusion

Ex. $\mathcal{A} =$



$\subset \mathbb{R}^2$

$L_{\mathcal{A}}$:



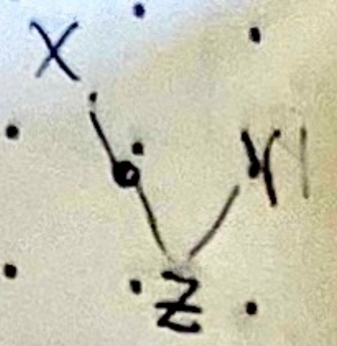
(no max elt.)

Lemma: If $\mathcal{A}$ is a central arrangement, then $L = L_{\mathcal{A}}$ is a lattice, called the <u>intersection lattice</u>

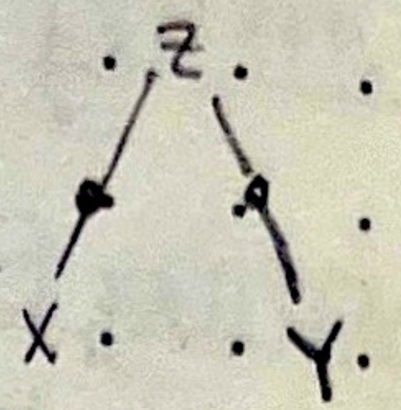
(Lattice defined next page)

Def: Lattice : $\forall X, Y \in L$. $\exists$

meet: $X \wedge Y$ = the unique max elt. Z s.t. $Z \le X$ & $Z \le Y$



join: $X \vee Y$ = the unique min. elt Z s.t. $Z \ge X$ & $Z \ge Y$.



Explicitly, $X \vee Y = X \cap Y$ ← may be empty for non-central lattice

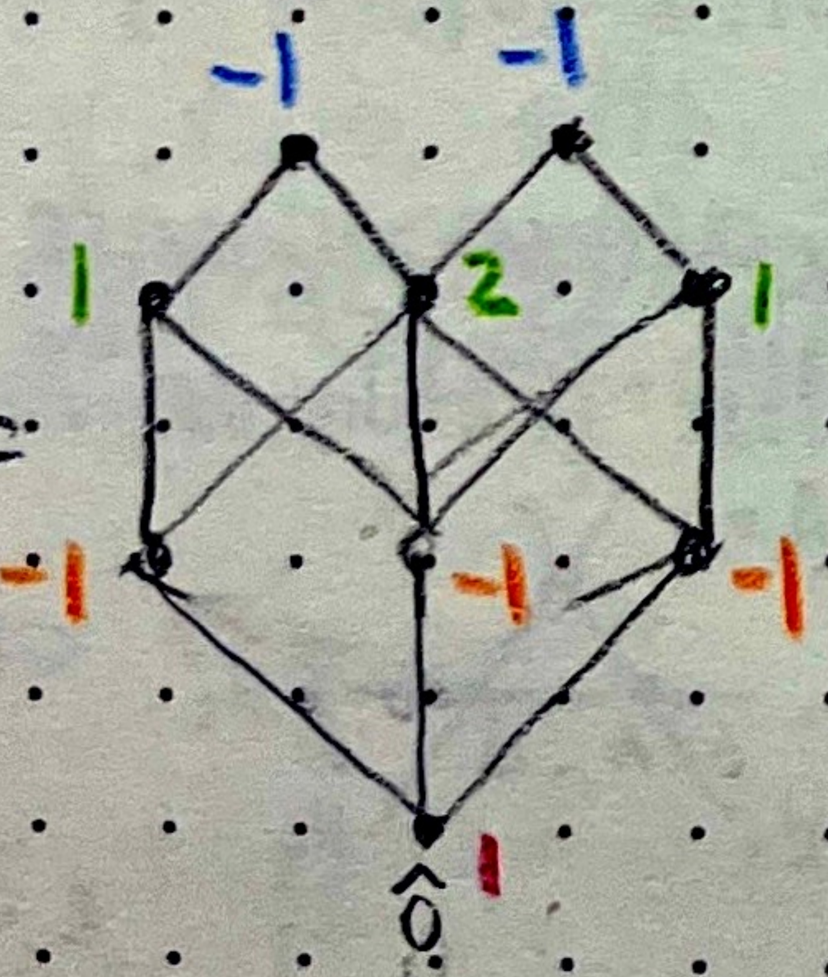
$X \wedge Y = \bigcap_{\{i \mid H_i \supset X \ \& \ H_i \supset Y\}}$ ← Exists for any arrangement.

