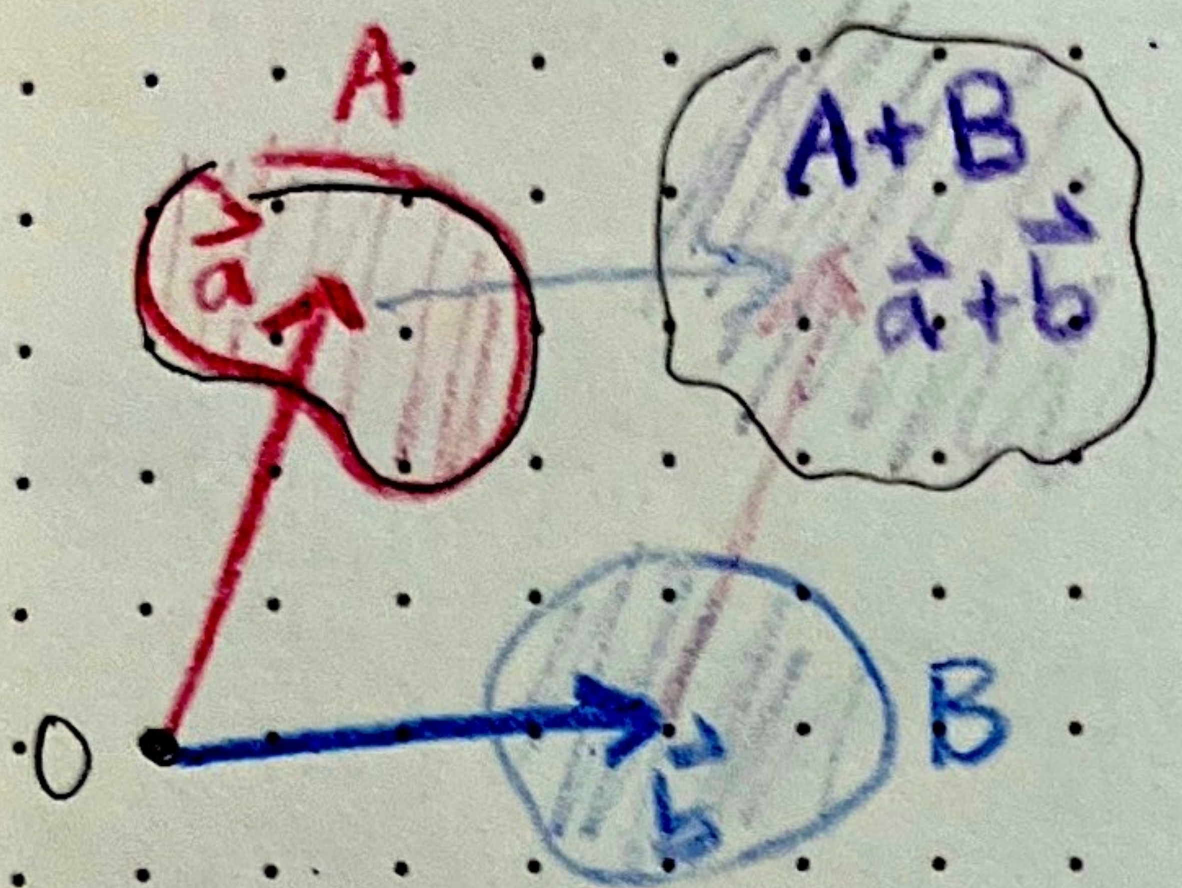
LECTURE 4 Wed 9/11

Minkowski Sums (and differences)

