18.212 Lecture 33 05/10/2021

last time:

<u>Flajolet's Fundamental Lemma</u>

$a_1, a_2, a_3, \ldots$; $b_1, b_2, b_3, \ldots$ some weights (or just formal variables)

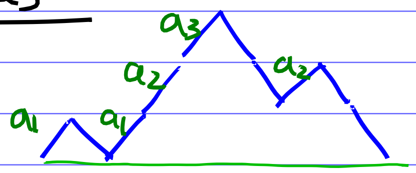
For a Dyck path P, define

$$wt(P) := \prod_{\substack{s \text{ is an} \\ \text{up step in } P}} a_{ht(s)}$$

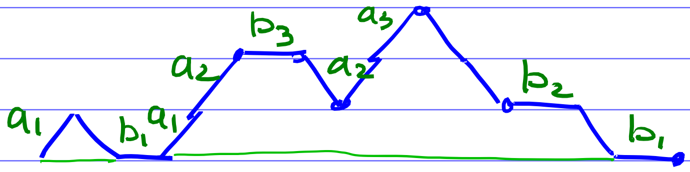
More generally, for a Motzkin path P, define

$$wt(P) := \prod_{\substack{s \text{ is an} \\ \text{up step}}} a_{ht(s)} \prod_{\substack{\tilde{s} \text{ is a} \\ \text{horizontal} \\ \text{step}}} b_{ht(\tilde{s})}$$

<u>Examples</u>

- $P =$ 

$$wt(P) = a_1 a_1 a_2 a_3 a_2$$

- $P =$ 

$$wt(P) = a_1 b_1 a_1 a_2 b_3 a_2 a_3 b_2 b_1$$

Theorem (P. Flajolet, 1980)

We have the following identities for formal power series:

- $$\sum_{n\geq 0}\Big(\sum_{\substack{P \text{ Dyck} \\ \text{path with} \\ 2n \text{ steps}}} wt(P)\Big)x^n =$$

$$= \cfrac{1}{1-\cfrac{a_1 x}{1-\cfrac{a_2 x}{1-\cfrac{a_3 x}{1-\dots}}}}$$

- $$\sum_{n\geq 0}\Big(\sum_{\substack{P \text{ Motzkin} \\ \text{path with} \\ n \text{ steps}}} wt(P)\Big)x^n =$$

$$= \cfrac{1}{1-b_1x-\cfrac{a_1 x^2}{1-b_2x-\cfrac{a_2 x^2}{1-b_3x-\cfrac{a_3x^2}{1-\dots}}}}$$

(The first formula is a special case of the second for $b_1=b_2=\dots=0$ and $x^2 \to x$.)

Examples. 1. $a_1 = a_2 = \dots = 1$, $b_1 = b_2 = \dots = 0$

$$\sum_{n\geq 0} C_n x^n = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-\dots}}}}$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}} = \varphi := \frac{1+\sqrt{5}}{2}$$

(the golden ratio)

It satisfies the eqn. $\varphi = 1 + \frac{1}{\varphi}$.

So the "alternating sum of the Catalan numbers" is φ^{-1}.

"$C_0 - C_1 + C_2 - C_3 + \dots$ "

$$= \text{"}\left(\sum_{n\geq 0} C_n x^n\right)\Bigg|_{x=-1}\text{"}$$

these series diverge

:= the value of the analytic continuation of

$$\sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

at $x = -1$

$$= \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}} = \varphi^{-1} = \frac{\sqrt{5}-1}{2}$$

Remark.
Recall, that the golden ratio φ is also related to the Fibonacci numbers F_n

($F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, $F_0 = 0$, $F_1 = 1$)

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \varphi$$

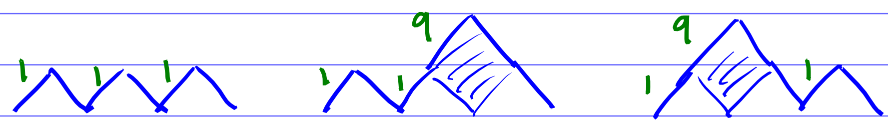
2. $a_i = q^{i-1}$, $b_1 = b_2 = \dots = 0$

$$\sum_{n \geq 0} C_n(q)\, x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{qx}{1 - \cfrac{q^2 x}{1 - \cfrac{q^3 x}{1 - \dots}}}}}$$

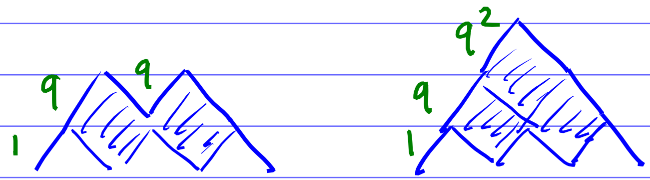
where $C_n(q) :=$

$$= \sum_{P \text{ Dyck path}} q^{\text{Area below } P}$$

e.g., for $n = 3$,



$1 \qquad q \qquad q$



$q^2 \qquad q^3$

$$C_3(q) = 1 + 2q + q^2 + q^3$$

3. $a_i = i\cdot(i+1)$ $\qquad b_1 = b_2 = \dots = 0$

$$\sum_{n\geq 0} A_{2n+1} x^n =$$

$$= \cfrac{1}{1 - \cfrac{1\cdot 2\cdot x}{1 - \cfrac{2\cdot 3\cdot x}{1 - \cfrac{3\cdot 4\cdot x}{1 - \dots}}}}$$