⇒ In general (for any affine arrang.)
$L_{\mathcal{A}}$ is a meet semi-lattice (i.e. $X \wedge Y$ exists.)

⇔ Any interval $[\hat{0}, X]$ in $L_{\mathcal{A}}$ is a lattice

Def: For a finite poset P, the **Mobius Function** $\mu(X, Y)$ for $X, Y \in P$, $X \le Y$
is defined by
(1) $\mu(X, X) = 1$
(2) For $X \lneq Y$, $\sum_{Z:\, X \le Z \le Y} \mu(X, Z) = 0$
⇕
(2') $\mu(X, Y) = -\sum_{Z:\, X \le Z \lneq Y} \mu(X, Z)$

Ex.

$P =$ 

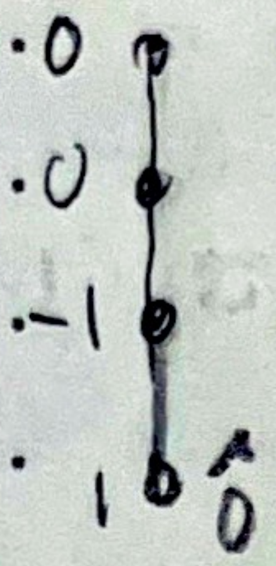
Assume $X = \hat{0}$, find each $\mu(\hat{0}, Y)$
To get value for a given elt, take sume of everything in interval below it & reverse the sign.

Consider 2 $|P| \times |P|$ matrices
$M = (\mu(X, Y))$
$Z = (\zeta(X, Y))$ where $\zeta(X, Y) := \begin{cases} 1 & \text{if } X \le Y \\ 0 & \text{otherwise} \end{cases}$

Claim: $M \cdot Z = I$
$M = Z^{-1}$

Note: Not Obvious! Proved in Lauren Williams' class

Ex. $P =$ 

$\longrightarrow$ 

$\mu(\hat{0}, Y)$.

$$Z = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M = Z^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

__Def__: The characteristic polynomial of $\mathcal{A}$

$$\chi_{\mathcal{A}}(t) := \sum_{X \in L_{\mathcal{A}}} \mu_{L_{\mathcal{A}}}(\hat{0}, X)\, t^{\dim X}$$

__Def__: $\mathcal{A} = \{H_1, \dots, H_N\}$ in $\mathbb{R}^n$ is __essential__ if the normal vectors $\vec{v}_i \perp H_i$ span $\mathbb{R}^n$.

braid arrang. $x_i - x_j = 0$ not essential in $\mathbb{R}^n$ but its restriction to $\{x_1 + \dots + x_n = 0\} \simeq \mathbb{R}^{n-1}$ is essential.

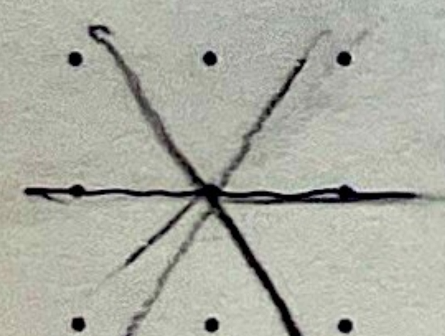
__Thrm__: Zaslavsky's Thrm.

