Last Time: Operators acting on $\mathbb{C}[x_1, \ldots, x_n]$

- Div. diff operators $d_i : f \mapsto x_i \frac{\partial}{\partial x_i} f$
- Demazure operators $D_i : f \mapsto \frac{1}{x_i} x_i \frac{\partial}{\partial x_i} f$

Let $x_i : f \mapsto x_i f$

Then $D_i = x_i \circ D_i$ i.e. $D_i f = x_i (D_i f)$

For $w = s_{i_1} \cdots s_{i_k} \in S_n$ reduced decomp. of $w$

$D_w = D_{i_k} \cdots D_{i_1}$ and $D_w = D_{\theta_w}$

2 formulas for Schur polynomials

$S_\lambda(x_1, \ldots, x_n) = \omega = \omega_\lambda(x^{\lambda+\delta}) = \omega_{\lambda}(x^{\lambda})$

Thm: $\omega = \omega_{\lambda}(x^{\lambda+\delta})$ and $D_w = D_w(x^{\lambda+\delta})$

Example: $n = 3$, $w_0 = s_1 s_2 s_3 \in S_3$

$2, 2, 2, x_1, x_2, x_3 = 2, 2, 2, x_1, x_2, x_3$

Problem: $\theta_i$'s don't commute with $D_j$'s

Lemma: $D_j x_i = x_i D_j$ if $j \notin \{i, i+1\}$

$D_j x_i = (x_i x_{i+1}) \theta_i$

Why? $D_j$ only affects $x_i$ and $x_{i+1}$, doesn't affect any others so for other $j$ we can commute.

$D_j$ is symmetric wrt $x_i$ and $x_{i+1}$

Example:

$2, 2, 2, x_1 x_2 x_3$

$= 2, 2, 2, x_1 x_2 x_3$

Proof (for general $n$):

- Write $w_0 = (s, s_2 \cdots s_n, (s, s_2 \cdots s_{n-2}) \cdots (s, s_2) s_1)$ reduced decomp.
- Wiring diagram:

  $\begin{array}{c}
  2, 2, 2, x_1, x_2, x_3 \\
  \vdots \\
  2, 2, 2, x_1, x_2, x_3 \\
  2, x_1, x_2, x_3 \\
  (2, x_1) \\
  \end{array}$

Can move all $X$'s in each line to the right

Now all combined $X$ terms symmetric wrt operad on the right, so we can commute them to get exactly what we want.
**Example**

$n = 3$

\[ w_0 : \text{Dw}_0 = 2w_1x_1^2x_2 \]

\[ s_1s_2 : a_1x_1a_2x_2 = 2a_2x_1x_2 \]

But

\[ s_2s_1 : 2_2x_2x_1 \neq 2_2x_1x_2 \]

Claim: Can write the $a_i$ first then $x_i$ iff $w$ is $312$ avoiding

Schubert Poly \( G_w = D_w(x^5) \)

Demazure char \( \chi_{aw} = D_w(x^\lambda) \)

**Question:** When is \( G_w = \chi_{aw} \)?

\( \chi = (o_1 \ldots 2\lambda) \), \( w \in S_n \)

**Theorem 1:** If \( w \) is a $312$-avoiding perm in \( S_n \), then \( \chi_{aw} = G_w \) for some perm \( u \in S_m \).

**Theorem 2:** If \( u \) is a $2143$-avoiding perm in \( S_m \), then \( G_u = \chi_{uw} \) for some \( \lambda, \), \( w \)

**Combinatorial Def of \( S_n \):**

\[ S_n(x_1 \ldots x_n) = \sum_{T \in SSYT of shape \lambda} x^{\text{weight}(T)} \]

**Example:**

\[ 11111 \ 3 \ 2 \ 2 \ 3 \ 2 \ 1 \]

**Weight:** \((\#1's, \#2's, \ldots, \#n's)\)

**Combinatorial formula for \( G_w \):**

RC-graphs aka pipe dreams

[Formin-Stanley, Billey-Jakushi-Stanley]