<u>Def</u>: The <u>Minkowski</u> sum of $A, B \subset \mathbb{R}^n$ is

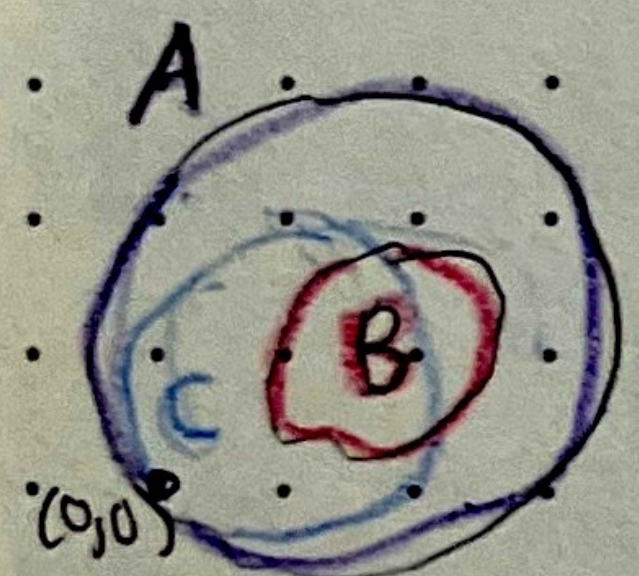
$$A+B = \{\vec{a} + \vec{b} \mid \vec{a} \in A, \vec{b} \in B\}$$



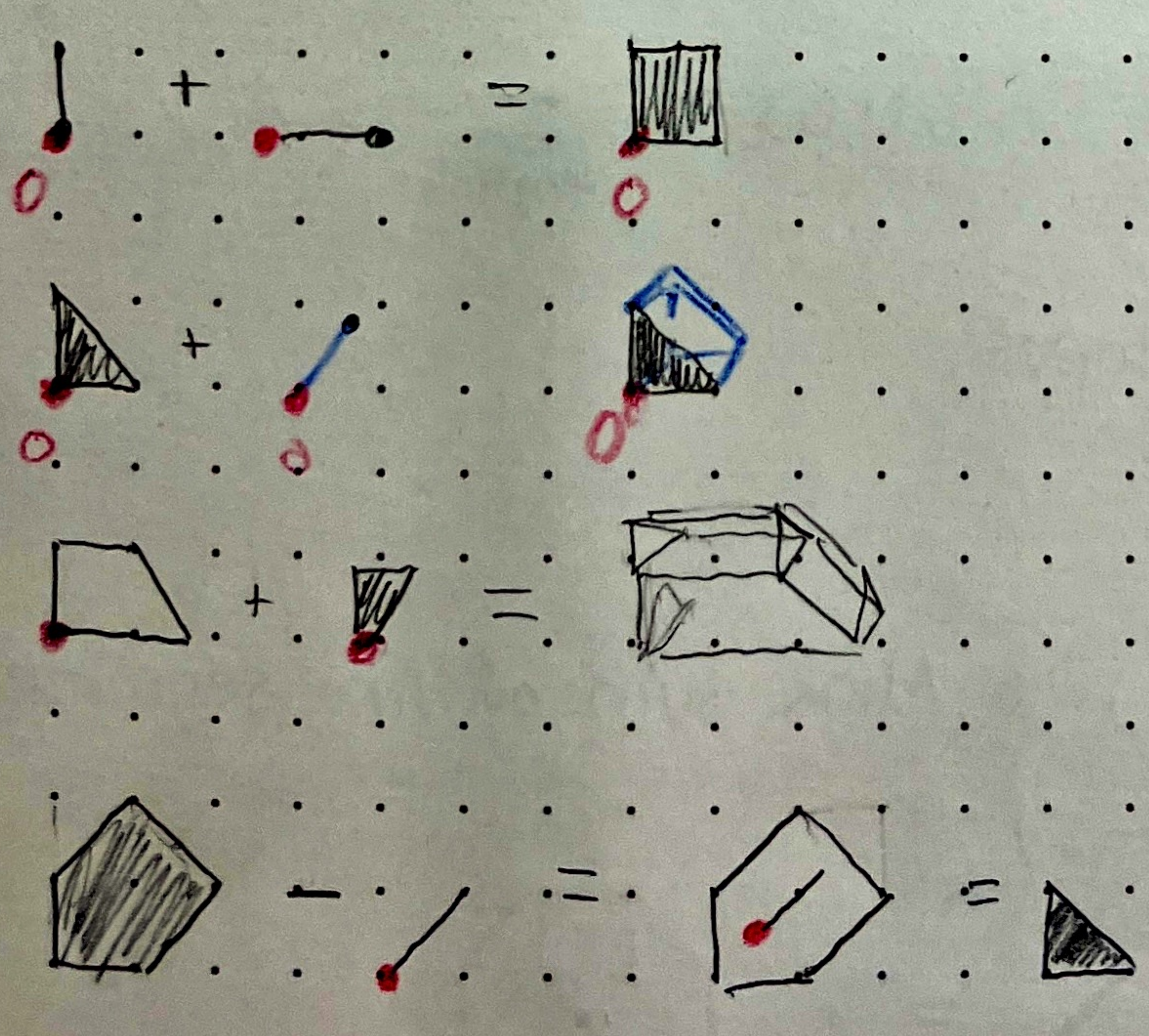
<u>Observation</u>: Parallel translations of A, B
⇝ Parallel translations of A+B

