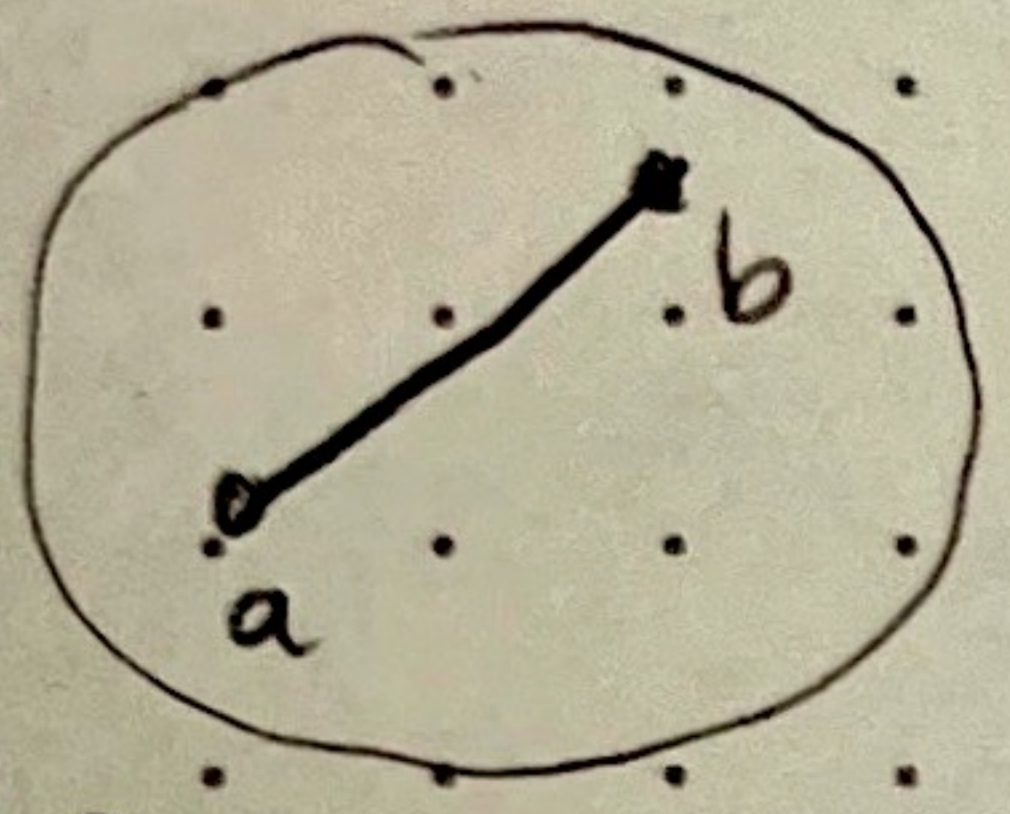
LECTURE 2 Fri 9/6

Polytopes: Basic Defs

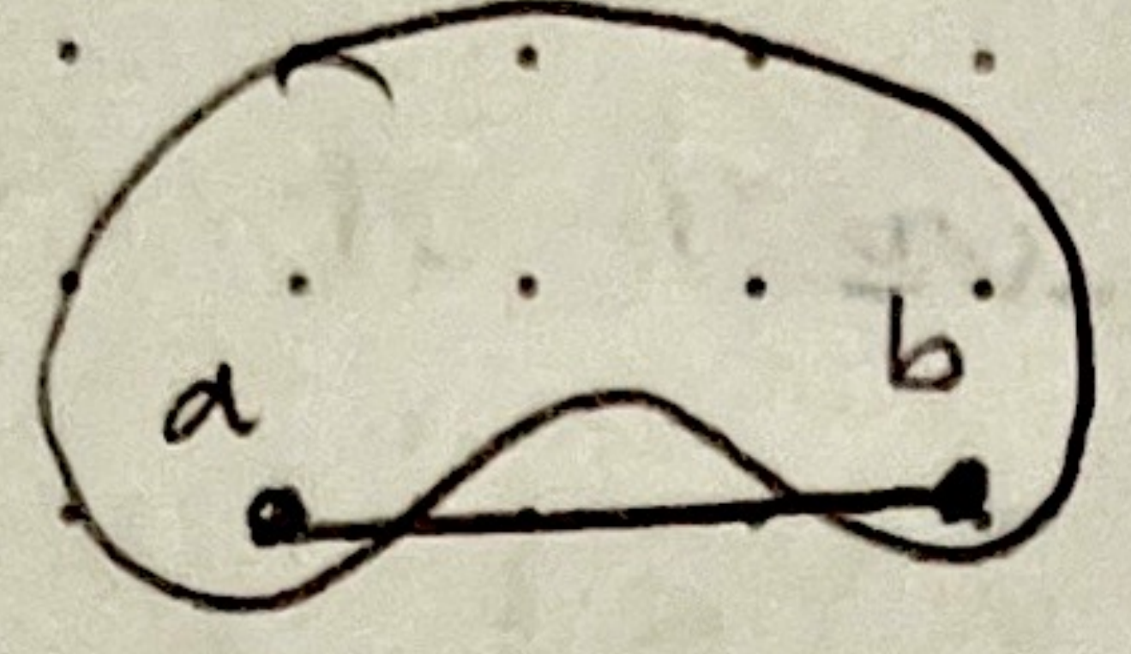
Ziegler: Lectures on Polytopes

Def (1): A set $A \subseteq \mathbb{R}^n$ is convex if $\forall\, a, b \in A$ the line segment from a to b
$$[a,b] := \{ ta + (1-t)b \mid 0 \le t \le 1 \}$$
is contained in A.

