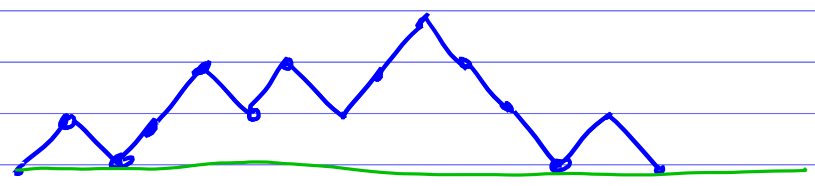
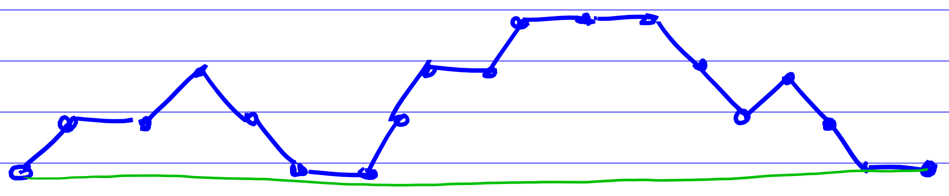
Weighted path enumeration & Françon-Viennot bijection

Recall, Dyck paths



Motzkin Paths are similar to Dyck paths, but they might also contain horizontal steps (in addition to up and down steps).

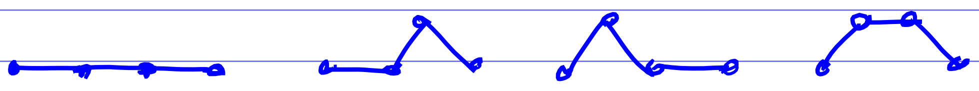
Example



a Motzkin path with 18 steps

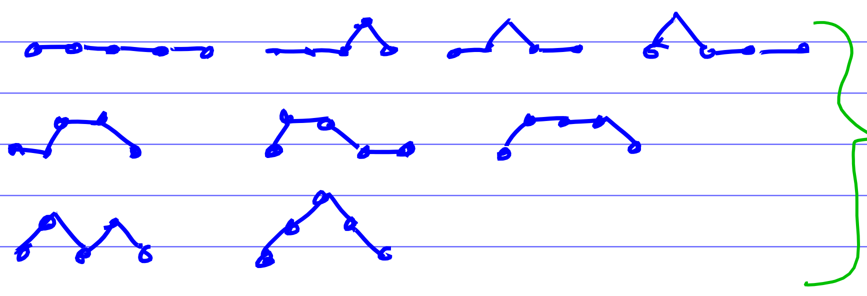
The Motzkin number M_n is # of Motzkin paths with n steps.

Example. $M_3 = 4$



4 Motzkin paths with 3 steps.

$M_4 = 9$



9 Motzkin paths with 4 steps

n	0	1	2	3	4	5	6	7	...
M_n	1	1	2	4	9	21	51	127	...

Clearly,

$$M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \underbrace{\binom{n}{2k}}_{\text{counts \# ways to pick } 2k \text{ non-horizontal (i.e. up or down) steps}} \underbrace{C_k}_{\text{the Catalan number}}$$

counts # ways to pick $2k$ non-horizontal (i.e. up or down) steps

the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$

We will now consider <u>weighted versions</u> of Dyck & Motzkin paths...

Define the <u>height</u> ht(s) of a step s in a Motzkin or Dyck path as 1 + the y-coordinate of its initial point.