4. $a_i = i^2$ $\qquad b_1 = b_2 = \dots = 0$

$$\sum_{n\geq 0} A_{2n} x^n =$$

$$= \cfrac{1}{1 - \cfrac{1^2 x}{1 - \cfrac{2^2 x}{1 - \cfrac{3^2 x}{1 - \dots}}}}$$

5. $a_i = i(i+1)$, $b_i = 2i$ Same weights as in Françon–Viennot bijection

$$\sum_{n\geq 0} (n+1)!\, x^n =$$

$$= \cfrac{1}{1-2x-\cfrac{1\cdot 2\cdot x^2}{1-4x-\cfrac{2\cdot 3\cdot x^2}{1-6x-\cfrac{3\cdot 4\cdot x^2}{1-\cdots}}}}$$

Remark. Examples 3, 4, 5 are exceptions to the rule that we should use exponential generating series for labelled objects. (Permutations and alternating permutation are labelled objects. Earilier we discussed exponential generating functions

$$\sum_{n\geq 0} A_{2n} \frac{x^{2n}}{(2n)!} = \sec(x),$$

$$\sum_{n\geq 0} A_{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \tan(x).)$$

Notice that the series in Examples 3, 4, 5 diverge for all non-zero values of x.

These identities are for formal power series. They are not analytic functions.

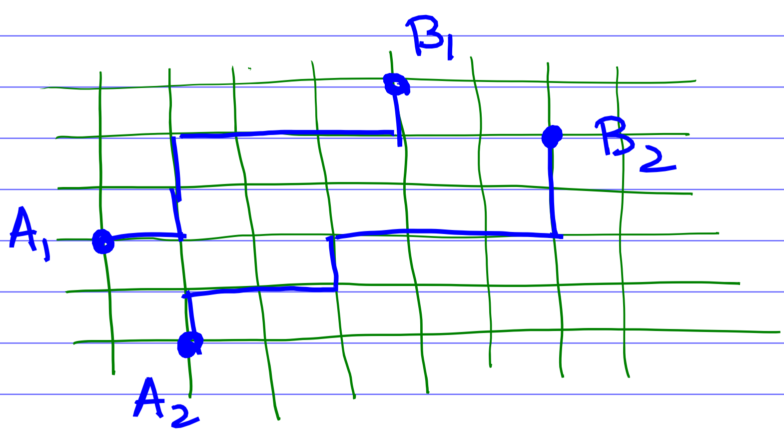
Let's move to the next topic ...

Lindström's Lemma

a.k.a. Gessel – Viennot method

Counting non-crossing paths.

A typical problem:

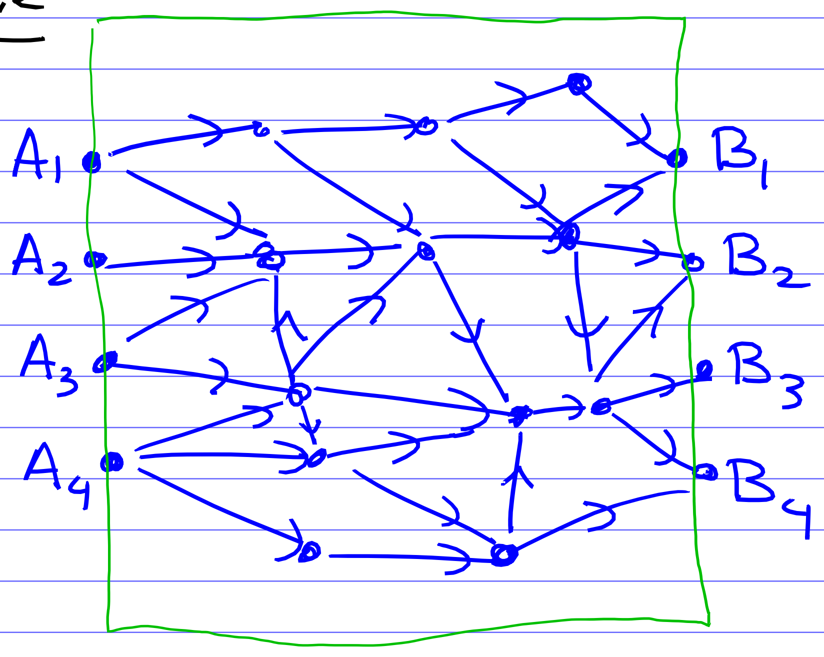


Calculate # of pairs of non-crossing lattice paths connecting points A_1 & A_2 with B_1 & B_2.