<u>Def</u>: Minkowski difference

$$A-B := \{\vec{c} \in \mathbb{R}^n \mid \{\vec{c}\} + B \subseteq A\}$$



<u>E.x.</u>



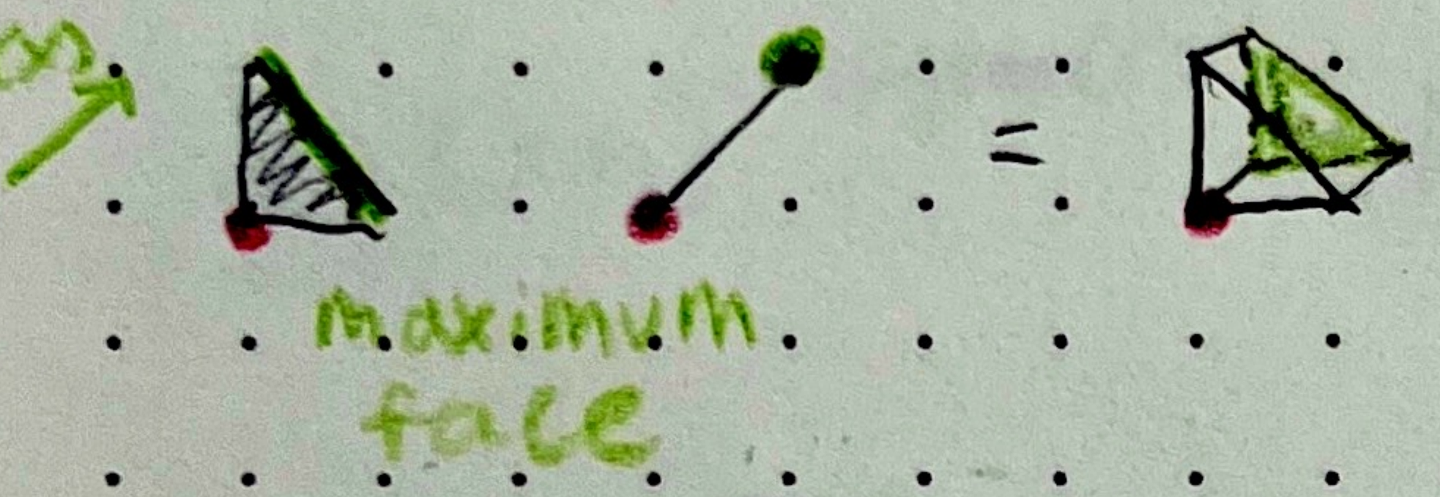
Lemma 1: Mink. sum. of polytopes is a polytope.

$$\mathrm{conv}(\vec{u}_1, \dots, \vec{u}_k) + \mathrm{conv}(\vec{v}_1, \dots, \vec{v}_\ell) = \mathrm{conv}(\{\vec{u}_i + \vec{v}_j\})$$

Proof: not written but "should be easy to figure out"

Lemma 2: P, Q polytopes
$a(x)$ lin. fcn on $\mathbb{R}^n$
The supporting faces of $P+Q$