E.x.



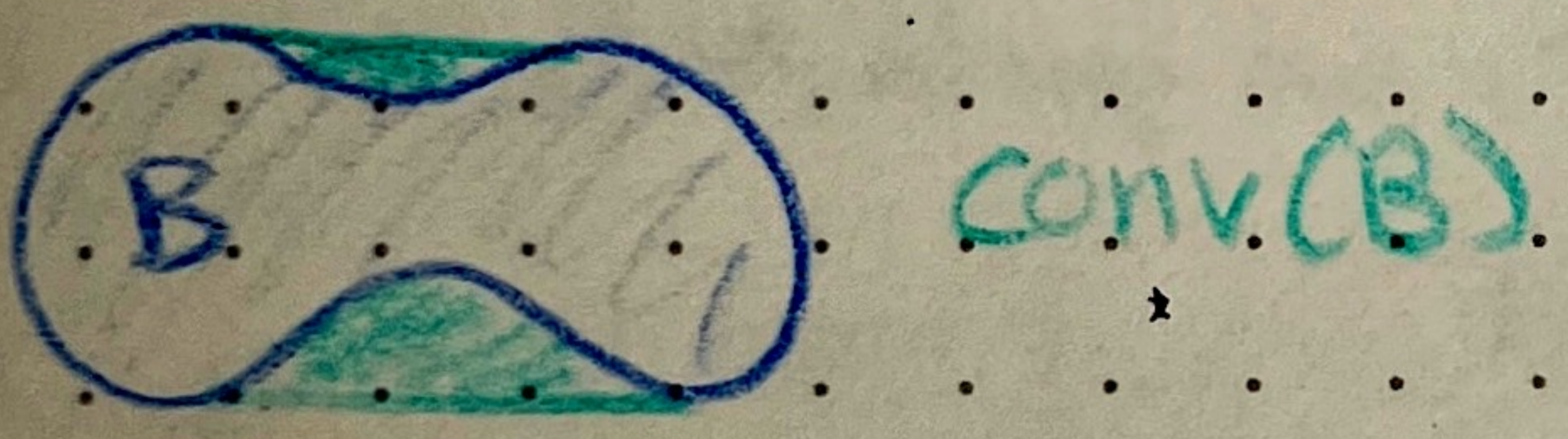
Convex



NOT convex

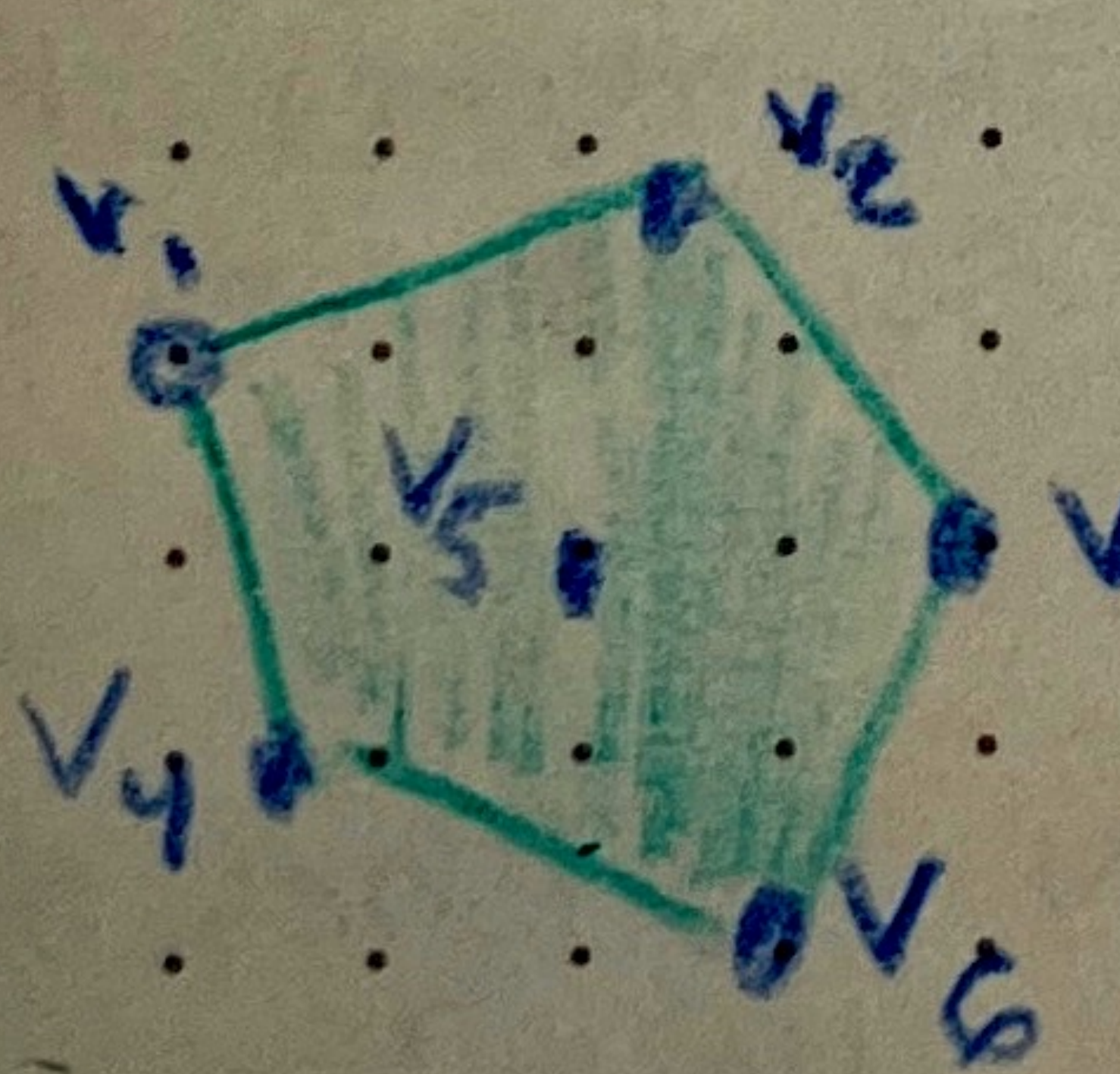
Def(2): For any $B \subset \mathbb{R}^n$, the convex hull of B is
$$\text{conv}(B) := \{ \alpha_1 b_1 + \dots + \alpha_k b_k \mid \forall\, b_i \in B \text{ and } \alpha_1, \dots, \alpha_k \in \mathbb{R}_{\ge 0},\ \alpha_1 + \dots + \alpha_k = 1 \}$$
= the minimal convex set containing B

E.x.



Def(3): A (convex) polytope is the convex hull of a finite set of points in $\mathbb{R}^n$.

