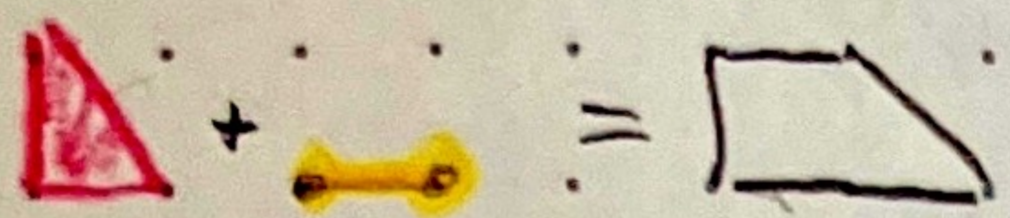
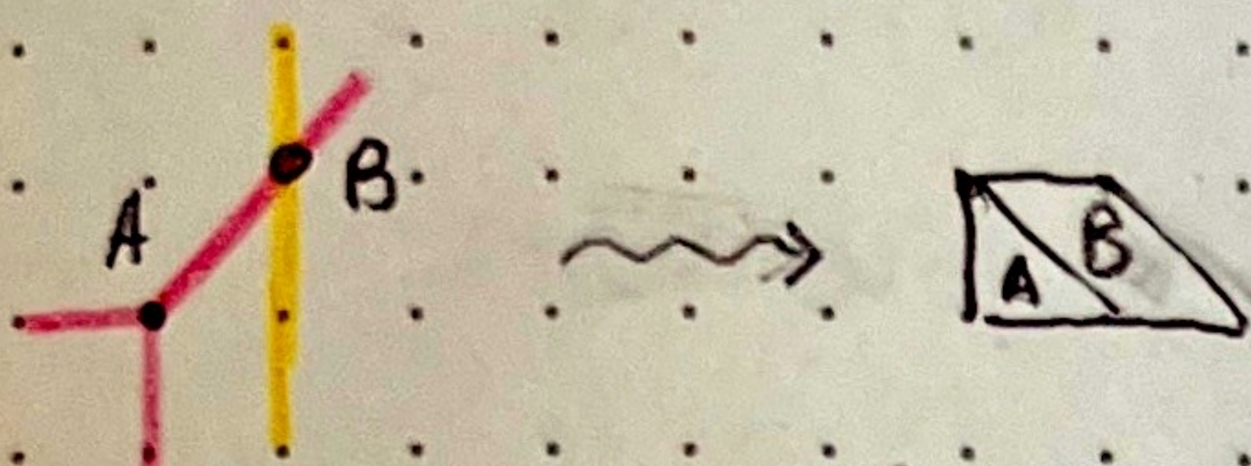
LECTURE 10 FRI 9/27

<u>Last Time</u>: Regular mixed subdivisions of $P = P_1 + \dots + P_N$ $\longleftrightarrow$ affine fan arrangements
(In particular, reg. zonotopal tilings $\longleftrightarrow$ hyperplane arrang.)