Let G be a directed graph such that:

- G has no directed cycles
- G is drawn on the plane
- All edges e of G have positive weights x_e.
- There are n selected vertices $A_1, \dots, A_n$ on the "left side" of G, and n selected vertices $B_1, \dots, B_n$ on the "right side" of G. Both $A_1, \dots, A_n$ and $B_1, \dots, B_n$ are arranged from top to bottom.

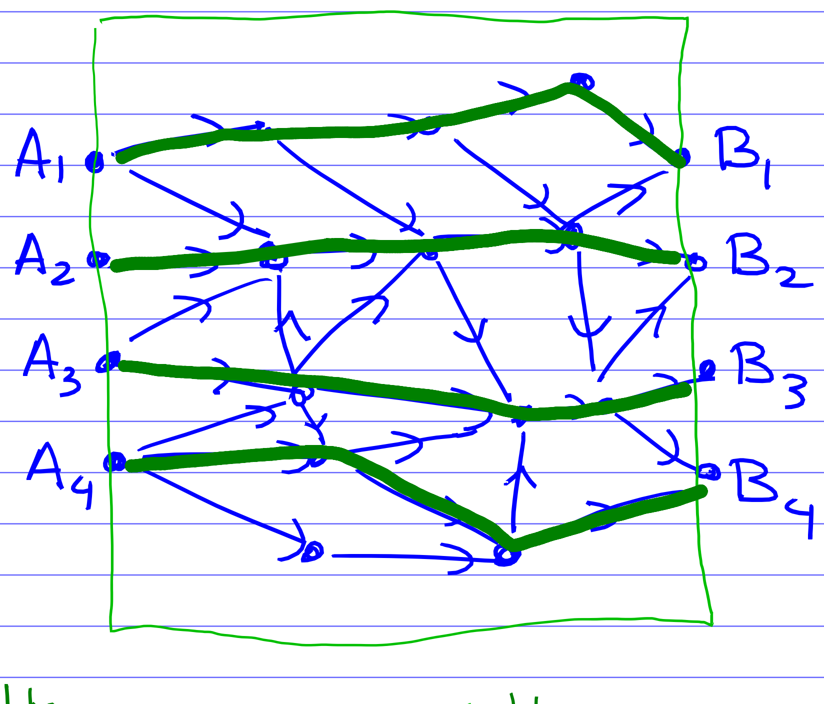
Example

$G =$ 

- a graph drawn inside a rectenlgle
- vertices $A_1, \dots, A_n$ on the left side of the boundary
- vertices $B_1, \dots, B_n$ on the right side of the boundary

We want to count # ways (or the weighted sum over) all collections of non-crossing paths $P_1, \dots, P_n$ that connect vertices $A_1, \dots, A_n$ with $B_1, \dots, B_n$

Definition. A collection of directed paths $P_1, P_2, \dots, P_n$ is non-crossing if P_i & P_j have no common vertices for any $i \neq j$.



Here is one possible way to connect A_i's with B_i's by non-crossing paths.

For a directed path P in G, let

$$\text{weight}(P) := \prod_{\substack{e \text{ edge} \\ \text{in } P}} x_e$$

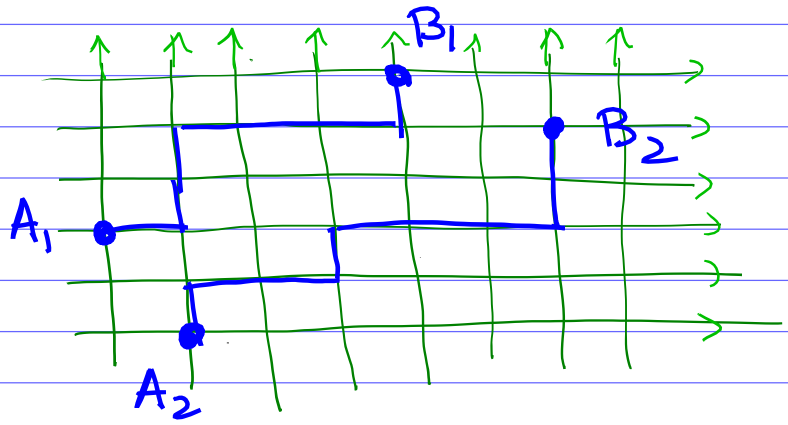
Let $C = (c_{ij})$ be the $n \times n$ matrix such that

$$c_{ij} = \sum_{\substack{P \text{ directed} \\ \text{path from} \\ A_i \text{ to } B_j}} \text{weight}(P)$$

<u>Lindström-Gessel-Viennot Lemma</u>

$$\sum_{\substack{(P_1, \ldots, P_n) \\ \text{collection of} \\ \text{non-crossing} \\ \text{directed paths in } G, \\ \text{s.t. } P_i : A_i \to B_i}} \prod_{i=1}^{n} \text{weight}(P_i) = \det(C)$$

Example



G square grid graph with edges directed right & up

All edge weights x_e are 1.