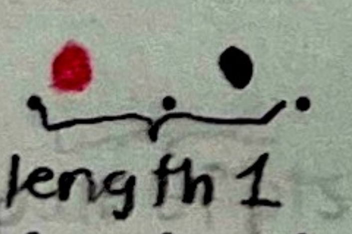
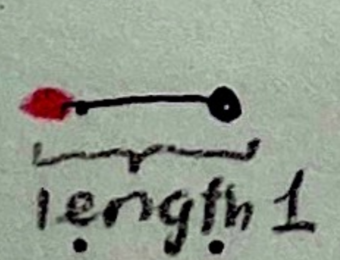
$$F_{a,P+Q} = F_{a,P} + F_{a,Q}$$

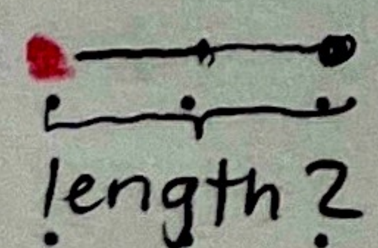
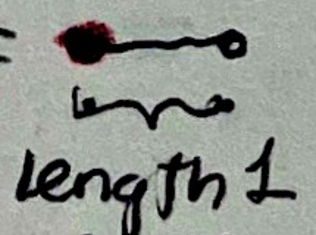
Ex.


Lemma 3: $A+B = B+A$
$(A+B)+C = A+(B+C)$

Lemma 4 $(A+B)-B \supseteq A$
$(A-B)+B \subseteq A$
} can be inequalities

counter Ex.: If B larger than A,
$A-B = \emptyset$ so $(A-B)+B = \emptyset + B = \emptyset \subsetneq A$