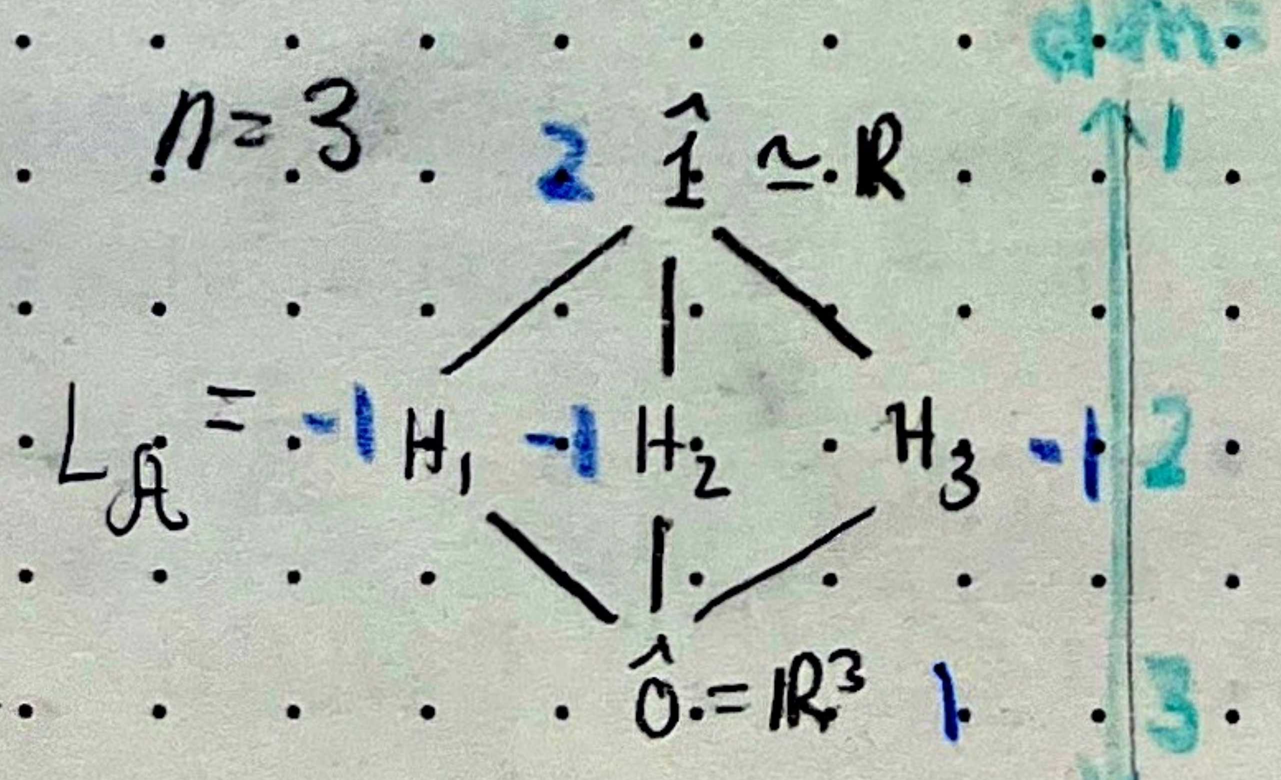
(1) $\forall$ arrangements $\mathcal{A}$ in $\mathbb{R}^n$,
   \# regions in $\mathcal{A} = (-1)^n \chi_{\mathcal{A}}(-1)$.

(2) $\forall$ essential arrang. in $\mathbb{R}^n$
   \# bounded regions $= (-1)^n \chi_{\mathcal{A}}(1)$

__Note__: If not essential, no bounded regions, b/c every region contains vertical line. (corresp. to vector in $\mathbb{R}^n$ not in the span)

__Braid arrang.__ $n = 3$





$\chi_{\mathcal{A}}(t) = t^3 - 3t^2 + 2t$
$= t(t-1)(t-2)$

$\chi_{\tilde{\mathcal{A}}}(t) = t^2 - 3t + 2 = (t-1)(t-2)$

essent. arrang. where we intersect w/ $\{x_1 + x_2 + x_3 = 0\}$

Can Prove Zaslavinsky by generalization of deletion-contraction:
Deletion-Restriction of hyperplane arrang.

__Lemma__: $H \in \mathcal{A}$ $\boxed{\chi_{\mathcal{A}}(q) = \chi_{\mathcal{A} \setminus H}(q) - \chi_{\mathcal{A} \cap H}(q)}$