directed paths

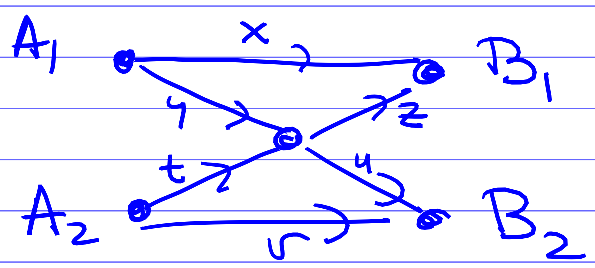
- from A_1 to B_1 is $\binom{7}{3}$
- from A_1 to B_2 is $\binom{8}{2}$
- from A_2 to B_1 is $\binom{8}{5}$
- from A_2 to B_2 is $\binom{9}{4}$

So # non-crossing pairs of paths connecting A_1 & A_2 with B_1 & B_2 equals

$$\det \begin{bmatrix} \binom{7}{3} & \binom{8}{2} \\ \binom{8}{5} & \binom{9}{4} \end{bmatrix}$$

$$= \binom{7}{3}\binom{9}{4} - \binom{8}{2}\binom{8}{5}.$$

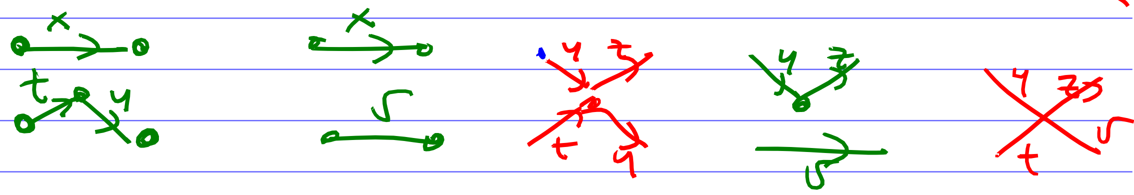
Another Example (with weights)



$$\det(C) = \begin{vmatrix} x + yz & yu \\ t\cdot z & t\cdot u + s \end{vmatrix}$$

$$= (x+yz)(t\cdot u + s) - t\cdot z\cdot y\cdot u$$

$$= x\cdot tu + x\cdot s + \cancel{yz\cdot tu} + yz\cdot s - \cancel{tz\cdot yu}$$



Notice that all terms in the expansion correspond to all possible ways to connect A_1 & A_2 with B_1 & B_2 by paths (including crossing paths). But the term corresponding to crossing pairs of paths cancel each other, and only non-crossing pairs of paths remain.

Proof of Lindström - Gessel - Viennot's Lemma.

We'll use the involution principle.

By the definition of det:

$$\det(C) =$$

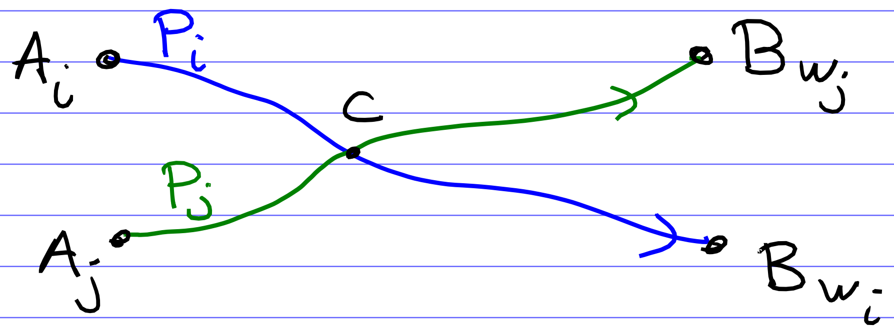
$$= \sum_{\substack{w = w_1 \dots w_n \\ \text{permutation in } S_n}} (-1)^{\ell(w)} c_{1w_1} c_{2w_2} \cdots c_{nw_n}$$

$$= \sum_{w \in S_n} (-1)^{\ell(w)} \sum_{\substack{P_1 : A_1 \to B_{w_1} \\ P_2 : A_2 \to B_{w_2} \\ \dots \\ P_n : A_n \to B_{w_n}}} \prod_{i=1}^{n} \text{weight}(P_i)$$

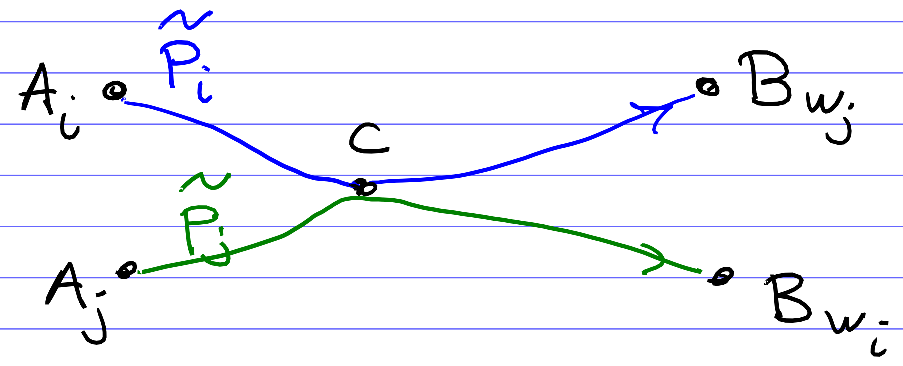
Here we have an signed sum over arbitrary collections of directed paths $P_1, \dots, P_n$ connecting $A_1, \dots, A_n$ with $B_1, B_2, \dots, B_n$.