**Def:** Pipe Dream

\[ \begin{align*}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 4 & 5 & 3 \\
2 & 3 & 4 & 5 & 1 \\
2 & 3 & 4 & 5 & 1 \\
3 & 4 & 5 & 1 & 2 \\
3 & 4 & 5 & 1 & 2 \\
4 & 5 & 1 & 2 & 3 \\
4 & 5 & 1 & 2 & 3 \\
5 & 1 & 2 & 3 & 4 \\
5 & 1 & 2 & 3 & 4 \\
\end{align*} \]

No double crossings

Any two wires intersect at most once

**Def:** weight \( (p) = (\beta_1, \beta_2, \ldots, \beta_n) \)

\( \beta_i = \# \text{ crossings in } i^{th} \text{ row} \)

**Theorem:** \( G_w = \sum x^{\text{weight}(p)} \)

**Example:**

\[ n = 3 \]

\[ G_w = x_1^2x_2 \]

\[ G_{s_1} = x_1x_2x_3 \]

\[ G_{s_2} = x_1^2x_3\]

\[ G_{s_3} = x_1x_2 \]

**Two options:** \( G_{s_3s_2} = x_1 + x_2 \)
Schubert Polynomials \( G_w \), \( w \in S_n \\n
- Defined div. diff. operators \( \partial_i \): 
  1. \( G_w = x_1 \cdots x_{\lambda_i - 1} x_{\lambda_i} \) 
  2. \( G_w = \partial_i(G_{w^i}) \) if \( \lambda(ws_i) = \lambda(w) + 1 \) 
- A combinatorial rule for \( G_w \) via pipe dreams 

Thm. [Billey-Tokush-Stanley] or [Fomin-Stanley] 
\[
G_w = \sum_{\text{pipe dream for } w} x^{\text{weight}(p)}
\]

\[\text{Ex.} \quad 3 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \]

\[\text{it should be reduced} \quad (\text{any 2 wires intersect at most once}) \]

\[w = 31542, \quad x^{\text{weight}(p)} = x_1^3x_2x_3\]

Nil Coxeter algebra \( N_n \) (over \( \mathbb{C} \)) 
- generators \( u_1, u_2, \ldots, u_{n-1} \) 
- relations 
  1. \( u_i^2 = 0 \) 
  2. \( u_i u_j = u_j u_i \) if \( |i - j| \geq 2 \) 
  3. \( u_i u_1 u_i u_{i+1} = u_{i+1} u_i u_1 u_i \) 

Today: Want to prove Thm. 
Show formula satisfies relation 
(1) is "obvious". Just draw pipe dream diagram for \( w_0 \) 
(There is exactly 1 way to do this)

Linear basis of \( N_n \) 
\( \{uw, w \in S_n\} \)

\[w = s_1 s_2 \cdots s_{i_k} \quad \text{(reduced decomp)} \]

\[uw = u_1 u_2 \cdots u_i \]

NOTE: If \( s_1 \cdots s_{i_k} \) is not reduced then \( u_1 \cdots u_{i_k} = 0 \).

\[v, w \in S_n \]

\[uv \cdot uw = \begin{cases} uw & \text{if } \lambda(uvw) = \lambda(u) + \lambda(w) \\ 0 & \text{otherwise} \end{cases} \]

Commutative variables \( x_1, x_2, \ldots, x_n \) that also commute with \( u_i \)'s

\[h_i(x) := 1 + x u_i \]

\[A_i(x) := h_{n-i}(x) h_{n-2}(x) \cdots h_i(x) \]

\[G := A_1(x_1)A_2(x_2) \cdots A_n(x_n) \]
Thm: [F.S.]

\[ G = \sum_{w \in S_n} G_w(x_1, x_2, \ldots, x_n) U_w \]

\[ \xi_n \]

\[ n = 3 \]

\[ G = A_i(x_i) A_2(x_2) \]

\[ = h_3(x_i) h_4(x_2) h_2(x_2) \]

\[ = (1 + x_1 U_1)(1 + x_2 U_2)(1 + x_2 U_2) \]

\[ = 1 + x_1 U_1 + (x_1 + x_2) U_2 + x_1 x_2 U_1 U_2 + x_1^2 U_2 U_1 + x_1^2 x_2 U_1 U_2 \]