<u>E.x.</u>



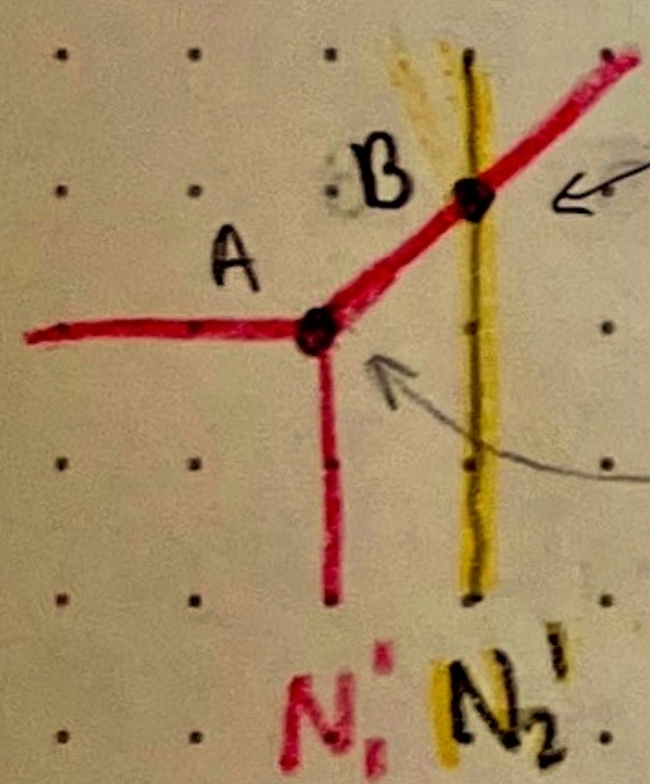


Each tile has the form $F_1 + \dots + F_N$ where F_i a face of P_i

<u>Lemma</u>: TFAE

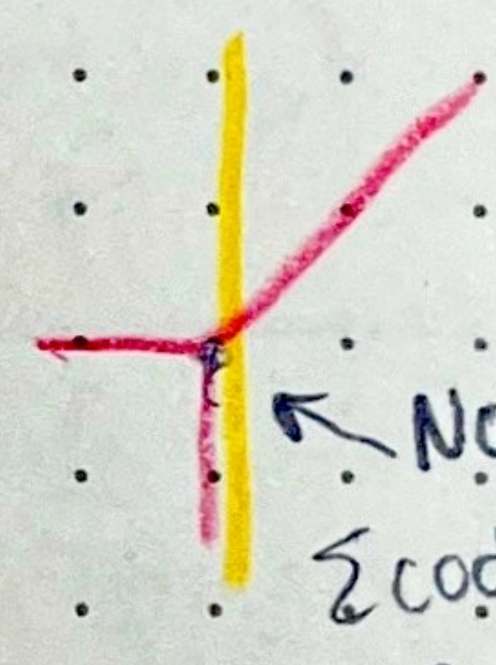
(1) A tile $F_1 + \dots + F_N$ is <u>fine</u> (i.e. cannot be subdivided into smaller tiles)
(2) $F_1 + \dots + F_N \simeq$ the direct product $F_1 \times \dots \times F_N$.
(3) $\dim(F_1 + \dots + F_N) = \sum_i \dim(F_i)$.
"Proof is pretty simple, won't do it here"

<u>Lemma</u>: For a fan arrangement with the generic lin. translations of fans, the corresponding mixed subdivision is fine.



$\sum_i \operatorname{codim}\left(\begin{matrix}\text{cone of } N_i' \\ \text{containing } B\end{matrix}\right) = 1+1 = 2$

$\sum \operatorname{codim}(\text{cones}) = 2+0 = 2$



Not fine
$\sum \operatorname{codim}(\text{cone}) = 2+1 > 2$

$P(t_1, \dots, t_N) = t_1 P_1 + \dots + t_N P_N$

$\rightsquigarrow$ mixed subdiv. with tiles $t_1 F_1 + \dots + t_N F_N$.

<u>Lemma</u>: For a fine tiling, we have

$$\operatorname{vol}(t_1 F_1 + \dots + t_N F_N) = t_1^{d_1} t_2^{d_2} \cdots t_n^{d_n} \operatorname{vol}(F_1 + \dots + F_N)$$

$\Rightarrow$ <u>Thrm</u>: $\operatorname{Vol}(t_1 P_1, \dots, t_N P_N)$ is a polynomial in t_i's.

Polyhedral Subdivisions & Triangulations

$A = \{a_1, \dots, a_N\} \subset \mathbb{R}^d$

$P = \mathrm{conv}(A)$ $\quad$ $\dim P = d$

<u>Def</u>: A polyhedral subdiv. of P is <u>proper</u> subdivision of P into tiles of the form $\mathrm{conv}(B)$ for $B \subset A$.
It is a <u>triangulation</u> if every triangle is a simplex.

<u>Regular</u> polyhedral subdivision/triangulations:
Pick some heights $h_1, \dots, h_N \in \mathbb{R}$