We want to cancel all terms corresponding to collections of paths $P_1, \ldots, P_n$ with at least one crossing. (i.e. $(P_1, \ldots, P_n)$ s.t. $\exists\, i \neq j$ s.t. P_i & P_j have a common vertex.)

- Find the "first intersection point" c of some P_i & P_j



- "Swap the tails" of P_i & P_j at c, i.e. replace P_i & P_j with $\tilde{P}_i$ & $\tilde{P}_j$:



- Don't change the other paths P_k, $k \neq i, j$.

$$\sigma : (P_1, \ldots P_i \ldots P_j \ldots P_n) \mapsto$$
$$\to (P_1 \ldots \tilde{P}_i \ldots \tilde{P}_j \ldots P_n)$$

Notice that the weight $\prod_{i=1}^{n} \text{weight}(P_i)$ does not change, but the sign $(-1)^{\ell(w)}$ reverses.

So the terms corresponding to the two collections of paths $(P_1, \ldots, P_n)$ and $(P_1 \ldots \tilde{P}_i \ldots \tilde{P}_j \ldots P_n)$ cancel each other.

The only subtle part of the construction of the "Swap of tails" operation σ is how to define the "first intersection point" c.

We want to consistently define the point c for any collection of paths $(P_1, \dots, P_n)$ with at least one crossing such that

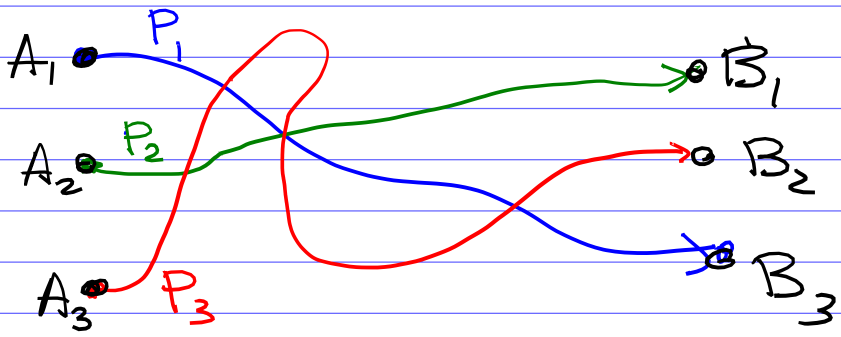
σ is an involution i.e. if

$$\sigma : (P_1 \dots P_n) \mapsto (P_1 .. \tilde{P}_i .. \tilde{P}_j .. P_n)$$

then $\sigma : (P_1 .. \tilde{P}_i ... \tilde{P}_j ... P_n) \mapsto (P_1 ... P_n)$

this collection of paths should have the same point c as for the original collection $(P_1, \dots, P_n)$.

Example

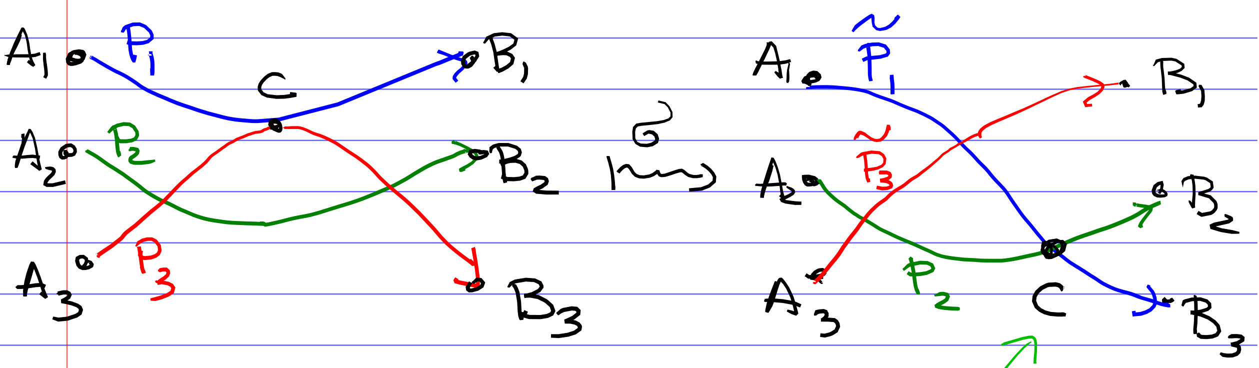


What intersection point of P_i & P_j should we take?

Let's try:

- Find the lexicographically minimal pair (i,j) s.t. P_i & P_j have a common vertex
- Find the first common vertex c of P_i & P_j

This does not work...



a different "first intersection point" c.

