18.212 Problem Set 3 (due Friday, May 6, 2022)

Solve 3 or more problems.

Problem 1. Prove the following result (which we mentioned in class). Let $f = (f_1, \dots, f_n)$ be a sequence of integers between 1 and n. The following 3 conditions are equivalent to each other:

(1) f is a parking function.
(2) $\#\{i \mid f_i \geq n-k+1\} \leq k$ for $k = 1, \dots, n$.
(3) There is a permutation $w_1, \dots, w_n$ of $1, \dots, n$ such that $f_i \leq w_i$ for $i = 1, \dots, n$.

Problem 2. Fix two positive integers n and k. We say that a sequences $(f_1, \dots, f_n)$ of positive integers is a *generalized* (n,k)*-parking function* if there exists a permutation $w_1, w_2, \dots, w_n$ of $1, 1+k, 1+2k, 1+3k, \dots, 1+(n-1)k$ such that $f_i \leq w_i$ for $i = 1, \dots, n$. (This is a generalization of condition (c) from the previous problem.)

Show that the number of generalized (n,k)-parking functions equals $(1+kn)^{n-1}$.

Problem 3. In class, we mentioned the Y-Delta transform of electrical networks. Recall that this is the transformation of some large electrical network composed of many resistors where we replace 3 resistors with resistances R_1, R_2, R_3 connected in Y-shape by 3 resistors with resistances R_1', R_2', R_3' connected in ∇-shape. (∇ is upside down Delta.)

Prove that there exist unique resistances R_1', R_2', R_3' such that all effective resistances in the resulting network are the same as the effective resistances in the original network. Find expressions for R_1', R_2', R_3' in terms of R_1, R_2, R_3.

Problem 4. Let $\mathrm{Cube}_d = K_2 \times K_2 \times \cdots \times K_d$ (d terms) be the graph which is the 1-skeleton of the d-dimensional hypercube. Consider Cube_d as an electrical network where every edge has resistance 1 Ohm. Find the effective resistance between a pair of opposite vertices in Cube_d.

Problem 5. In class, we showed that the number of spanning trees in Cube_d equals $2^{2^d-d-1}\prod_{k=1}^{d} k^{\binom{d}{k}}$.

Let G be the graph obtained from Cube_d be adding one extra edge connecting a pair of opposite vertices. Find an explicit formula for the number of spanning trees in G.

(For example, for $d = 2$, the graph G is obtained from the square $K_2 \times K_2$ by adding one diagonal edge. We used this graph in many examples in the lectures. Recall that this graph has 8 spanning trees.)

Problem 6. Let $G = [4]\times[4]$ be the 4×4-grid graph. It has 4^2 vertices corresponding to pairs (i, j), $i, j \in [4]$. Let us mark two opposite corners $A = (1, 1)$ (the "house") and $B = (4, 4)$ (the "cliff") of the graph G.

Consider the random walk on the graph G such that, at each step, we go from a vertex v to any of the neighbors of v with probability $1/\deg(v)$. A walk stops when we arrive one of the vertices A or B.

For any initial vertex (i, j), find the probability that a random walk starting at (i, j) stops at vertex A.

Problem 7. In class, we showed that the numbers A_n of alternating permutations in S_n satisfy the recurrence relation:

$$A_n = \sum_{k\in[n],\ k \text{ is even}} \binom{n-1}{k-1} A_{k-1}\, A_{n-k},$$

for $n \geq 1$. And $A_0 = 1$.

Consider the following two exponential generating functions:

$$T(x) = \sum_{k\geq 0} A_{2k+1}\, x^{2k+1}/(2k+1)! \quad \text{and} \quad S(x) = \sum_{k\geq 0} A_{2k}\, x^{2k}/(2k)!$$

(a) Deduce from the recurrence relation that $T(x)$ satisfies the differential equation $T'(x) = 1+(T(x))^2$ with the initial condition $T(0) = 0$.

(b) Also show that $S(x)$ satisfies the differential equation $S'(x) = S(x)T(x)$ and $S(0) = 1$.

(c) Now deduce that $T(x) = \tan(x)$ and $S(x) = \sec(x)$.

Problem 8. The Euler-Bernoulli triangle is the triangular array of numbers

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 1 & & 1 & & 0 & & & \\
 & & 0 & & 1 & & 2 & & 2 & & \\
 & 5 & & 5 & & 4 & & 2 & & 0 & \\
0 & & 5 & & 10 & & 14 & & 16 & & 16 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Each odd/even row of this triangle is obtained by adding the numbers in the row above it starting from the right/left.

Show that the numbers $1, 1, 2, 5, 16, \dots$ that appear on the sides of this triangle are the numbers A_n of alternating permutations in S_n.

Problem 9. We say that a tree T on the vertex set $0, 1, \dots, n$ is an *increasing odd tree* if

(1) T is an increasing tree (i.e., vertex labels increase as we go away from the vertex 0 (root)), and
(2) degrees of all vertices in T are odd.

Notice that, for even n, there are no increasing odd trees on $n + 1$ vertices.

Show that, if n is odd, then the number of increasing odd trees on $n+1$ vertices equals the number A_n of alternating permutations in S_n.

Problem 10. Calculate the number of permutations w in S_n such that w is 123-avoiding and alternating.

Problem 11. Let

$$I_n(x) := \sum_{T \text{ is a spanning tree of } K_{n+1}} x^{\mathrm{inv}(T)}$$

be the tree inversion polynomial. In class, we discussed the values $I_n(1) = (n + 1)^{n-1}$, $I_n(0) = n!$, and $I_n(-1) = A_n$.

Prove that the value $I_n(2)$ equals the number of connected subgraphs of K_{n+1}.

Problem 12. Calculate the determinant of the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = C_{i+j-2}$, for $i, j \in [n]$. Here $C_k = \frac{1}{k+1}\binom{2k}{k}$ are the Catalan numbers.