$\tilde{a}_i = (a_i, h)$

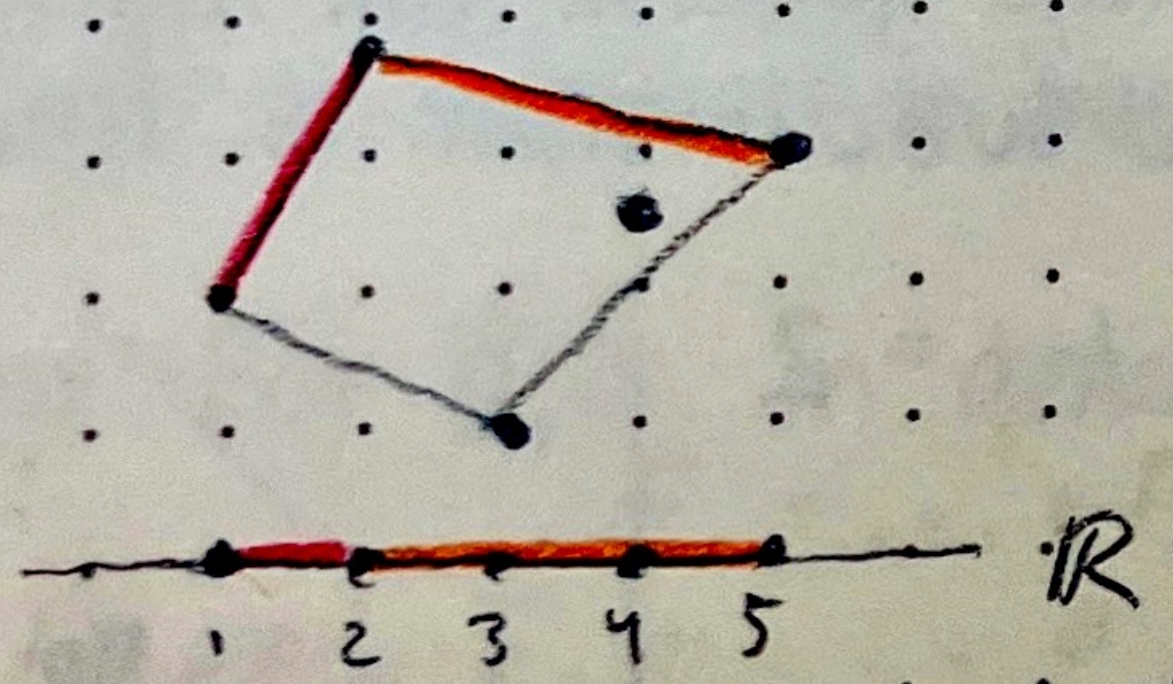
$$\begin{matrix} \tilde{P} = \mathrm{conv}(\tilde{a}_1, \dots, \tilde{a}_N) \subset \mathbb{R}^{d+1} \\ \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \\ P \qquad\qquad\qquad\qquad\qquad \mathbb{R}^d \end{matrix}$$

The corresp. regular subdiv. of P is the projection of the upper boundary of $\tilde{P}$ onto P.

Ex. $d=1$, $A = \{(1), (2), (3), (4), (5)\}$ $\quad$ $P = [1, 5]$

$\tilde{P} =$

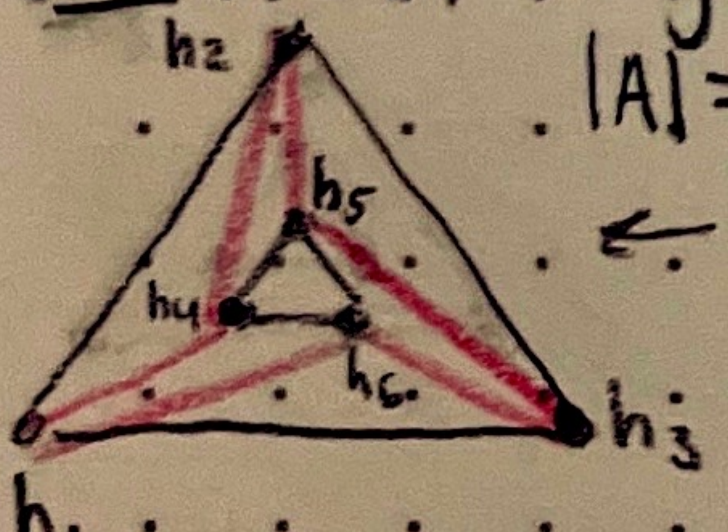
$P =$



Both regular polytopal subdivisions & reg. zonotopal tilings are special case of <u>Fiber Polytopes</u>, come from projection $P \to Q$

But first...