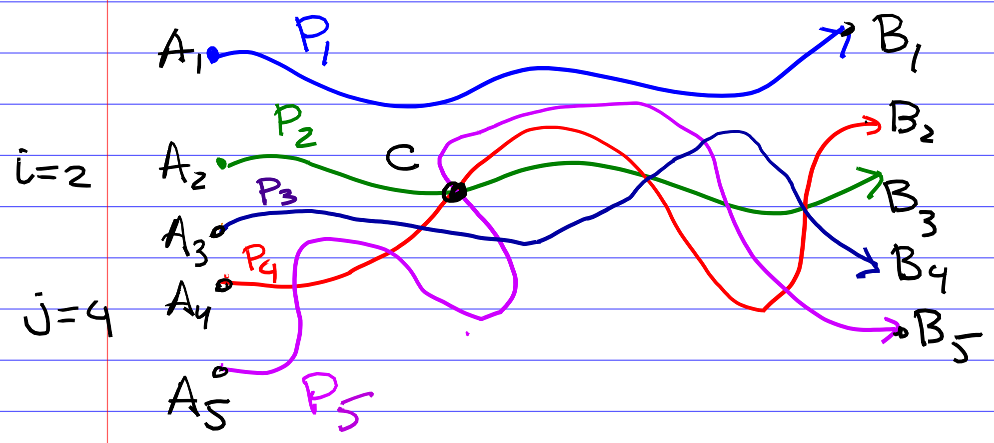
Here is the correct construction of map σ:

- Find the minimal i such that P_i has a common vertex with another path.
- Find the first vertex c of P_i that belongs to another path
- Find the minimal $j \neq i$ such that P_j passes through vertex c.

Then swap the tails of P_i & P_j at vertex c.

$$\sigma : (P_1, .., P_i, .., P_j .. P_n) \mapsto (P_1, .., \tilde{P}_i, .., \tilde{P}_j .. P_n)$$

Example



It is easy to see that σ defined like this is an involution.

So it matches all the "bad guys" in pairs that cancel each other.

Only terms corresponding to non-crossing collections of paths $P_1, \ldots, P_n$ remain.

All non-crossing collections correspond to the identity permutation $w = 1, 2, \ldots, n$, i.e. $P_i : A_i \rightsquigarrow B_i \ \forall i$

the sign of this permutation is $+1$. So we get the needed identity:

$$\det(C) = \sum_{\substack{(P_1 \ldots P_n) \\ \text{non-crossing}}} \prod_i \text{weight}(P)$$

$\square$

Plane Partitions

Fix $m, n, k \geq 1$

A <u>plane partition</u> (of shape $m \times n$) is a filling of the $m \times n$ rectangle by non-negative integers $\in \{0, 1, \dots, k\}$ (equiv. a $m \times n$ matrix) such that the entries weakly decrease in rows & columns.

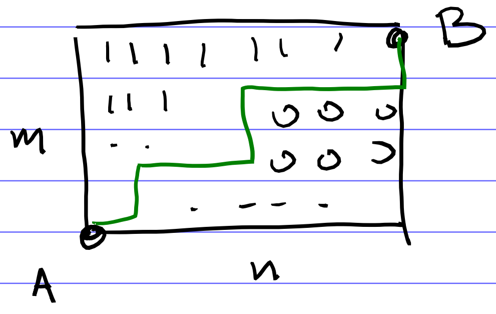
<u>Example</u> $m = 3, \; n = 4, \; k = 2$

$$\begin{array}{cc} & \geq \\ \geq & \begin{array}{|c|c|c|c|} \hline 2 & 2 & 1 & 1 \\ \hline 2 & 1 & 1 & 0 \\ \hline 2 & 1 & 1 & 0 \\ \hline \end{array} \end{array}$$

a plane partition of shape 3×4 with entries $\in \{0, 1, 2\}$

Find the number of plane partition (for given m, n, k).