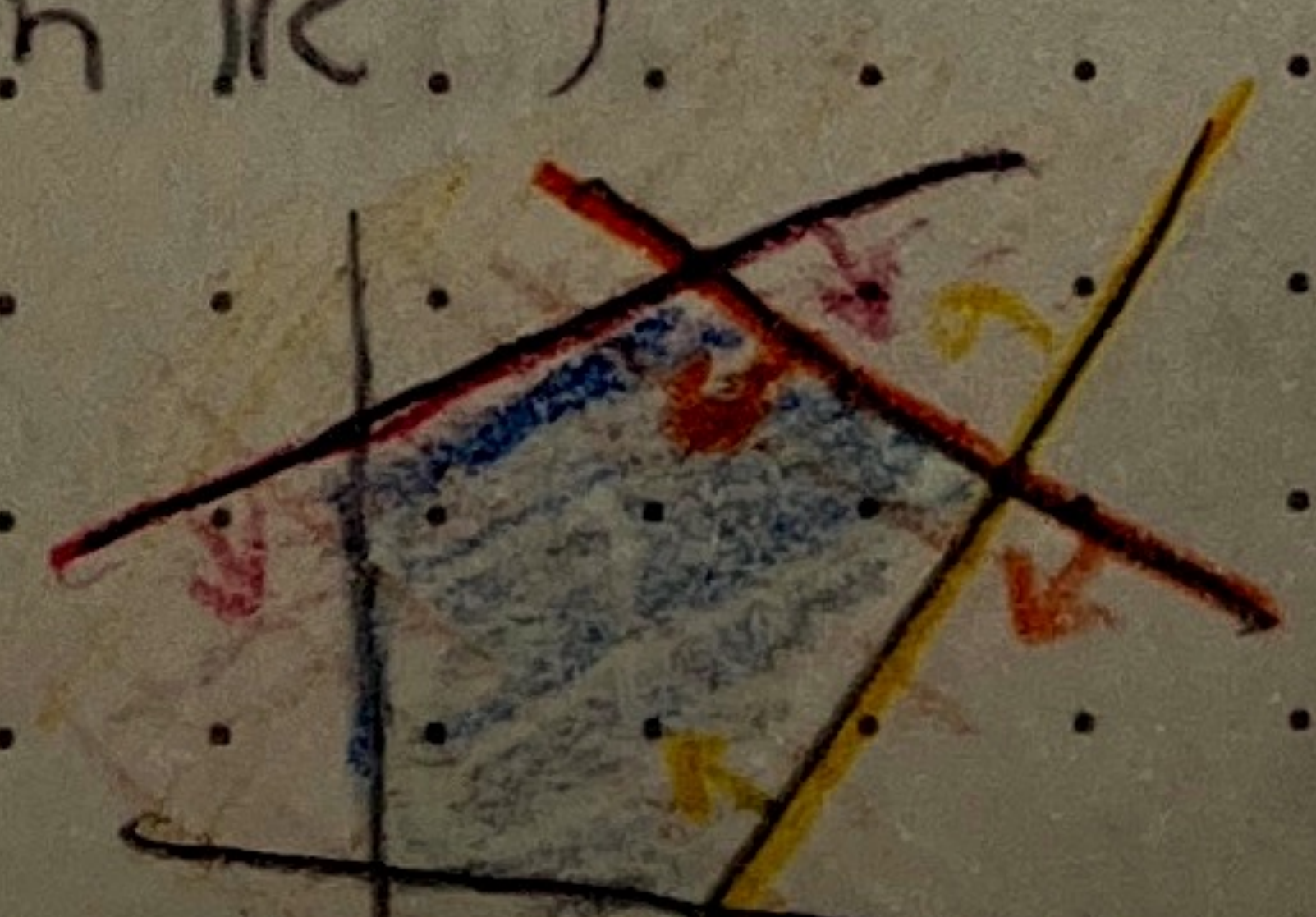
E.x. P =



$= \text{conv}(v_1, \dots, v_6)$.

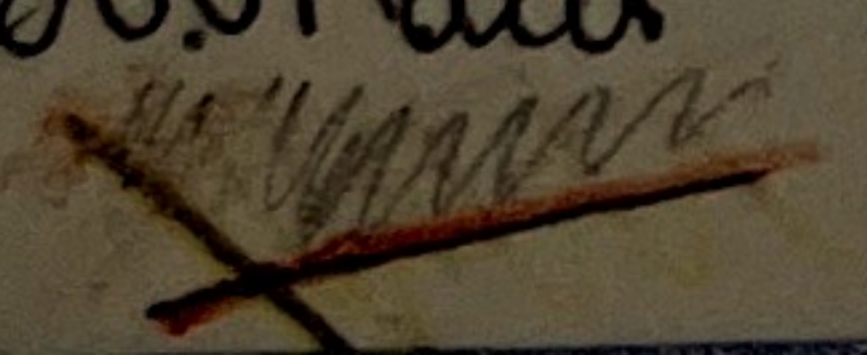
Def(4): A (convex) polyhedra $P \subseteq \mathbb{R}^n$ is a set given by finitely many weak linear inequalities ($\Rightarrow$ intersection of finite # of halfspaces in $\mathbb{R}^n$).

E.x. $P = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \begin{matrix} a_{11}x_1 + \dots + a_{1n}x_n \le b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \le b_m \end{matrix} \right\}$



Note: Not always bounded

e.g.



The Fundamental Thrm of Complex Polytopes

Polytopes are exactly bounded polytopes

V-polytopes in Ziegler's book . . . H-polytopes in Ziegler's book

$$P = \operatorname{conv}(v_1, \dots, v_k) = \{\vec{x} \mid A\vec{x} \le \vec{b}\}$$

Polor Duality

Def: $A \subseteq \mathbb{R}^n$. The polor dual of A is

$$A^* = \{\vec{x} \in \mathbb{R}^n \mid (\vec{a}, \vec{x}) \le 1 \ \forall \ \vec{a} \in A\}$$