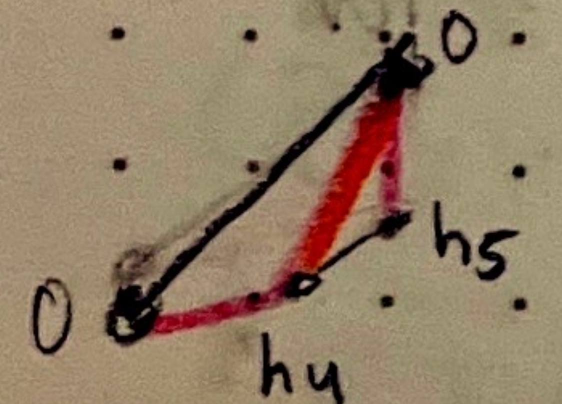
<u>Ex 1</u> Non-regular triangulation



$|A| = 6$

$\leftarrow$ Can never be proj. of higher dim. polytope

Why: WLOG assume $h_1 = h_2 = h_3 = 0$



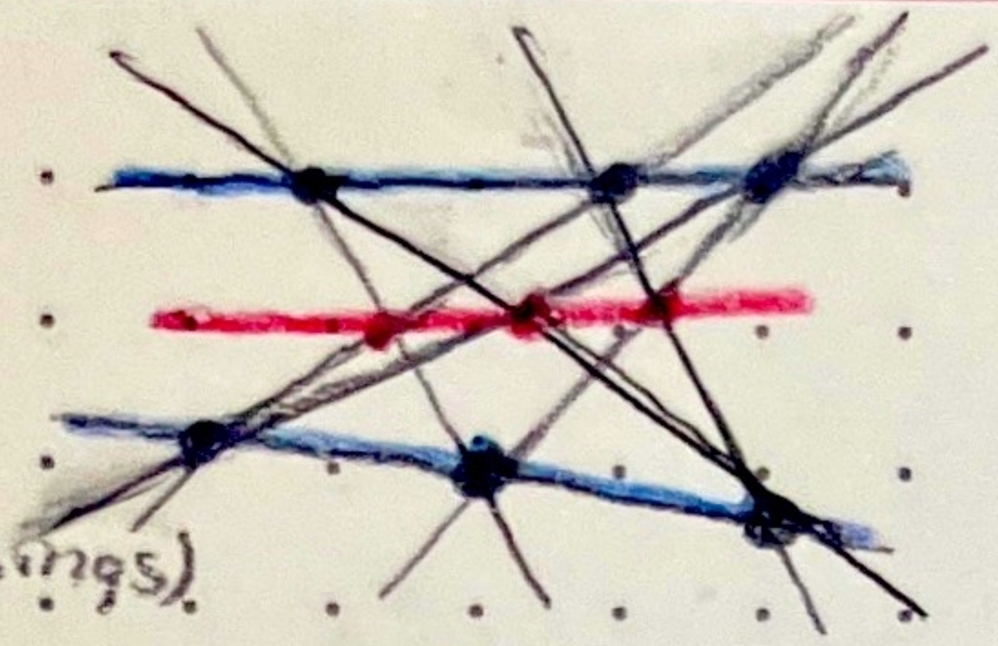
To see orange diage. (and have convex shape) need to lift h_4 more than h_5.

$\Rightarrow h_4 > h_5$

but by symmetry also $h_5 > h_6$, $h_6 > h_4$

$\Rightarrow$ NOT possible

Ex. 2: Pappus' Thrm.

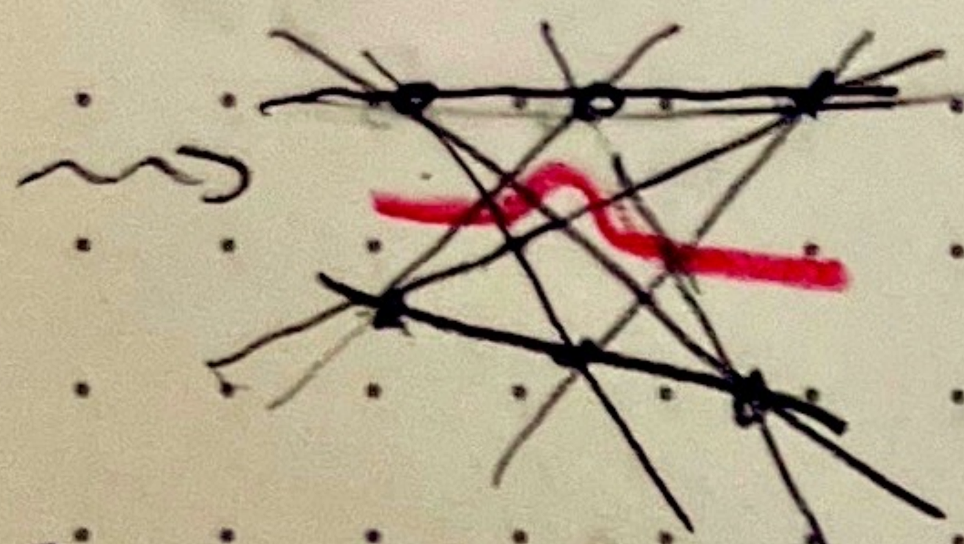


Get zonotopal tiling w/
9 hexegons (from triples crossings)
& 5 rhombuses (from double crossings. Some happen outside of blue lines)

Pick 3 pts each on blue lines. Draw lines connected as shown (in pencil). Then 3 intersection pts lie on same line (shown in red).

