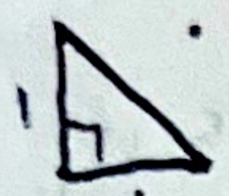
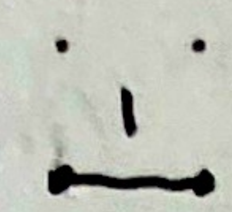
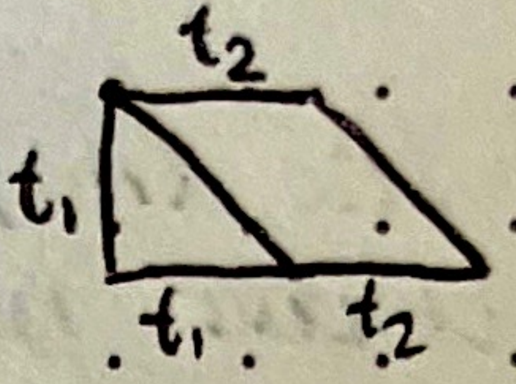
LECTURE 8 MON 9/23

Polytopes $P_1, \dots, P_n \subset \mathbb{R}^d$

Minkowski sum of dilated poly.

$f(t_1, \dots, t_N) = \mathrm{Vol}(t_1P_1 + \dots + t_nP_n)$, $t_i \geq 0$

<u>Thrm 1</u>: $f(t_1, \dots, t_N)$ is a polynomial in $t_1, \dots, t_N$

<u>Ex.</u> $P_1 =$  $P_2 =$  $t_1P_1 + t_2P_2 =$ 

$f(t_1, t_2) = \frac{1}{2}t_1^2 + t_1t_2$

Why do we care?
- Mixed volumes & Bernstein-Khovoriski-Kushirenko thrm
  ↳ result about # of sltns to system of equations of d-vars in dim d
- Lorentzian polynomials

Special case: Zonotopes

$\vec{v}_1, \dots, \vec{v}_N \in \mathbb{R}^d$ (spanning $\mathbb{R}^d$)

a vector configuration

$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_N \\ | & & | \end{bmatrix}$ $d \times N$ matrix of rank d

Zonotope: $Z = Z(\vec{v}_1, \dots, \vec{v}_N) = \sum_{i=1}^{N} [0, \vec{v}_i]$

<u>Thrm 2</u>: $\mathrm{Vol}(Z) = \sum_{B \in \binom{[N]}{d}} \underbrace{|\det(\vec{v}_i)_{i \in B}|}_{\text{maximal minors of } A}$ ← abs. value

$= \sum_{B \text{ bases}} |\det(\vec{v}_i)_{i \in B}|$

<u>Rmk</u> This implies Thrm 1 in zonotope case
Also polynomial in moving the $\vec{v}_i$ as long as we preserve signs of minors a.k.a fixing oriented matroid & as long as you don't change oriented matroid structure, you get polynomials in all vars ($\vec{v}_i$'s not just t_i's).

Tody: Want to present explicit construction of fine zonotopal tiling that makes Thrm 2 obvious
(And this idea generalizes for Thrm 1)

Even more special case: Graphical zonotopes Z_G

Connected graph $G=(V,E)$ $|V|=n$, $|E|=N$

$$Z_G = \sum_{(i,j)\in E} [0, \vec{e}_i - \vec{e}_j]$$ Note: This is a translation of our previous def to line things up at 0.

$d = n-1 = \dim Z_G$

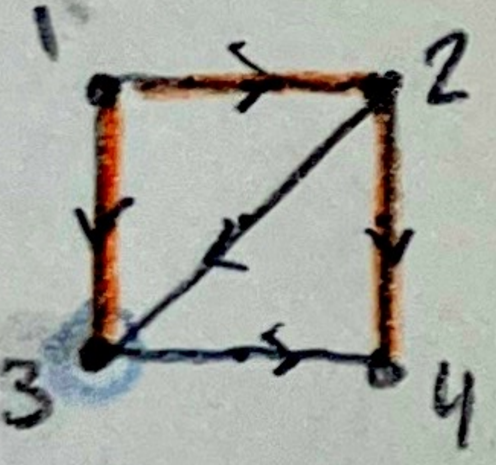
$\mathrm{Vol}(Z_G)$ = the $(n-1)$-dim volume of the image of Z_G under projection $\mathbb{R}^n \to \mathbb{R}^{n-1}$
$(x_1,\dots,x_n) \mapsto (x_1,\dots,x_{n-1})$

<u>Cor</u> of Thm 2: $\mathrm{Vol}(Z_G) = \#$ spanning trees in G

- In particular, $\mathrm{Vol}(n^{th}$ permutohedron$) = n^{n-2} = \#$ spanning trees is complete graph
- Moreover, in this case

$$f(t_1,\dots,t_N) = \sum_{\substack{T \text{ spanning} \\ \text{trees in } G}} \prod_{\substack{e \text{ edge} \\ \text{of } T}} t_e$$

<u>Ex.</u> G:



$n=4$
$N=5$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{-1} \end{bmatrix}$$ (remove to project into $\mathbb{R}^3$)

<u>Lemma</u>: (1) All maximal minors of $A \in \{\pm 1, 0\}$
(2) Non-zero max minors $\xleftrightarrow{\text{bij}}$ spanning trees

<u>Proof sketch</u>: (2) If there is cycle $\Rightarrow$ lin. dependence of corresp. columns $\Rightarrow$ minor is 0

(1) <u>Ex.</u> (Pick tree in orange) $\xrightarrow{\text{minor}}$ $\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$

Note: Will always have some row w/ only one ± 1.
(corresp. to one of the leaves)
Remove that leaf to get new minor of one size smaller, etc, keep going $\Rightarrow \pm 1$ at each step, so multiplying get ± 1.