For \( w = \) And the for each crossing can undo it (corresponds to the 1 term) or keep (corresponds to \( x_1 U_1 \) term)

Let \( G = \sum_{w \in S_n} \tilde{G}_w(x_1, \ldots, x_n) U_w \)

\[ \text{WTS } \tilde{G}_w = G_w \]

Need to check \( (1) \) \( \tilde{G}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \)

\[ (2) \tilde{G}_{w} = \left( \tilde{G}_{w_{S_1}} \right) \text{ if } l(w S_i) = l(w) + 1 \]

We need to prove the identity

\[ (x) \tilde{G} \Leftarrow (G) = G : U_i \]

\[ a_i(G) = \sum_{w \in S_n} \tilde{G}_w U_i U_w \]

\[ G U_i \Leftarrow \sum_{w \in S_n} \tilde{G}_w U_i U_{w_S_i} \]

Proof of \( (x) \) \( \tilde{G} \Leftarrow (A_i(x_i) A_2(x_2) \cdots A_i(x_i) A_{i+1}(x_{i+1}) \cdots A_{i-1}(x_1) U_i) \]

\[ \text{ETS } \tilde{G} \Leftarrow (A_i(x_i) A_{i+1}(x_{i+1})) = A_i(x_i) A_{i+1}(x_{i+1}) U_i \]

\[ (**) \]

Lemma 1:

1. \( h_i(x) h_i(y) = h_i(x+y) \quad h_i(0) = 1 \)
2. \( h_i(x) h_j(y) = h_j(y) h_i(x) \) if \( |i-j| > 2 \)
3. \( h_i(x) h_{i+1}(x+y) h_i(y) = h_{i+1}(y) h_i(x+y) h_{i+1}(x) \)

\( \text{Yang-Baxter relations} \)

Lemma 2: \( A_i(x) \) & \( A_i(y) \) commute with each other

\[ A_i(x) A_i(y) = A_i(y) A_i(x) \] (expression symmetric in \( x \) & \( y \))

Lemma 3: \( A_i(x) A_{i+1}(y) U_i = A_i(y) A_{i+1}(x) U_i \)

Lemma 4: \( A_i(x) A_{i+1}(y) - A_i(y) A_{i+1}(x) = (x-y) A_i(x) A_{i+1}(y) U_i \)
Lemma 4 \Rightarrow (\star\star)

Ex. of Lemma 2 \quad n=3, i=2
\[ h_n(x)h_i(y) = h_n(x+y) = h_n(y) + h_n(x) \]

\[ h_2(x)h_1(x)h_2(y)h_1(y) \]
\[ A_i(x) \quad A_i(y) \]
\[ = h_2(x)h_1(x)h_2(y)h_1(y) \]
\[ = h_2(x)h_1(x)h_2(y)h_1(y) \]
\[ = h_2(x)h_1(x)h_2(y)h_1(y) \]
\[ = h_2(x)h_1(x)h_2(x)h_1(x) \]
Schub Polys $\rightarrow$ pipe dreams

$\lambda = (\lambda_1, \ldots, \lambda_k) \in k \times (n-k)$ rectangle

Partition

$w(\lambda) \in S_n$ Grassmannian perm., $w(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_{k+1}$

Then $S_\lambda(x_1, \ldots, x_k) = G_w(\lambda)$

Comb. Formula for $S_\lambda(x_1, \ldots, x_k)$

Thm: $S_\lambda(x_1, \ldots, x_k) = \sum_{T \in SSYT(\lambda)} \prod_{\text{# boxes in box } T} x_{\text{# entries in box } T}$

Recall: $SSYT(\lambda) =$ semi-standard Young tableaux of shape $\lambda$

Pipe dream $\rightarrow$ SSYT formula for $S_\lambda$

(want to see SSYT formula as special case of pipe dreams)

Claim: Pipe dreams $\pi$ for $w(\lambda)$ are in bij. with $SSYT(\lambda)$

Rule: Order the wires of $\pi$ from top to bottom by their left end

For $i = 1, \ldots, k$,

# crossings of $i$th wire = # entries $k+1-j$ in $k+1-i$th row of $\pi$

Example: $n = 12, k = 3$

$\lambda = (9, 5, 2)$

$w(\lambda) = 3, 7, 12, 1, 2, 4, 5, 6, 8, 9, 10, 11, 12$

Each grid point either $+$ or $-$

(Should be drawn inside boxes, oops)