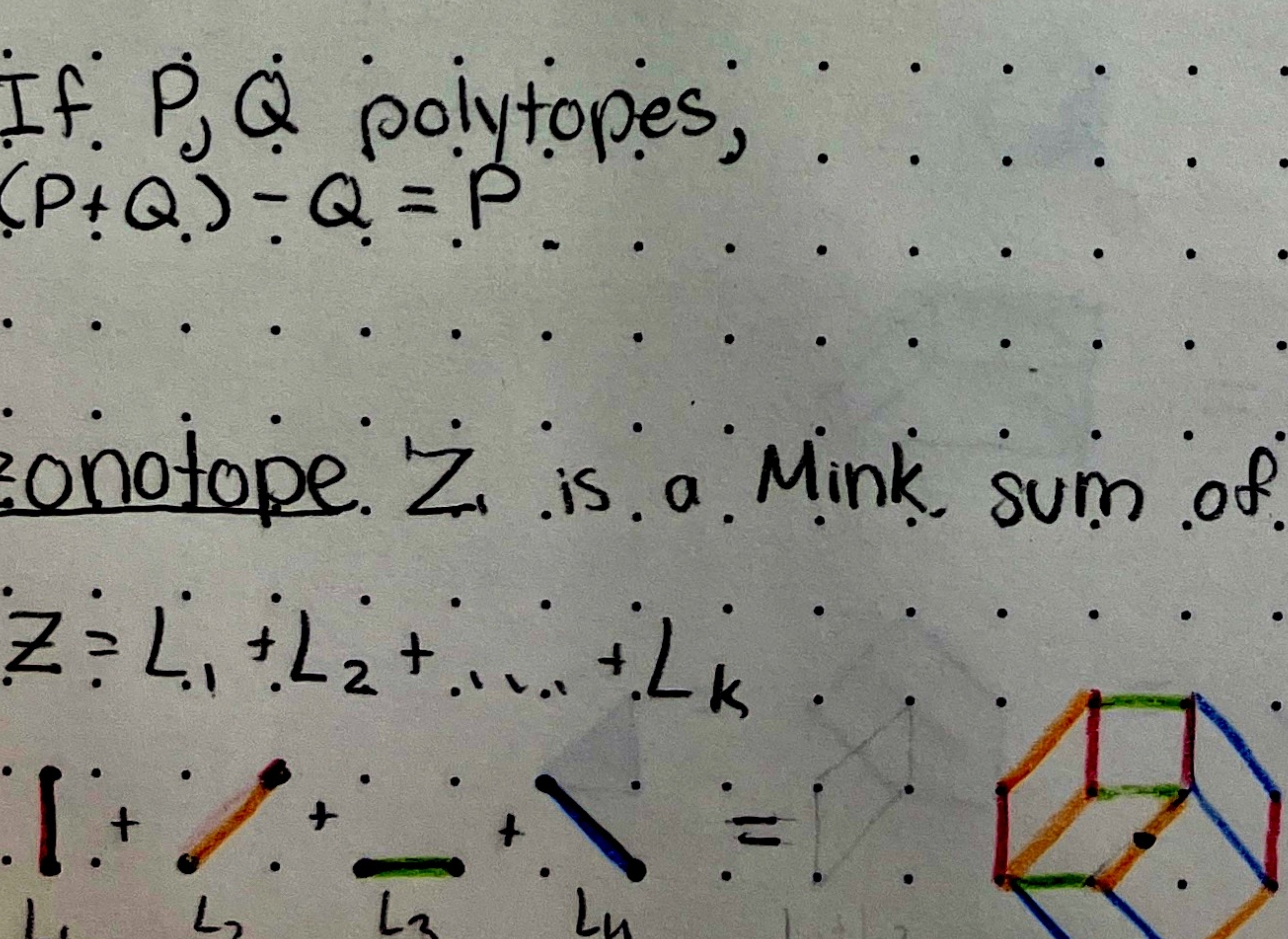
counter Ex. $A =$  $B =$ 

$A+B =$  $\Rightarrow (A+B)-B =$  $\supsetneq A$

Lemma 5: If P, Q polytopes,
$(P+Q)-Q = P$

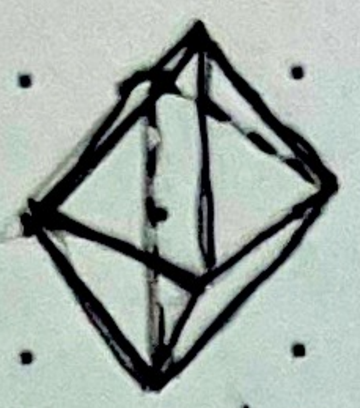
Def: A zonotope Z is a Mink sum of line segments L_i.

$$Z = L_1 + L_2 + \dots + L_k$$

Ex. $Z =$


Prop: 2-dim zonotopes are exactly centrally symmetric $2n$-gons

Note: NOT true in higher dim

counter E.x.  $:= \text{conv}((\pm 1,0,0), (0,\pm 1,0), (0,0,\pm 1))$

octahedron. Centrally symmetric, but has faces which are triangles $\Rightarrow$ not a zonotope

