Solve 6 or more problems.

Problem 1. For a finite poset P, prove that the maximal cardinality of a chain in P equals the minimal number of antichains in P whose union contains all elements of P. (This result is called Mirsky's theorem.)

Problem 2. Fix two positive integers m and n. Prove that any permutation of $m \cdot n + 1$ letters contains either an increasing subsequence of $m + 1$ letters or a decreasing subsequence of $n + 1$ letters.

(This result is called Erdős-Szekeres theorem. In class, we mentioned that it easily follows from Greene's theorem. In this problem, you need to prove it directly without using Greene's theorem.)

Problem 3. The *partition lattice* Π_n consists of all set partitions of $[n] := \{1, 2, 3, \dots, n\}$ ordered by refinement, i.e., $\pi \le \sigma$ iff any block of π is contained in a block of σ. The poset Π_n has the unique minimal element $\hat{0} = (1|2|3|\dots|n)$ and the unique maximal element $\hat{1} = (1, 2, 3, \dots, n)$.

Find the number of saturated chains from $\hat{0}$ to $\hat{1}$ in the partition lattice Π_n.

Problem 4. A *walk* on Young's lattice $\mathbb{Y}$ is a sequence of Young diagrams $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(m)}$ such that $\lambda^{(i)}$ and $\lambda^{(i+1)}$ differ by a single box, for any i. (We can have either $\lambda^{(i)} \subset \lambda^{(i+1)}$ or $\lambda^{(i)} \supset \lambda^{(i+1)}$.)

Show that, for $m = 2n$, the number of walks on $\mathbb{Y}$ from $\lambda^{(0)} = \emptyset$ to $\lambda^{(m)} = \emptyset$ equals $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)$.

Hint: One approach to this problem is based on the up and down operators U and D acting on $\mathbb{R}[\mathbb{Y}]$ and the identity $D\,U - U\,D = Id$.

Problem 5. The *Fibonacci poset* $\mathbb{F}$ is the differential poset defined on pages 14–18 of the notes for Lecture 16 (March 9). Show that the poset $\mathbb{F}$ is a lattice.

Hint: First, you can try to find a non-recursive description of $\mathbb{F}$.

Problem 6. Construct a bijection between partitions of n with odd parts and partitions of n with distinct parts.

Problem 7. Prove the identity for q-binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{s=0}^{\min(k,n-k)} q^{s^2} \begin{bmatrix} k \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ s \end{bmatrix}_q$$

Problem 8. Prove the identity

$$\prod_{i=1}^{n} \frac{1}{1-xq^i} = \sum_{k=0}^{\infty} (qx)^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$$

Problem 9. Prove the identity

$$\prod_{n\geq 1} \frac{1-q^n}{1+q^n} = \sum_{k\in\mathbb{Z}} (-1)^k\, q^{k^2}.$$

Problem 10. Let us say that a tree T is *trivalent* if any non-leaf vertex of T has degree 3. In other words, for any vertex v, $\deg_T(v) \in \{1, 3\}$.

Show that the number of trivalent trees T on $2n$ labelled vertices equals $\frac{(2n)!}{(n+1)!}\,(2n-3)!!$. (Recall that $(2n-3)!! := 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)$.)

Problem 11. More generally, let us say that a tree T is *d-valent* if any non-leaf vertex in T has degree d. Find an explicit formula for the number of labelled d-valent trees on n vertices.

Problem 12. Find an explicit formula for the number of spanning trees in the complete bipartite graph $K_{m,n}$.

Problem 13. Let L be any symmetric $n \times n$ matrix with zero row sums. Let $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of L. Prove that any cofactor L^{ij} of L equals $\frac{1}{n}\lambda_2 \cdots \lambda_n$.

Problem 14. Let x, y, z be three variables. Prove the identity

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}\, y\,(y+kz)^{k-1}\,(x-kz)^{n-k}.$$

Problem 15. Let $K_{m,n,k,l}$ be the complete 4-partite graph. It has $m+n+k+l$ vertices subdivided into 4 blocks of sizes m, n, k, and l. Two vertices are connected by an edge if and only if they belong to two different blocks.

Find an explicit formula for the number of spanning trees in $K_{m,n,k,l}$.

Problem 16. Find en explicit formula for the number of spanning trees of the product $K_3 \times K_3 \times \cdots \times K_3$ of n copies of the complete graph K_3.