Always get $+$ or $-$ have choices (calculate rows from bottom up)
Thm: $S_n(x_1, x_2, \ldots, x_k) = \sum_{(i:j) \in S_n} x_i^{a(i)}$

Lemma: $S_n = (x_1 \ldots x_k) = G_{(0)}$ is symmetric in $x_1, \ldots, x_k$

Lemma: $G_w$ is symmetric in $x_i$ if $\ell(ws_i) = \ell(w) + 1$

**Double Schubert Polynomials**

$G_w(x_1, y_1, \ldots, x_n, y_n) \quad w \in S_n$

Almost do the same as for usual Schub

(1) $G_w(x_i, y_j) = \prod_{(i:j) \in w} (x_i - y_j)$

(2) $G_w(x_i, y_j) = 2^i \cdot G_{ws_i}(x_i, y_j)$ if $\ell(ws_i) = \ell(w) + 1$

Prop. Positivity: $G_w(x_1, \ldots, x_n, y_1, \ldots, y_n)$ has positive int. coeffs.

**Pipe dream formula for $G_w(x_i, y_j)$**

Thm: $G_w(x_i, y_j) = \sum \prod_{P \text{ pipe dream}} (x_i - y_j)$

for $w$ crossing in the $i$th row

Cor (Symmetry): $G_w(x_1, \ldots, x_n, y_1, \ldots, y_n) = G_w(-y_1, \ldots, -y_n, -x_1, \ldots, -x_n)$

**Algebra (over $\mathbb{C}$)**

Generators: $u_1, \ldots, u_n, x_1, x_2, \ldots, x_n$

Relations: $x_i, y_j$ commute with each other and $u_i$'s

(1) $u_i^2 = 0$

(2) $u_i u_j = u_j u_i$ if $|i - j| \geq 2$

(3) $u_i u_i + u_i u_i = u_i u_i u_i$

Prop. Satisfy YB relations

$G_{x_i, y_j} = \prod_{(i:j) \in \text{order (ij) as }} (x_i - y_j)$

Order (ij) as $(1, n-1)(1, n-2) \ldots (1, 1)$

$(2, n-2)(2, n-3) \ldots (2, 1)$

$(3, n-3)(3, n-4) \ldots (3, 1)$

$(n-1, 1)$
Cauchy identity for $s_{\alpha}$

$$\sum_{\lambda} s_{\lambda}(x_1, ..., x_n) s_{\lambda}(y_1, ..., y_n) = \prod_{i,j} \frac{1}{1-x_i y_j}$$

let $x^\alpha y^\beta = x_1^{\alpha_1} ... x_n^{\alpha_n} y_1^{\beta_1} ... y_n^{\beta_n}$

coeff of $x^\alpha y^\beta = \# \{(P, Q) | P, Q \text{ SSYT of shape } \lambda \text{ s.t. } \alpha = \text{wt}(P) \}$

LHS: $[x^\alpha y^\beta] = \# \{(P, Q) | P, Q \text{ SSYT of shape } \lambda \}$

RHS: $\prod_{i,j} \frac{1}{1-x_i y_j} = \prod_{(x_i, y_j) \in A} \prod_{i,j=1}^{n \times n} (x_i y_j)^{A_{i,j}}$

= $\sum_{A=(A_{i,j})} \prod_{i,j=1}^{n \times n} (x_i y_j)^{A_{i,j}}$

$\text{row sums: } a+b = f, \text{ col. sum: } a+c = e, \text{ row sums: } a+b = f$

Robinson-Schensted-Knuth correspondence (RSK)

Robinson-Schensted correspondence

special case where $\alpha = (\beta = (\lambda_1, ..., \lambda_n))$

$P, Q$ are SYT's of the same shape

$A$ is permutation matrix

$S_n \xrightarrow{RS} \{(P, Q) \text{ SYT's of same shape } \lambda \}$

Example of RSK $n=2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{RSK} (P, Q)$

$P = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{RS} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 \end{pmatrix}$

$Q = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{RS} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{RSK} \begin{pmatrix} \lambda_1 & \lambda_1 - e \\ \lambda_2 & \lambda_2 \end{pmatrix}$

$\text{col. sum: } a+c = e, \text{ row sums: } a+b = f, \text{ col. sum: } a+c = e, \text{ row sums: } a+b = f$

These 4 eqn are not lin. independent $\Rightarrow$ Not enough to solve for RSK correspondence

Answer: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{RSK} \min(b, c)$

In general can view RSK as piecewise linear transform of matrices
Classical Construction of RSK

Ex.

\[ A = \begin{pmatrix} \ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \rightarrow \text{biword } w \text{ with } a_{ij} \text{ entries } \left( \begin{array}{c} i \\ j \end{array} \right) \text{ arranged lexicographically} \]

\[ w = \left( \begin{array}{ccc} 2 & 1 & 4 \\ 1 & 2 & 3 \end{array} \right) \]

1 in pos \(2,1\) \(2,3\) \(2,4\)

\(1,2\) is pos \(1,3\) \(1,4\)

will insert entries into \(P \times Q\) using Schnested insertion algorithm

\[ \begin{array}{cccc} 1 & 1 & 1 \text{ (3,3)} & 3 & 3 & 8 \\ 2 & 2 & 2 & \text{ (4,5)} & \rightarrow & j = 2 \\ 3 & 4 & 6 & 6 & 7 \\ 5 & & & & \end{array} \]

1st entry in 1st row which \(> j\)

1st entry in 2nd row which \(> 3\)

1st entry in 3rd row which \(> 9\)

\[ \begin{array}{cccc} 1 & 1 & 1 & 2 & 3 & 8 \\ 2 & 2 & 2 & 3 & 5 \\ 3 & 4 & 4 & 6 & 7 \\ 5 & 6 & \end{array} \]

Back to Ex. from top

\( (2) \) \( (2) \) \( (3) \) \( (2) \) \( (4) \) \( (3) \) \( (2) \)

\[ \begin{array}{cccc} P & \text{ insert}^2 & 1 & 2 \\ \text{ insert}^1 & 1 & 2 \end{array} \]

\[ Q & \text{ insert}^1 & 1 & 2 \\ \text{ insert}^1 & 1 & 1 & 2 \]

* Insert \( j \) to \( P \) using Schnested insertion
* Insert \( i \) to \( Q \) in whatever box got added to \( P \)

Thm: (Knuth) This construction \( A \rightarrow (P, Q) \) is bijective with needed properties.

Proof: Bijection: Can undo algorithm.

Idea: Find max entry in \( Q \) tableau. That tells where we added in \( P \) tableau. Un-add via Schnested in reverse.
Thm: (Knuth) If $A \rightrightarrows (P, Q)$, then $A^T \rightrightarrows (Q, P)$

Not so clear why we should have this symmetry from this version of RSK.

Thm: (Knuth) If $\lambda$ is the shape of $P \& Q$ tableaux, then

$\lambda_1 = \text{length of a 'max weakly increasing subsequence' in } w$

$\lambda'_1 = \text{length of a 'max strictly decreasing subseq.' in } w$

Def: Weakly increasing subseq.

$w = (i_{a_1}) (i_{a_2}) \ldots (i_{a_\ell})$ st.

\[ i_{a_1} \leq i_{a_2} \leq \ldots \leq i_{a_\ell} \]

ja_1 \leq \ldots \leq ja_\ell

Strictly decreasing subseq

\[ i_{a_1} < \ldots < i_{a_\ell} \]

\[ ja_1 > \ldots > ja_\ell \]

Ex. In example before, $\lambda_1 = 3$, $\lambda'_1 = 2$

A different correspondence that lets us see Thm properties better.

SSYT $\leftrightarrow$ Gelfand-Tsetlin Patterns

Ex.

\[
\begin{array}{cc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & \text{SSYT} \\
& 2 & 2 & 3 & 3 & 3 & \text{GT Pattern} \\
\end{array}
\]

- Write # els in each row of T
- Cross out all of highest number
- Repeat for next row of GT-pattern