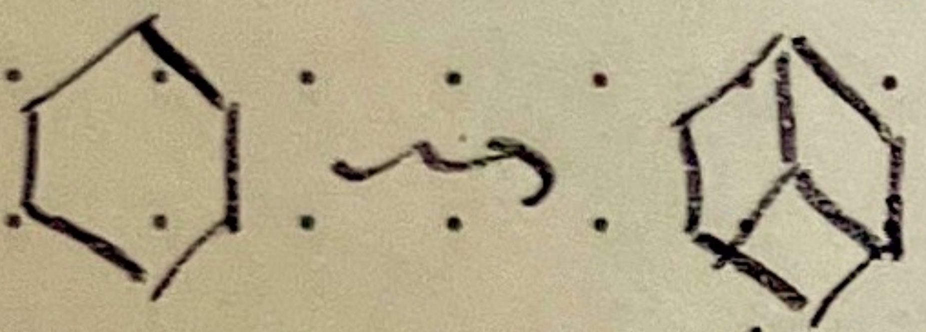
Zonotopal tilings given by these lines is regular but not fine.

Ex. 3 Try to replace triple crossing w/ 2 double crossings



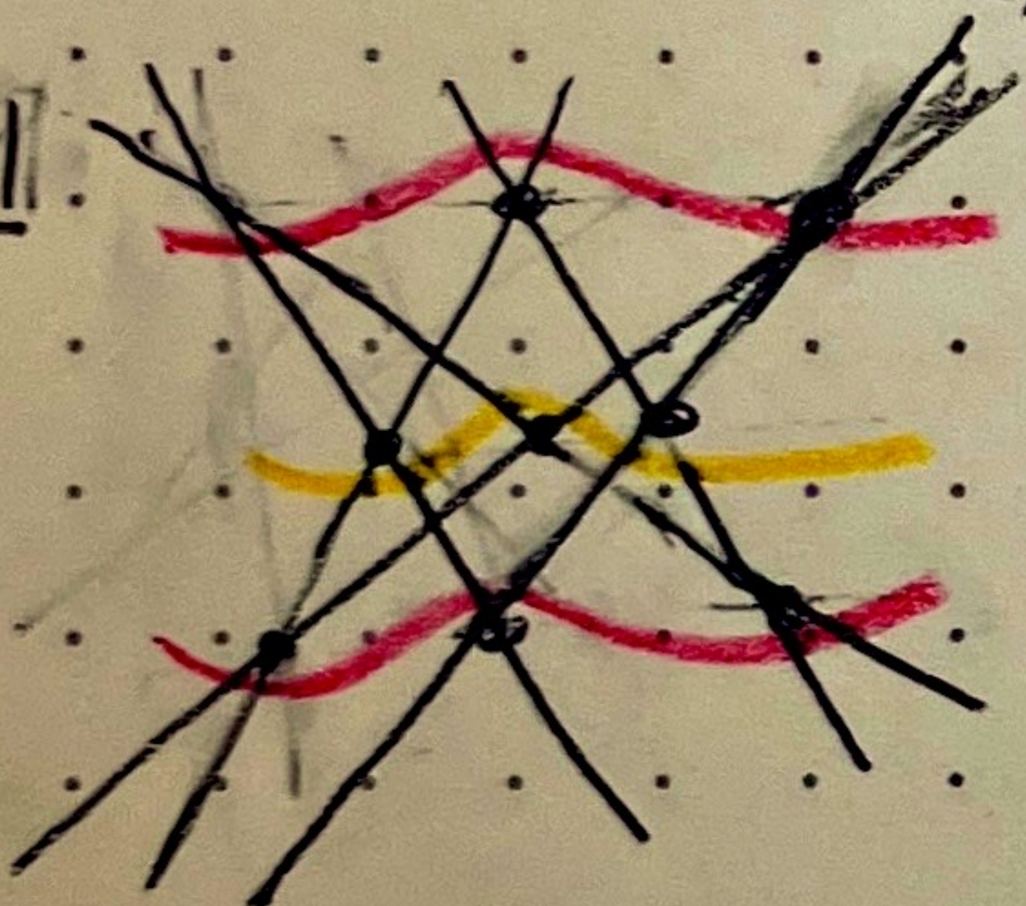
impossible by Pappus' Thrm, but get a "pseudo line configuration"

Can turn this into zonotopal tiling which is 18-gon. Where one of the hexegons from before subdivided into 3 rhombuses



Not regular.
Not fine (still have triple crossing ⟷ hexegons)

Ex. 4



⇒ In this set up, no way to straighten all the lines
⇒ fine, but non-regular tiling
← b/c only double crossings remain

Ex. n : Desagues' Thrm line configurations also lead to non-regular tilings in similar way

Can give rigorous def of psuedo-line arrangements using oriented matroids.

Thrm: Zonotopal tiling $\xleftrightarrow{\text{equiv}}$ lifts of oriented matroids.