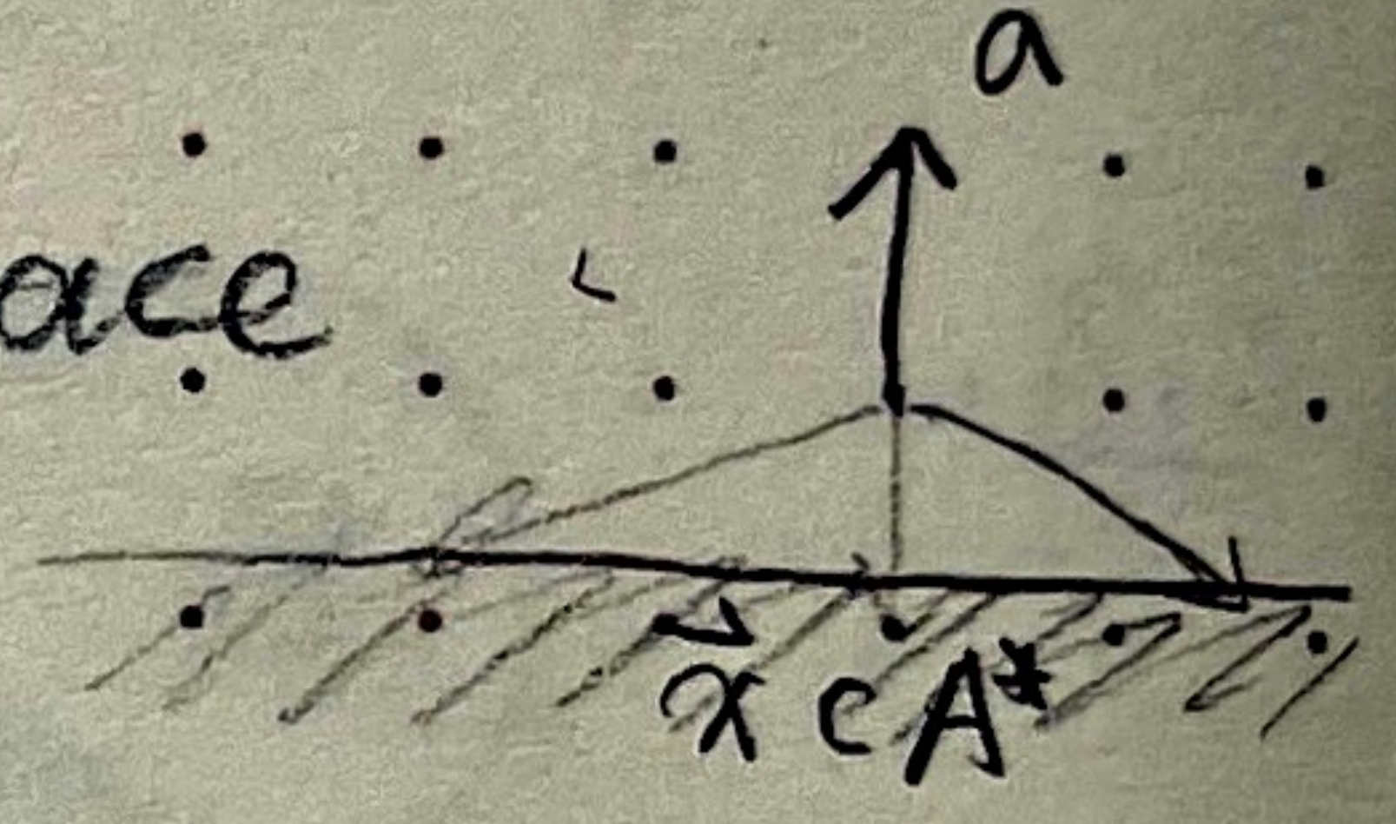
Observations: 0.) $(A^*)^* \supset A$

1.) $\vec{0} \in A^*$

b/c $(\vec{0}, \vec{x}) = 0 \ \forall \vec{x}$

2.) A^* is convex.

Why: For each $\vec{a}$, we get a half space



3.) A^* is closed

Exercise: Prove that for any polytope $P \subset \mathbb{R}^n$ s.t.