<u>Case $k = 1$.</u>

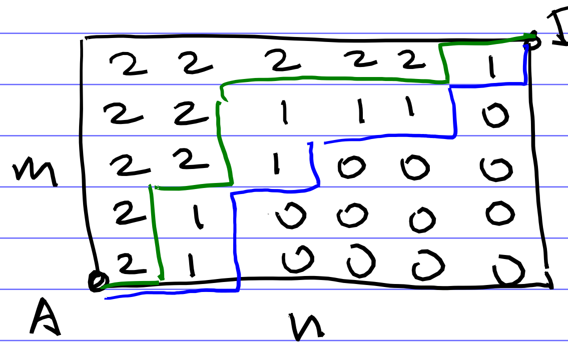


$\#$ plane partitions $=$

$= \#$ lattice paths from A to B

$$= \binom{m+n}{m}$$

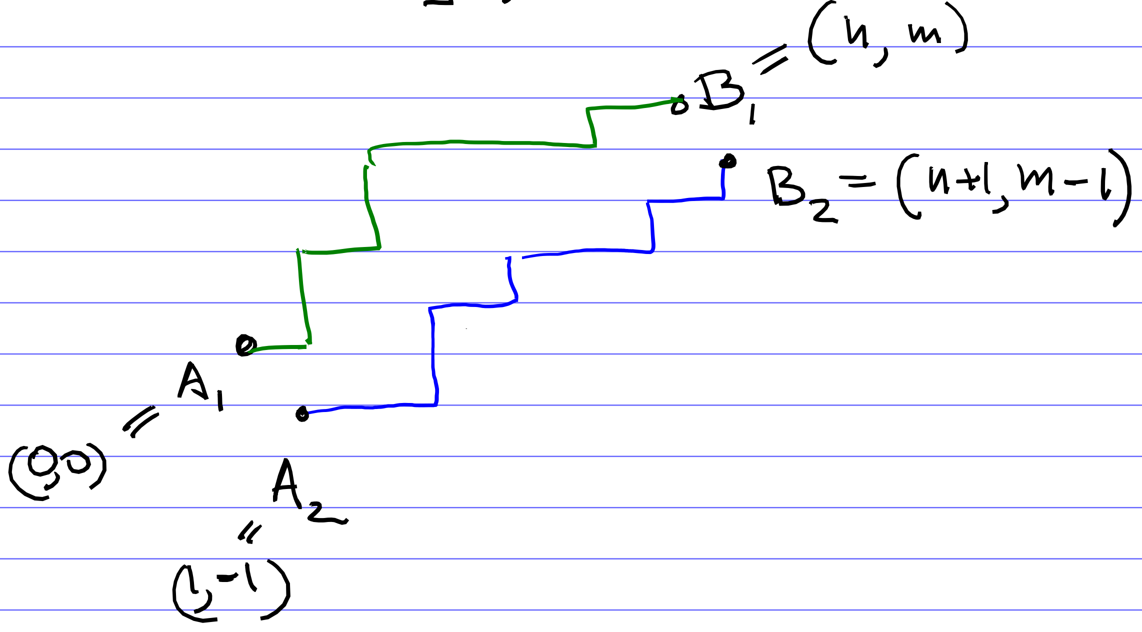
Case $k = 2$



plane partitions =

= # pairs of lattice paths from A to B that cannot cross each other (in the strict sense) but can "touch" each other.

Let's shift the second path by 1 step in the direction "↘".

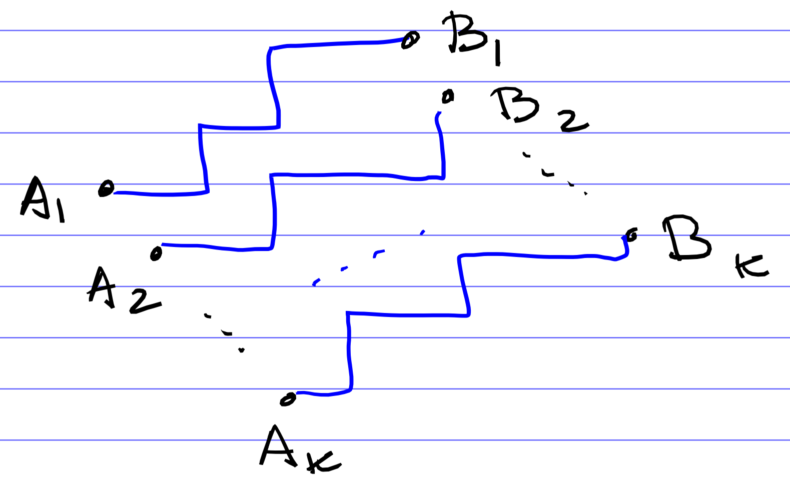


Now we get a pair of non-crossing paths from A_1 & A_2 to B_1 & B_2, which we can count using Lidström-Gessel-Viennot Lemma:

$$\begin{vmatrix} \binom{m+n}{m} & \binom{m+n}{m-1} \\ \binom{m+n}{m+1} & \binom{m+n}{m} \end{vmatrix}$$

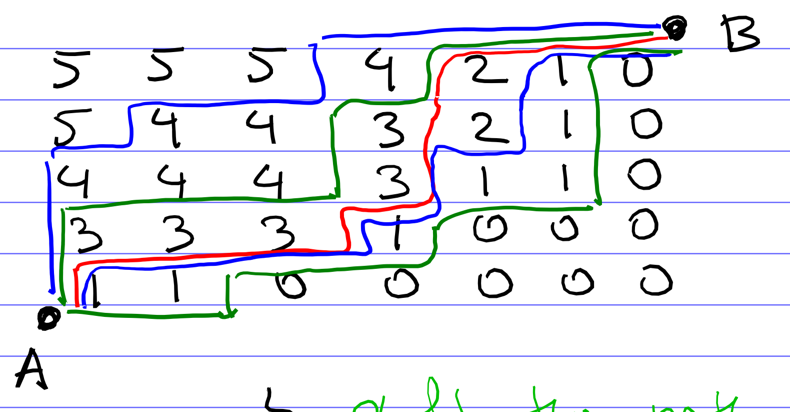
We can do a similar construction for any k: Shift the second path by $(1,-1)$, shift the third path by $(2,-2)$, etc.