<u>Regular zonotopal tilings</u> of $Z(\vec{v}_1,\dots,\vec{v}_N) \subset \mathbb{R}^d$
Pick "heights" $h_1,\dots,h_N \in \mathbb{R}$
"lifted" vectors $\tilde{v}_i = (\vec{v}_i, h_i) \in \mathbb{R}^{d+1}$
<u>Lifted zonotope</u> $\tilde{Z} = Z(\tilde{v}_1,\dots,\tilde{v}_N)$

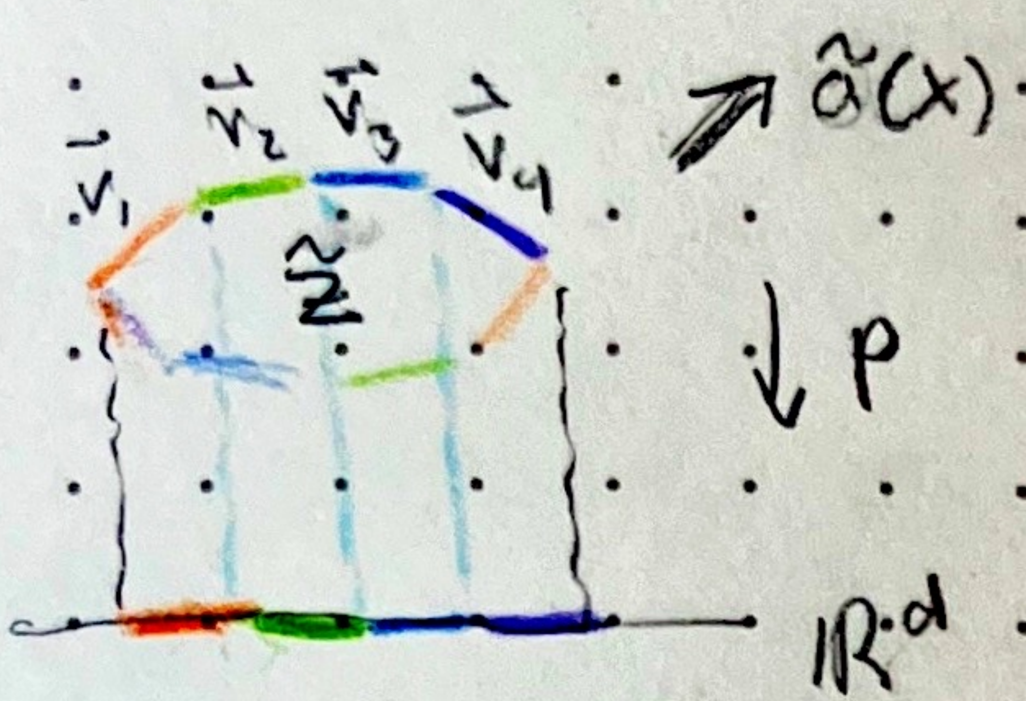
Projection $p : \mathbb{R}^{d+1} \to \mathbb{R}^d$

$(x_1, \dots, x_{d+1}) \mapsto (x_1, \dots, x_d)$

$\Rightarrow p(\tilde{Z}) = Z$

Def: The <u>regular zonotopal tiling</u> of Z (assosciated w/ heights $h_1, \dots, h_N$) is the tiling whose tiles are projections of d-dim face (facets) in the upper boundry of $\tilde{Z}$

E.x.



Q: Why is this a zonotopal tiling

A: Faces of zonotope $\tilde{Z}$ are zonotopes.
If you project them down, they are still zonotopes.

<u>Supporting faces of $\tilde{Z}$</u>

$F = F_{\tilde{\alpha}, \tilde{Z}}$ $\quad \tilde{\alpha}(x) = a_1 x_1 + \dots + a_{d+1} x_{d+1}$

linear function on $\mathbb{R}^{d+1}$

$F \subseteq$ upper boundary if $a_{d+1} > 0$.

Lets rescale a_{d+1} to 1

3 options: Max $\tilde{\alpha}(x)$ on $[0, \tilde{v}_i]$. Max reached at

(1) vertex $\tilde{v}_i$

(2) vertex 0

(3) segment $[0, \tilde{v}_i]$

$$F = \sum_i \begin{cases} \tilde{v}_i & \text{if } \tilde{\alpha}(\tilde{v}_i) > 0 \\ 0 & \text{if } \tilde{\alpha}(\tilde{v}_i) < 0 \\ [0, \tilde{v}_i] & \text{if } \tilde{\alpha}(\tilde{v}_i) = 0 \end{cases}$$

Tiles are $p(F)$

$$p(F) = \sum_{i=1}^{N} \begin{cases} \{\vec{v}_i\} & \text{if } a(\vec{v}_i) + h_i > 0 \\ 0 & \text{if } a(\vec{v}_i) + h_i < 0 \\ [0, \hat{v}_i] & \text{if } \tilde{a}(\vec{v}_i) + h_i = 0 \end{cases}$$

$a(x) = a_1 x_1 + \dots + a_d x_d$

function on $\mathbb{R}^d$