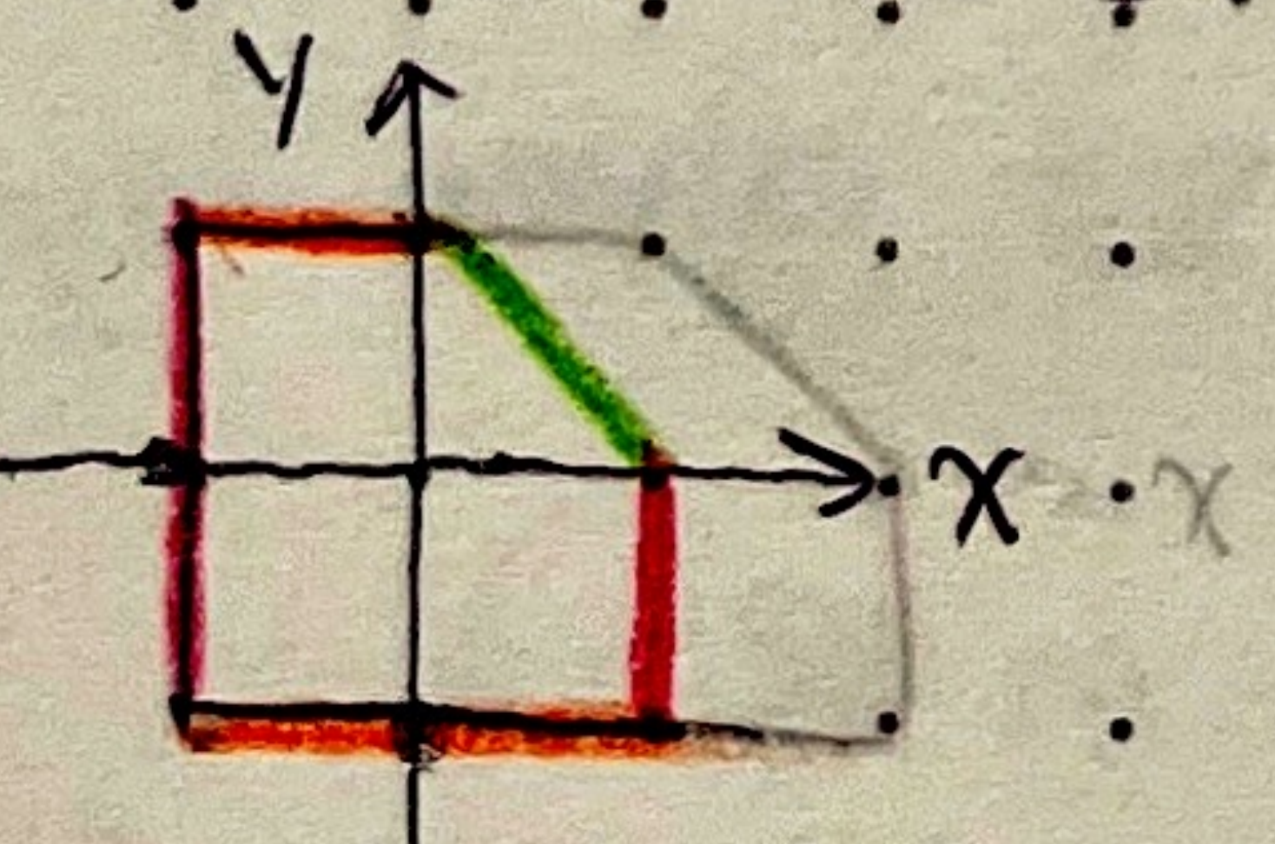
(i) P is full dimensional,

(ii) $\vec{0}$ is strictly inside (the interior) of P.

P^* is a polytope also satisfying (i) and (ii), then $(P^*)^* = P$

This notation of polor duality switches us from one presentation of polytope to the other

E.x.

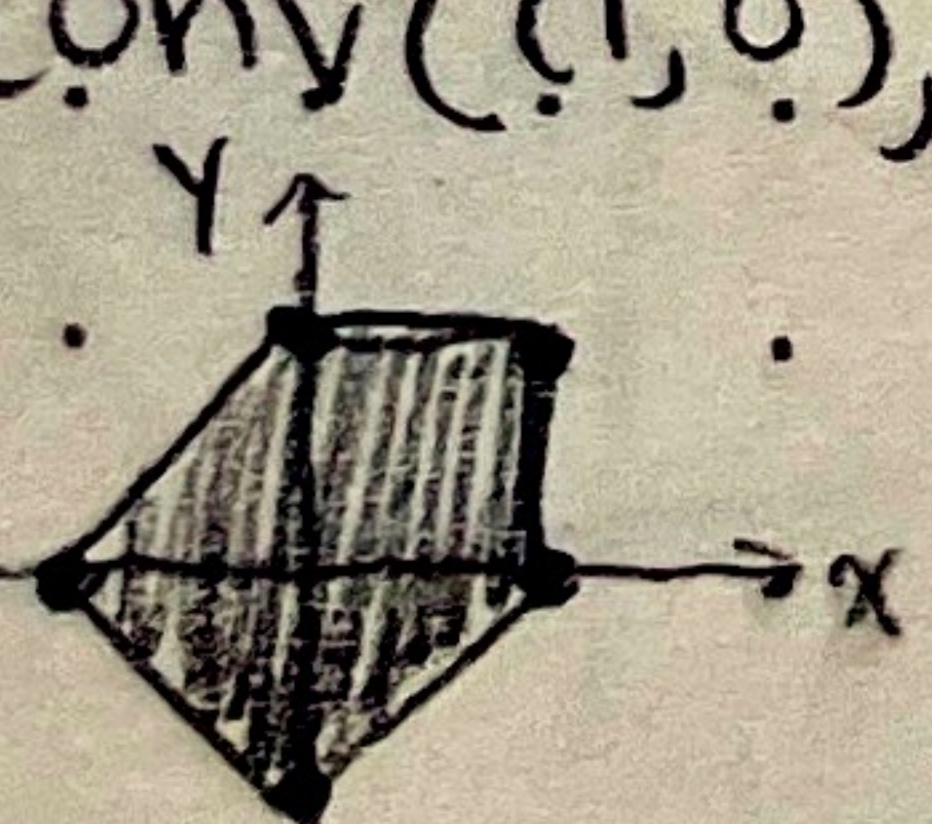


$$P = \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \\ x+y \le 1 \end{cases} = \begin{cases} -x \le 1 & (-1,0) \\ x \le 1 & (1,0) \\ -y \le 1 & (0,-1) \\ y \le 1 & (0,1) \\ x+y \le 1 & (1,1) \end{cases}$$