For general k, # plane partitions with entries $\in \{0,1,\dots,k\}$ equals # k-tuples of non-crossing paths connecting A_i's with B_i's

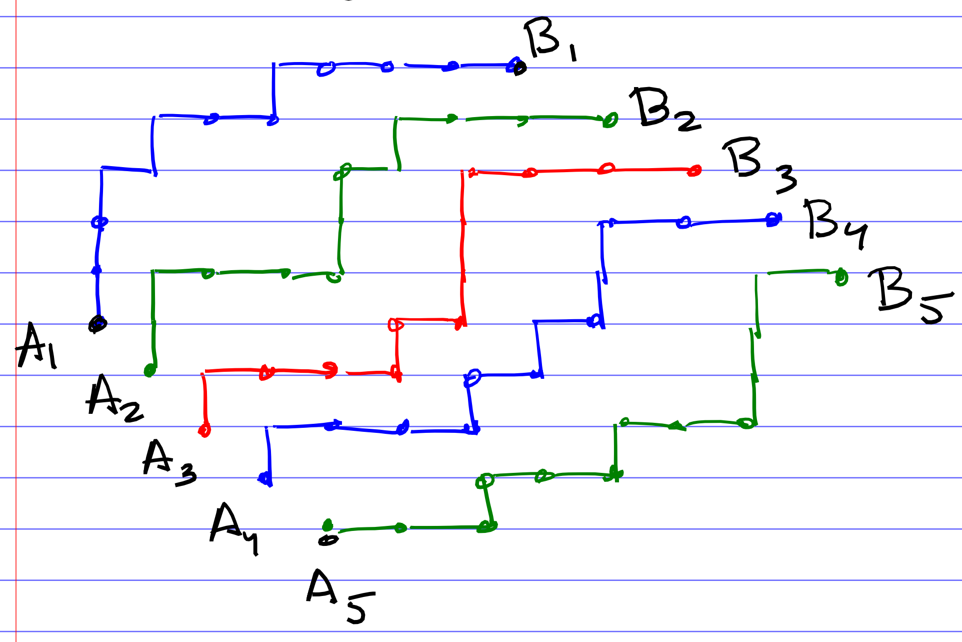


$A_i = (i-1, -i+1)$, $B_i = (n+i-1, m-i+1)$ for $i = 1, 2, \dots, k$

<u>Example</u> $(m, n, k) = (5, 7, 5)$



shift the paths



Linström-Gessel-Viennot Lemma implies:

<u>Theorem</u> # plane partitions of shape $m \times n$ with entries $\in \{0,1,\dots,k\}$ equals the determinant of $k \times k$ matrix

$$\det \begin{bmatrix} \binom{m+n}{m} & \binom{m+n}{m-1} & \cdots & \binom{m+n}{m-k+1} \\ \binom{m+n}{m+1} & \binom{m+n}{m} & \cdots & \binom{m+n}{m-k+2} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{m+n}{m+k-1} & \binom{m+n}{m+k-2} & \cdots & \binom{m+n}{m} \end{bmatrix}.$$

Actually there is a more explicit formula for # plane partitions.

<u>Theorem</u> (A. MacMahon 1916)

plane partitions of shape $m \times n$ with entries $\in \{0, 1, \dots, k\}$ equals

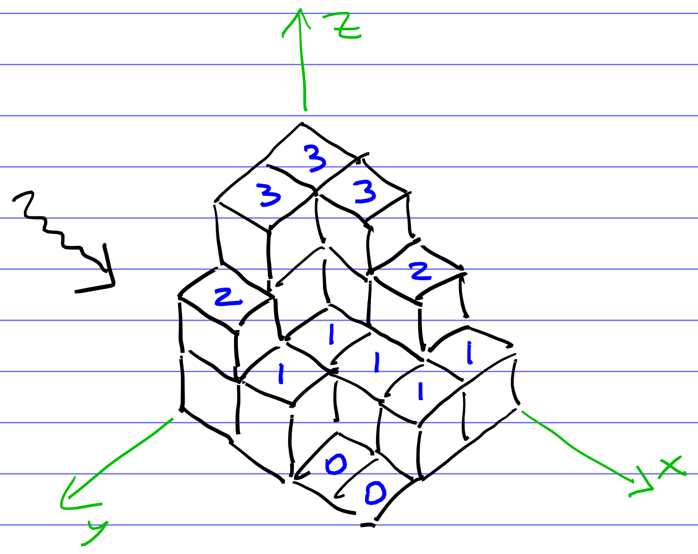
$$\prod_{a=1}^{m} \prod_{b=1}^{n} \prod_{c=1}^{k} \frac{a+b+c-1}{a+b+c-2}.$$

<u>Remark</u> Notice that the resulting formula is symmetric in m, n, k. This symmetry becomes clear if we represent plane partitions as "3-dimensional Young diagrams".

<u>Example</u> $(m, n, k) = (3, 4, 3)$

3	3	2	1
3	1	1	1
2	1	0	0

a plane partition



If a box of the plane partition is filled with ℓ, then put ℓ $1 \times 1 \times 1$ cubes above it.

We get a "3-dim Young diagram" that should fit inside the $m \times n \times k$ box.

It is now clear that # such diagrams should be symmetric in m, n, k.