Schubert Calculus is a tool to solve some algebraic geometric problems.

\[ \text{E.g.} \]

\[ \text{Borel:} \quad H^*(Fl_n, \mathbb{C}) \simeq \mathbb{C}[x_1, \ldots, x_n]/I_n \]

\[ I_n = \langle e_i(x_1, \ldots, x_n) \mid i = 1, \ldots, n \rangle \]

elementary symmetric polynomial

\[ e_i(x_1, \ldots, x_n) = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i} \]

\[ \text{Ex (n=3) consider coinvariant algebra} \]

\[ \mathbb{C}[x_1, x_2, x_3]/\langle x_1 + x_2 + x_3, x_1 x_2 + x_1 x_3 + x_2 x_3, x_1 x_2 x_3 \rangle \]
<table>
<thead>
<tr>
<th>Linear Basis</th>
<th>deg 0</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>deg 1</td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td></td>
<td>deg 2</td>
<td>$x_1x_2$</td>
<td>$x_1^2$</td>
</tr>
<tr>
<td></td>
<td>deg 3</td>
<td>$x_1x_2^2$</td>
<td>1</td>
</tr>
</tbody>
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so total dimension is 6

**Lemma** All monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}$ such that $0 \leq a_i \leq n-i$ for $i=1,\ldots,n$ form a basis of coinvariant algebra $\mathbb{C}[x_1,\ldots,x_n]/I_n$
Bernstein-Gelfand-Gelfand
defined divided difference operators
\[ D_i f = \frac{1}{x_i - x_{i+1}} (1 - s_i) f \]
more generally, if \( \omega = s_i \cdots s_i \) is a reduced decomposition, then
\[ D \omega = D s_i \cdots D s_i \]
They showed that Schubert basis is given by \( D \omega^{-1} \omega_0(f) \) for almost any \( f \) of degree \( \binom{n}{2} \).

Lascoux-Schitzenberger
Schubert Polynomials
\[ S_\omega = D \omega^{-1} \omega_0(x_1^{n-1} \cdots x_n) \]
Properties

(1) \{Sw, w \in Sn\} mod In is the linear basis of coinvariant algebra \( C[x_1, \ldots, x_n]/I_n \) given by Schur classes (BGG)

(2) (non-negativity) \( Sw \) have non-neg. integer coefficients.

(3) (Stability) \( Sn \rightarrow Sn+1 \)

\[ w = w_1 \cdots w_n \rightarrow \tilde{w} = w_1 \cdots w_n (n+1) \]

Then \( Sw(x_1, \ldots, x_n) = \tilde{Sw}(x_1, \ldots, x_{n+1}) \)
Last time we saw two definitions of Schur Polynomials:

(A) Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $0 \leq k \leq n$

$$S_{\lambda}(x_1, \ldots, x_k) = S_{\omega(\lambda)} = \omega(\omega(\lambda)^{-1}) \omega_0(x^\delta)$$

(B) $\mu = (\mu_1, \ldots, \mu_n)$

$$S_{\mu}(x_1, \ldots, x_n) = \omega_0(x^{\lambda+\delta})$$

**Question:** How to show that (A) $\iff$ (B)?

**Ans:** $x^{\lambda+\delta} = S_\mu$ for $u \in S_m$ for some $m > n$ and use stability.

This raises the following question:

**Question:** how to see that a monomial is Schubert?
**Theorem** Any monomial $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ such that

1. $x^\lambda$ divides $x^\delta$
2. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

is a Schubert polynomial $S_\omega$ for some $\omega \in S_n$.

$\#$ of such monomials equal to Catalan $C_n$
**Question:** when is $\mathcal{E}_i(x_i^{d_i} \cdots x_n^{d_n}) = \text{monomial}$

\[ \mathcal{E}_i(x_i^a x_{i+1}^b) = \begin{cases} 0 & \text{if } a = b \\ x_i^{a-1} x_{i+1}^{b-1} & \text{if } a > b \\ -(x_i^{b-1} x_{i+1}^{a-1} + \cdots + x_i^a x_{i+1}^{b-1}) & \text{if } a < b \end{cases} \]

**Ans:** $d_i = d_{i+1} + 1$

**Claim** if we only allow such $\mathcal{E}_i$ we will get all Schubert monomial

**Proof:** induction in a $(\lambda^\mathrm{t})-1\lambda^\mathrm{t}$

If $\lambda \neq \delta$ then we can always find $i$ s.t. $\lambda_i = \lambda_{i+1}$
we can add a box and apply induction hypothesis on $\mu$. Then $\exists i : x^\mu \rightarrow x^1$. \qed