All ≤ 1 on the right

$$P^* = \operatorname{conv}((1,0), (-1,0), (0,1), (0,-1), (1,1))$$

$P^* =$



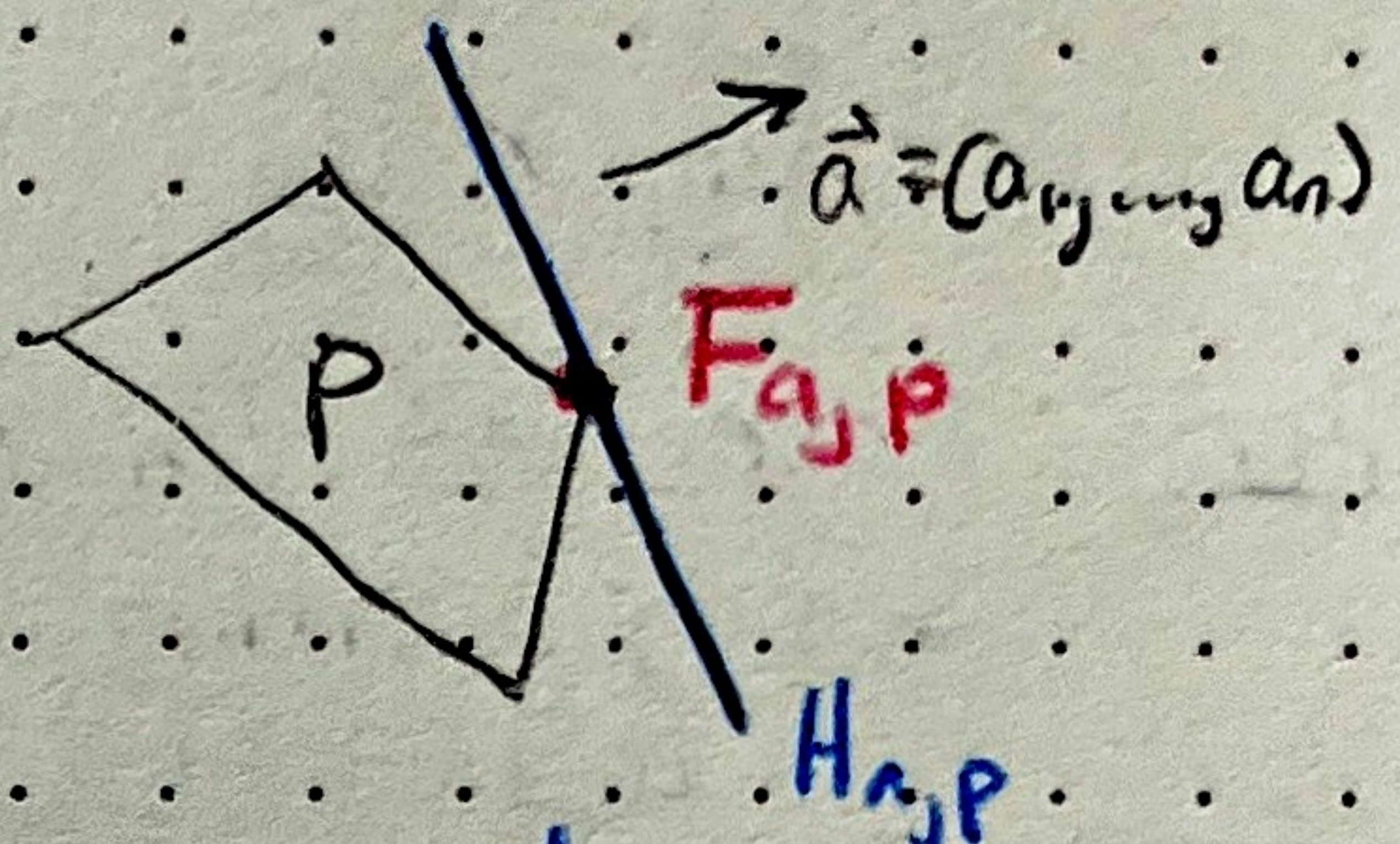
How to define faces of polytopes?

$a(x) = a_1 x_1 + \dots + a_n x_n$ a lin. function on $\mathbb{R}^n$
P polytope

$h = \sup(a(x) \mid x \in P) = \max(a(x) \mid x \in P)$
$= \max(a(v_1), \dots, a(v_m))$
if $P = \text{conv}(v_1, \dots, v_m)$