Obs: Any face of a zonotope must be a zonotope

Graphical Zonotopes

$G = (V,E)$ a graph. $V = [n] := \{1, 2, \ldots, n\}$

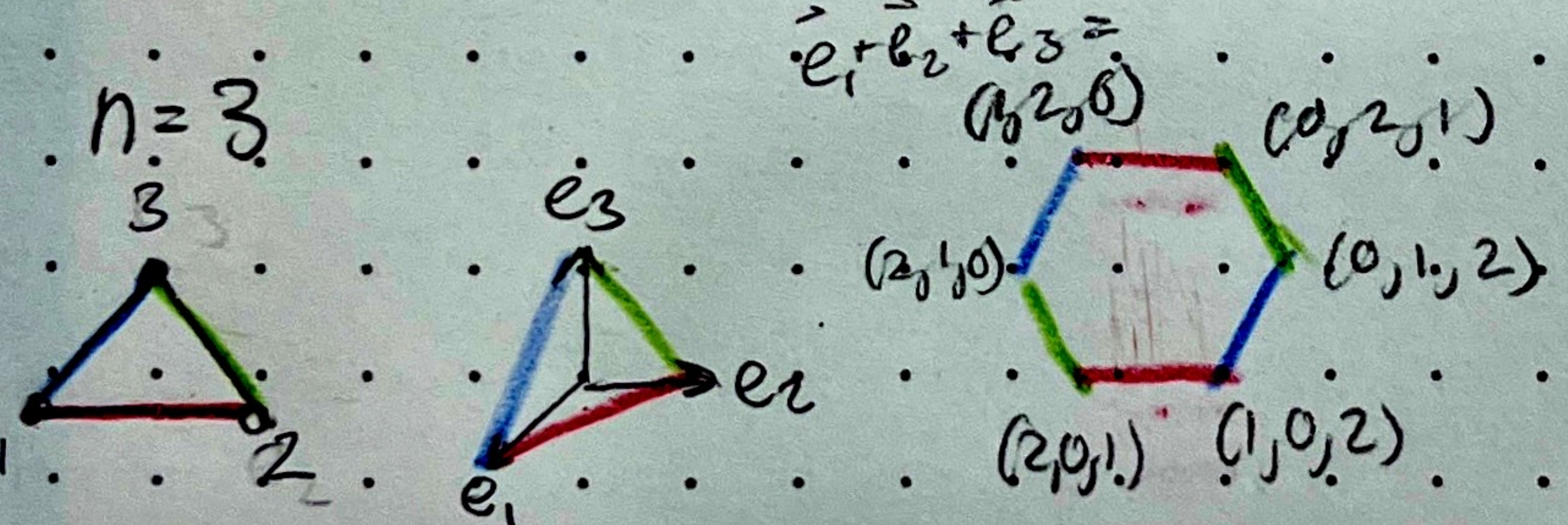
$\vec{e}_1, \ldots, \vec{e}_n$ std. coord vectors in $\mathbb{R}^n$

Def: $Z_G = \sum_{\{i,j\} \in E} [\vec{e}_i, \vec{e}_j]$

Mink. sum

Prop: K_n the complete graph. Z_{K_n} is the std. permutohedron (translated by $(-1,-1,\ldots,-1)$)

E.x. $n=3$



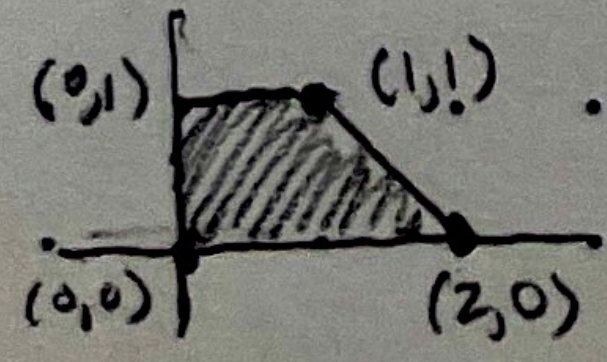
Def: Newton polytopes

$f(x_1, \ldots, x_n) = \sum c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$ i_j's can be negative

a (Laurent) polynomial

$\text{Newton}(f) := \text{conv}\{(i_1, \ldots, i_n) \in \mathbb{Z}^n \mid c_{i_1 \cdots i_n} \neq 0\}$

E.x. $\text{Newton}(1 - 3x^2 + 5y + 2024xy)$

$=$ 

Lemma 6: Newton$(f \cdot g)$ = Newton(f) + Newton(g)

True for vertices, need to check convex hulls also work

Def: Vandermonde determinant =

$$\det\begin{bmatrix} x_1^0 & \cdots & x_n^0 \\ x_1^1 & \cdots & x_n^1 \\ \vdots & & \vdots \\ x_1^n & \cdots & x_n^n \end{bmatrix} = \prod_{i<j}(x_i - x_j).$$

Why: $\det = 0$ if $x_i = x_j$. Can check via degree that these are all the roots.

Newton Polytope

$$\sum_{i<j}[\vec{e}_i, \vec{e}_j] = \text{Permutohedron}.$$

Let $A_n = \sum_{1 \le i \le j \le n} \operatorname{conv}(e_i, e_{i+1}, \dots, e_j)$

$$= \operatorname{Newton}\Big(\prod_{1 \le i \le j \le n}(x_i + x_{i+1} + \dots + x_j)\Big).$$

Exercise: Figure out the vertices of this polytope

$n=3$: