18.212 PROBLEM SET 1 (due Friday, March 10, 2023)

Each problem is 10 points.

Problem 1. Find an explicit formula for the number of Dyck paths of size n (i.e., Dyck paths with n up steps and n down steps) that start with 3 (or more) up steps.

For example, for $n = 3$, there is only one such Dyck path $UUUDDD$; and, for $n = 4$, there are 4 paths: $UUUUDDDD$, $UUUDUDDD$, $UUUDDUDD$, $UUUDDDUD$. (Here U and D denote up and down steps.)

Problem 2. In class, we mentioned that both binary trees on n vertices and plane trees on $n+1$ vertices are counted by the Catalan number C_n. Here "binary trees" are not necessarily "complete binary trees." A binary tree can have vertices with only one (left or right) child.

Prove bijectively that the number binary trees on n vertices equals the number of plane trees on $n+1$ vertices.

(You'll get a slightly reduced credit -2 points for a non-bijective proof, e.g., a proof based on recurrence relations.)

Problem 3. Prove bijectively that, for any $1 \le k \le n$, the number of non-crossing set partitions of $[n]$ with k blocks equals the number of non-crossing set partitions of $[n]$ with $n-k+1$ blocks.

(Again, -2 points for a non-bijective proof.)

Problem 4. Recall that the *major index* of a permutation $w = w_1 w_2 \cdots w_n$ is defined as

$$\operatorname{maj}(w) := \sum_{i\,:\,w_i > w_{i+1}} i\,.$$

Define the *modular major index* $\operatorname{modmaj}(w) \in \{0, 1, \ldots, n-1\}$ as the reside of $\operatorname{maj}(w) \bmod n$.

Prove bijectively that, for any $i, j \in \{0, 1, \ldots, n-1\}$,

$$\#\{w \in S_n \mid \operatorname{modmaj}(w) = i\} = \#\{w \in S_n \mid \operatorname{modmaj}(w) = j\}.$$

(-2 points for a non-bijective proof)

Problem 5. Find a bijective proof of the formula

$$\sum_{k=0}^{n} c(n,k)\, x^k = x(x+1)\dots(x+n-1)$$

using a bijection. Here $c(n,k)$ is the signless Stirling number of the first kind, i.e., the number of permutations $w \in S_n$ with exactly k cycles.

(Here is one possible approach to this problem: Assume that x is a positive integer. Give combinatorial interpretations of both sides of this equation; and construct a bijection between these combinatorial objects.)

Problem 6. Recall that the number of *exceedances* of a permutation $w = w_1 w_2 \dots w_n$ is defined as $\mathrm{exc}(w) := \{i \in [n] \mid w_i > i\}$. Define the number of *weak exceedances* as $\mathrm{wexc}(w) := \{i \in [n] \mid w_i \geq i\}$.

Prove bijectively that, for any $k \geq 0$,

$$\#\{w \in S_n \mid \mathrm{exc}(w) = k\} = \#\{w \in S_n \mid \mathrm{wexc}(w) = k+1\}.$$

(-2 points for a non-bijective proof)

Bonus Problems:

Problem 7. Recall that the Stirling number of the second kind $S(n,k)$ equals the number of set partition on $[n]$ with k blocks; and the Eulerian number $A(n,k)$ equals the number of permutations in $w \in S_n$ with k descents.

Prove the formula

$$\sum_{k=1}^{n} k!\, S(n,k)\, x^{n-k} = \sum_{k=0}^{n-1} A(n,k)\, (x+1)^k$$

Problem 8. Let $K_n = (V, E)$ be the *complete graph* on n vertices. Its set of vertices is $V = [n]$; and its set of edges E is the set of all pairs $\{i, j\} \subset [n]$, $i \neq j$.

For $n \geq 4$, construct a bijection $f : E \to E$ from the set of edges of K_n to itself such that, for any $e \in E$, the edges e and $f(e)$ have no common vertices.