Def: the supporting hyperplane

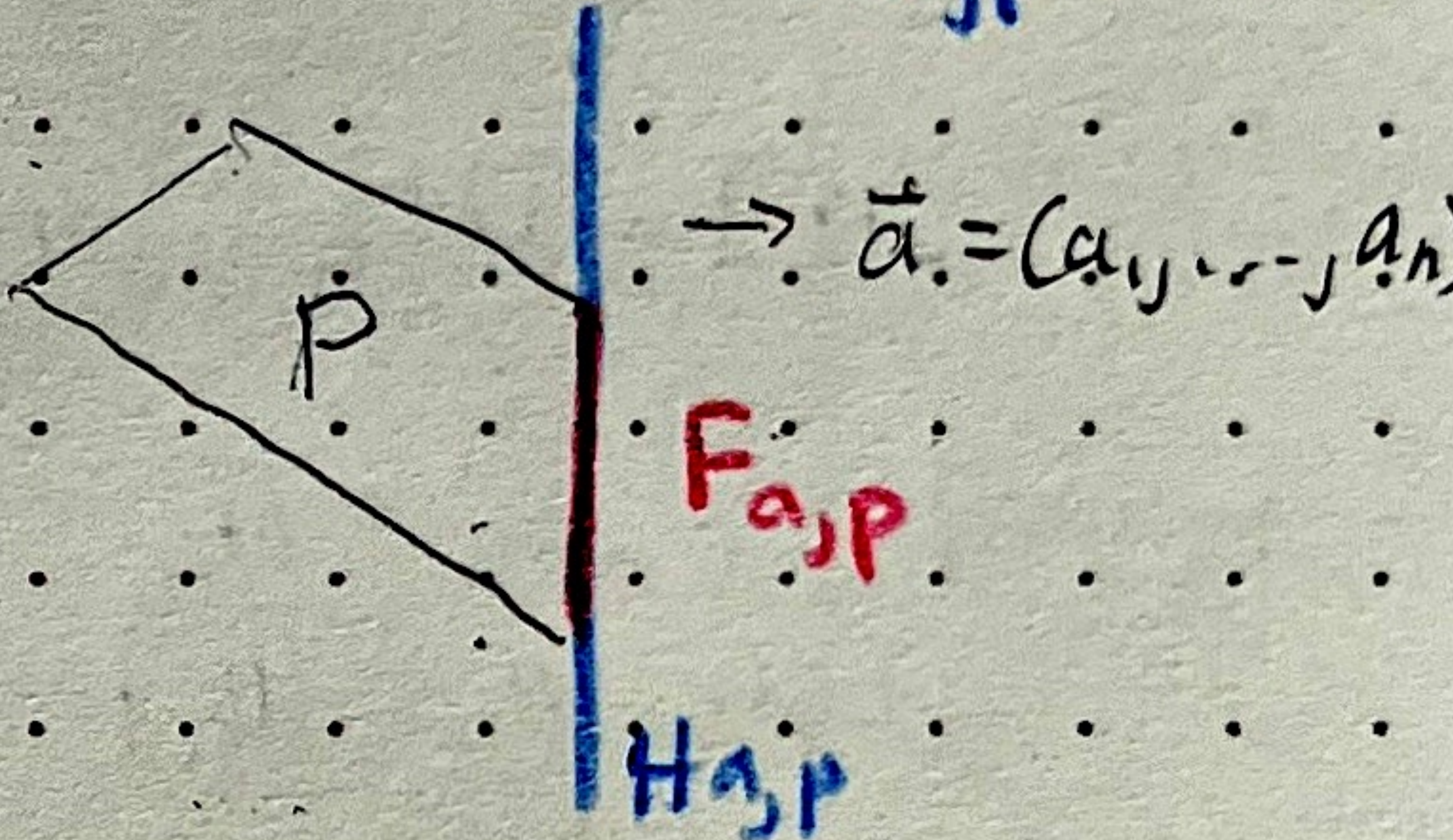
E.x.



$H_{a,P} = \{\vec{x} \in \mathbb{R}^n \mid a(x) = h\}$

The supporting face

$F_{a,P} := P \cap H_{a,P}$



Most degenerate case: $a(x) = 0$
Then $H_{a,P}$ is all of $\mathbb{R}^n$
and $F_{a,P}$ is all of P.

Def: vertices - 0-dim. faces
edges - 1-dim. faces
facets - faces of codim 1

Ex. The (standard) permutohedron

$\Pi_n := \text{conv}((w_1, \dots, w_n) \mid w_1, \dots, w_n \text{ a perm of } 1, \dots, n)$

$a(x) = a_1 x_1 + \dots + a_n x_n$

$h = \max_{w \text{ perm}} (a_1 w_1 + \dots + a_n w_n)$

WLOG can re-arrange so $a_1 \leq a_2 \leq \dots \leq a_n$
Then $h = a_1 \cdot 1 + a_2 \cdot 2 + \dots + a_n \cdot n$.

If $a_1 < a_2 < \dots < a_n$ (all strict) then $F_{a,\Pi_n} = \{(1, 2, \dots, n)\}$.
Inequalities give us faces F_{a,Π_n} of higher dimension.

$$\underbrace{a_1 = a_2 = \dots = a_{n_1}}_{n_1} < \underbrace{a_{n_1+1} = a_{n_1+2} = \dots = a_{n_1+n_2}}_{n_2} < \dots$$

$$F_{a,\Pi_n} = \mathrm{conv}\left((w_1, \dots, w_n) \;\middle|\; \begin{array}{l} w_1, \dots, w_{n_1} \text{ a perm of } 1, 2, \dots, n_1 \\ w_{n_1+1}, \dots w_{n_1+n_2} \text{ a perm of } n_1+1, \dots, n_1+n_2 \\ \text{etc} \end{array}\right\}$$

$$\simeq \Pi_{n_1} \times \Pi_{n_2} \times \dots \times \Pi_{n_k}$$

Prop: Faces of Π_n are in bijection with ordered set partition $\pi_1 = (B_1 | B_2 | \dots | B_k)$ of $[n] := \{1, 2, \dots, n\}$

$[n] = \cup B_i$ disjoint union of non-empty blocks

$F_\pi \simeq \Pi_{n_1} \times \dots \times \Pi_{n_k}$ where $n_i = |B_i|$ | $\dim F_\pi = n-k$

$\uparrow$ sum of $\dim \Pi_{n_i} = n_i - 1$