LECTURE 12 Wed 10/2

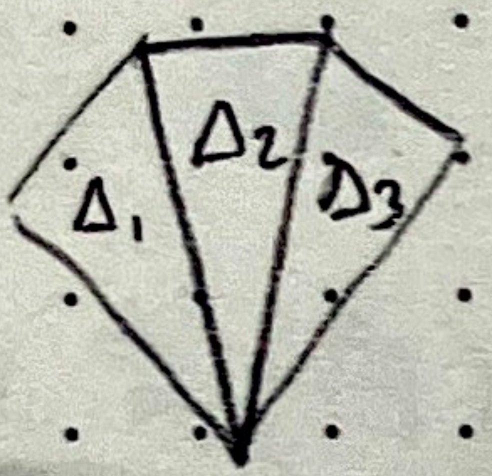
Ehrhat polynomial

$P \subset \mathbb{R}^d$ a polytope

$i_P(t) := \#(tP \cap \mathbb{Z}^d)$ ← # integer lattice pts. in dilated polytope tP, $t \in \mathbb{Z}_{\geq 0}$

Thrm/Def (Ehrhat): If P is a lattice polytope (i.e. $P = \mathrm{conv}(A)$, $A \subset \mathbb{Z}^d$), then $i_P(t)$ is a polynomial in t called the Ehrhat polynomial of P.

Proof Sketch:
1. Check case when P is a simplex
2. Consider a triangulation of P



$\rightsquigarrow$ 2 approaches → (I) Inclusion-Exclusion; → (II) Consider open simplices

(Can't just add $i_\Delta(t)$'s b/c would be over-counting shared faces)

Characteristic function:

$$ch_F : x \longmapsto \begin{cases} 1 & x \in P \\ 0 & x \notin P \end{cases}$$

(I). ch_F = alternating sums of $ch_{simplex}$

$$ch_P = \sum ch_{\Delta_i} - \sum ch_{L_j}$$

(II) Denote $P^\circ = P \setminus \partial P$ (open polytope)

∂P ← bdry of P

Lemma $ch_P = \sum_{F \text{ a face of triangulation}} ch_{F_\circ}$

FACT: $i_P(t) = \mathrm{Vol}(P)\, t^d$ + lower terms.

Ex.1) std cube. $[0,1]^d$

$$i_{[0,1]^d}(t) = (t+1)^d$$

Ex.2). Std. simplex

$\Delta^{d-1} = \mathrm{conv}(\vec{e}_1, \dots, \vec{e}_d)$.

$i_{\Delta^{d+1}}(t) = \#\{(a_1,\dots,a_d)\in\mathbb{Z}^d \mid a_i \geq 0,\ \sum a_i = t\}$

(weak composition of t with d parts).

$= \#\{(a_1',\dots,a_d')\in\mathbb{Z}^d \mid a_i' \geq 1,\ \sum a_i' = t+d\}$

strict composition of $t+d$ w/ d parts

$= \binom{t+d-1}{d-1}$

Why: Want to break $t+d$ into d parts

• ○|○ ⋯ • ○|○   Put in $d-1$ bars
Have $t+d-1$ choices of spots

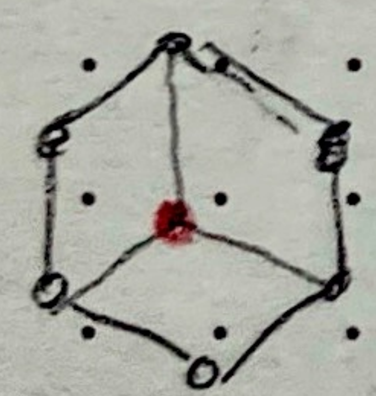
Thrm: For a graphical zonotope Z_G,

$$i_{Z_G}(t) = \sum_{\substack{F\subset G\\ \text{forest}}} t^{\#\text{ edges in } F}$$

Recall: $\mathrm{Vol}(Z_G) = \#$ trees
$Z_G \cap \mathbb{Z}^n = \#$ forests

both follow from Thrm

Ex. Std 2-dim permutohedron Z_{K_3}

$P =$ 

$i_{Z_{K_3}}(t) = 3t^2 + 3t + 1$ ← the empty forest

spanning trees; forest w/ one edge

(I) Inclusion-exclusion

$i_{Z_{K_3}}(t) = 3(t+1)^2 - 3(t+1) + 1$

From edges in overlaps; vertex in intersection of edges

(II) Consider open tiles:

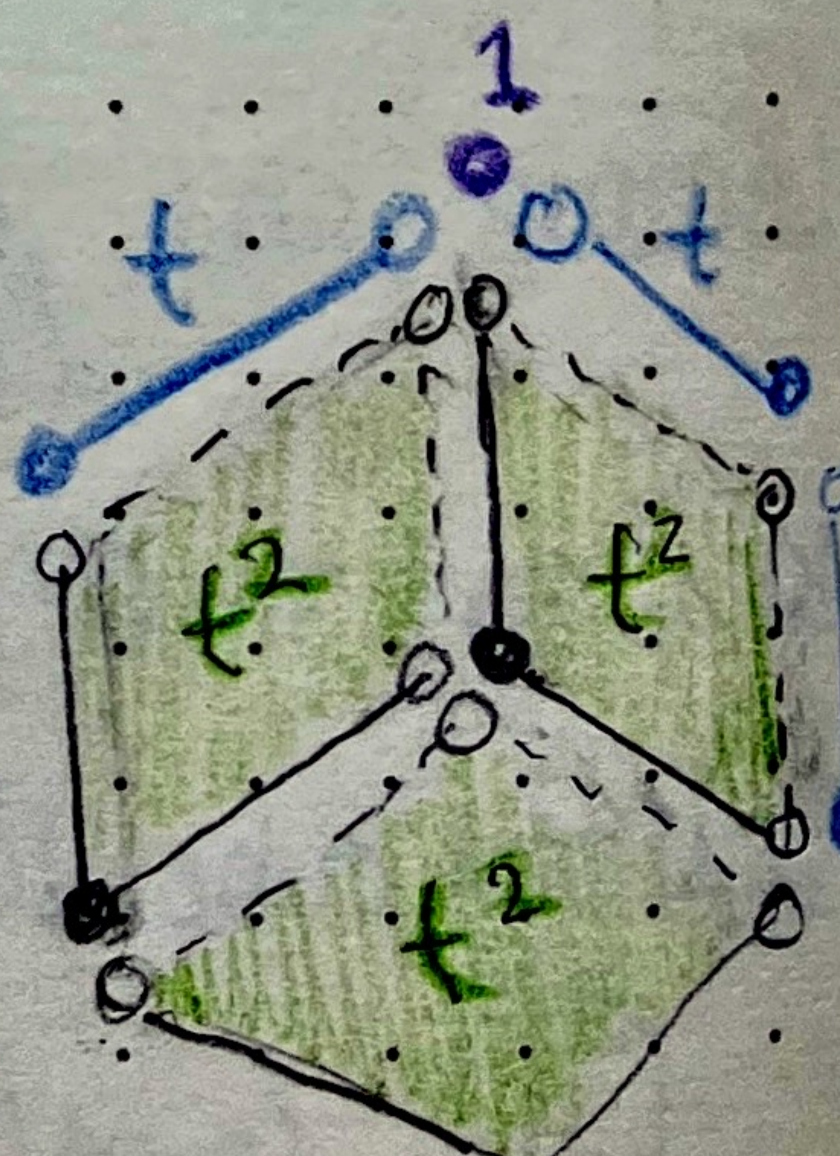
$3(t-1)^2 + 9(t-1) + 7$

all edges; all vertices

To avoid all minuses:

(III) Consider half-open tiles.

$3t^2 + 3t + 1$



Exercise: Extend construction (III) to Z_{K_n}.

Will be on PSET!

Tile w/ parallelpipeds. Decide which faces are we taking & which ones not.

Ehrhat Reciprocity

Thrm: For a lattice polytope P of dim $= d$.
$P^\circ = P \setminus \partial P$
$i_P(-t) = (-1)^d \, i_{P^\circ}(t)$ for $t \in \mathbb{Z}_{\geq 1}$

In particular $i_P(0) = 1$, $i_{P^\circ}(0) = (-1)^d$

Chromatic Polynomial

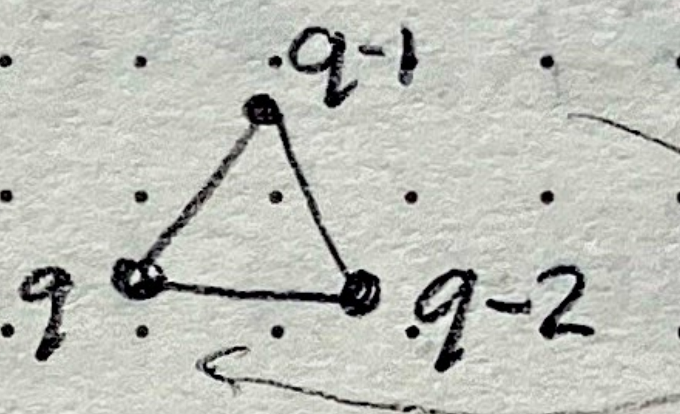
$G = (V, E)$ a graph (w/out loops ⊸)
A q-coloring of G is a function
$c : V \longrightarrow \{1, 2, \ldots, q\}$ (q colors) $q \in \mathbb{Z}_{\geq 0}$

Thrm/Def: $\exists!$ polynomial called the chromatic polynomial

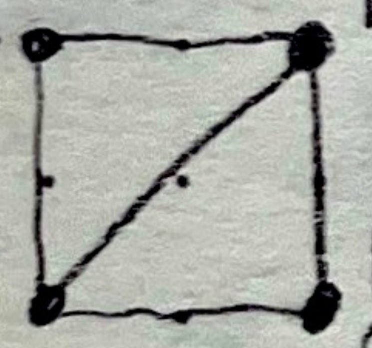
$$\chi_G(q) = \# \text{ q-colorings of } G \ \forall q \in \mathbb{Z}_{\geq 0}$$

(Note $\chi_G(0) = 0$)

Ex. $\chi_{K_3}(q) = q(q-1)(q-2)$



# of color options

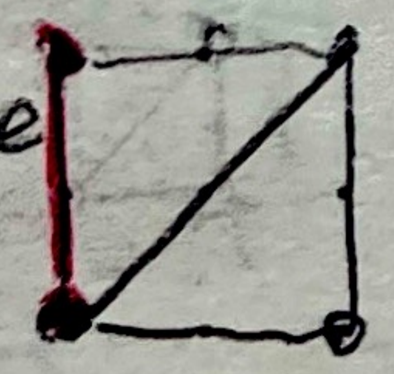


$\Rightarrow \chi_{\boxtimes}(q) = q(q-1)(q-2)^2$

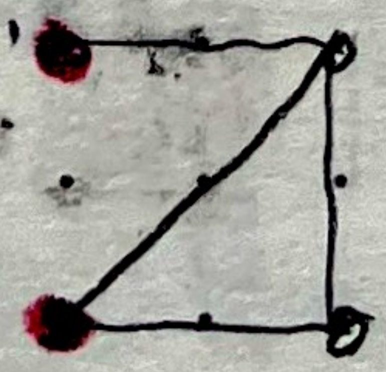
Thrm (Stanley): $\chi_G(-1) = (-1)^{|V|} \cdot \#\{\text{acyclic orientations of } G\}$

How to prove:
Induction on $|E|$ and deletion-contraction

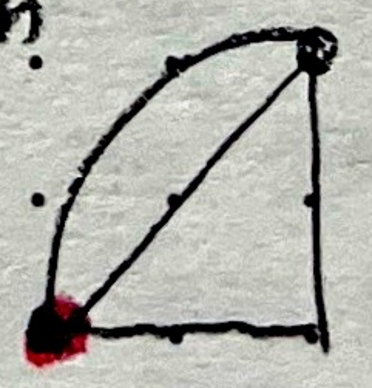
$e \in E$

$G =$ 

deletion of e $G \setminus e$



contraction of e



Lemma: (1) $\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G/e}(q)$

(2) $AO(G) = AO(G \setminus e) + AO(G/e)$
↖ # (acyclic orientations)

This gives "boring proof" of thrm, but there is more conceptual interpretation

More conceptual proof coming next lecture

Idea: $V = [n]$. $\chi_G(q) = \#\{(a_1, \ldots, a_n) \in \mathbb{Z}^d \setminus \bigcup_{H \in \mathcal{A}_G} H \mid 1 \leq a_i \leq q \ \forall i\}$