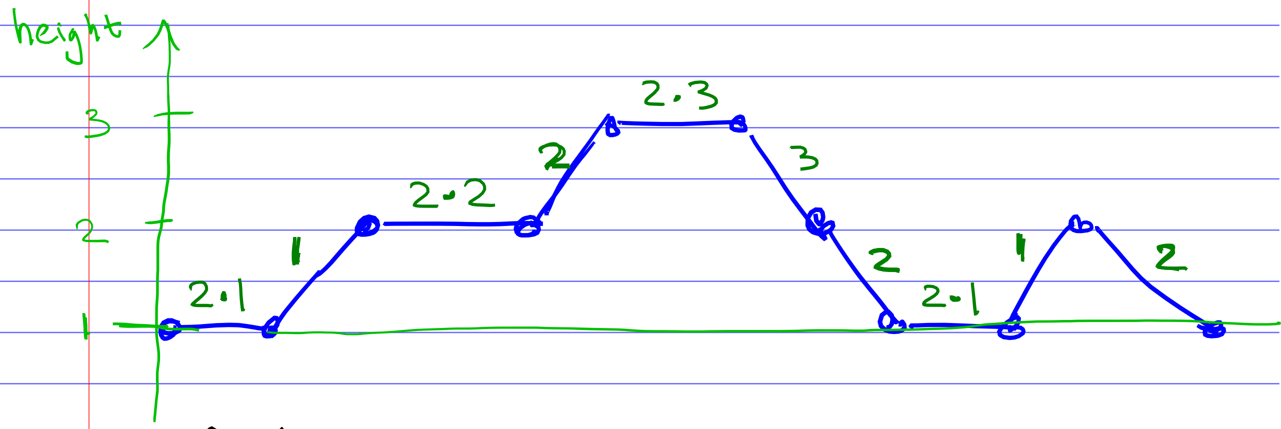
Now define the weight wt(s) of a step s, as follows:

- up or down step : $wt(s) = ht(s)$
- horizontal step : $wt(s) = 2 \cdot ht(s)$.

Define the weight of path P as

$$wt(P) := \prod_{\substack{s \text{ is a} \\ \text{step in } P}} wt(s)$$

<u>Example</u>



$$wt(P) = 2 \cdot 1 \cdot 4 \cdot 2 \cdot 6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 2$$

Theorem

- $$\sum_{\substack{P:\text{ Motzkin}\\ \text{path w/}\\ n\text{ steps}}} wt(P) = (n+1)!$$

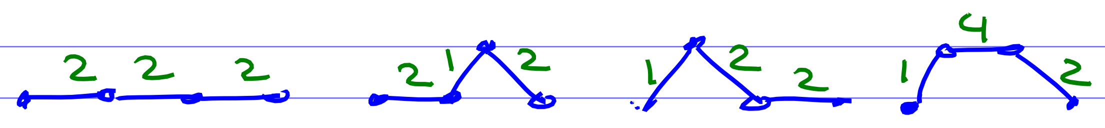
- $$\sum_{\substack{P:\text{ Dyck}\\ \text{path with}\\ 2n\text{ steps}}} wt(P) = A_{2n+1}$$

  \# alternating permutations

  $w = w_1 < w_2 > w_3 < \cdots > w_{2n+1}$

Examples

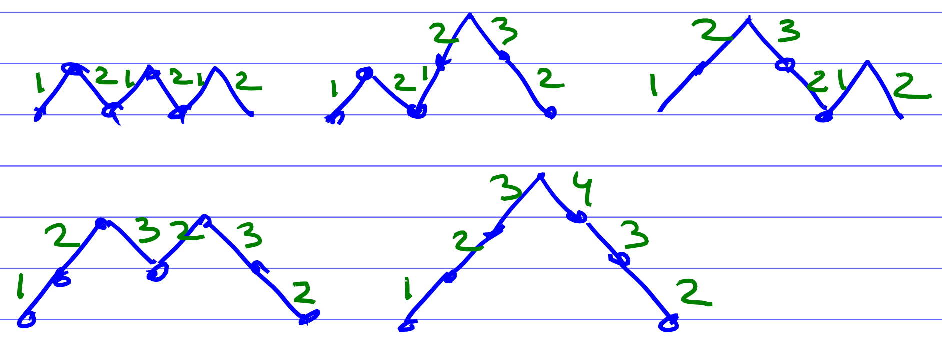
- weighted Motzin paths with 3 steps:



$2\cdot2\cdot2 + 2\cdot1\cdot2 + 1\cdot2\cdot2 + 1\cdot4\cdot2$

$= 24 = 4!$

- weighted Dyck paths with $2\cdot3$ steps:



$1\cdot2\cdot1\cdot2\cdot1\cdot2 + 1\cdot2\cdot1\cdot2\cdot3\cdot2 + 1\cdot2\cdot3\cdot2\cdot1\cdot2$

$+ 1\cdot2\cdot3\cdot2\cdot3\cdot2 + 1\cdot2\cdot3\cdot4\cdot3\cdot2$

$= 272 = A_7$

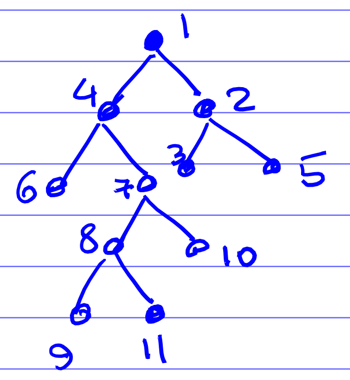
Recall (from the last lecture):

- $(n+1)!$ = # increasing binary trees on $n+1$ vertices

- The number A_{2n+1} of alternating permutations of size $2n+1$ equals

  # complete increasing binary trees with $2n+1$ vertices.

Example.



a complete increasing binary tree:

- labels increase downward
- each vertex has 2 or 0 children

Françon – Viennot bijection gives a bijective way to prove this theorem.

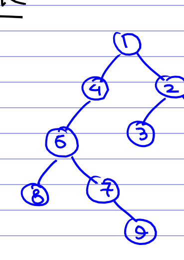
We'll construct a bijection between increasing binary trees and colored Motzkin paths:

- each up & down step s has "color" $\in \{1, 2, \dots, ht(s)\}$
  ht(s) choices
- each horizontal step s has "color" $\in \{1, 2, \dots, ht(s), 1', 2', \dots, ht(s)'\}$
  2 ht(s) choices

Clearly, # colorings of a path P equals wt(P).

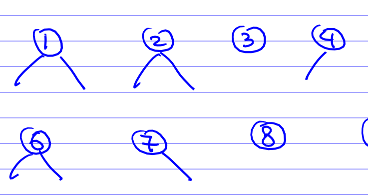
$$\left\{\begin{matrix}\text{increasing} \\ \text{binary} \\ \text{trees on} \\ n+1 \text{ vertices}\end{matrix}\right\} \overset{\text{Françon-Viennot bijection}}{\longleftrightarrow} \left\{\begin{matrix}\text{Colored} \\ \text{Motzkin} \\ \text{paths with} \\ n \text{ steps}\end{matrix}\right\}$$

__Example__

T = 

a (non-complete) increasing binary tree

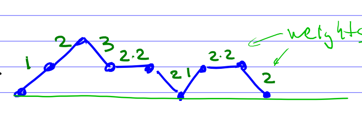
Let's "decompose" T into pieces consisting of vertices with placeholders for their children:



We'll ignore the last vertex n+1 because it will always be like this

In the corresponding Motzkin path P, for $i = 1, \dots, n$,

- if i ⇒ i^{th} step is "up"
- If i ⇒ i^{th} step is "down"
- if i or i ⇒ i^{th} step is horizontal

P = 

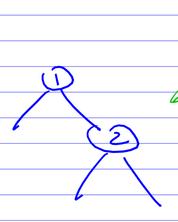
$$wt(P) = 1 \cdot 2 \cdot 3 \cdot (2 \cdot 2) \cdot 2 \cdot 1 \cdot (2 \cdot 2) \cdot 2$$

counts # ways to combine the pieces into a binary tree.

In our example:

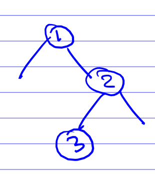
1. ① 1st step in P is "up"
   wt(s) = 1

2. ② 2nd step is "up", wt = 2.
   2 ways to attach it to one of the placeholders:
   the color of 2nd step = the position (from the left) of the placeholder to which 2nd vertex is attached.



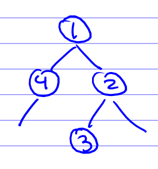
② is attached to the 2nd placeholder, so the corresp step is colored "2"

3. ③ 3rd step is "down", wt = 3
   3 ways to attach to one of the placeholders



③ is attached to 2nd placeholder so the color is 2

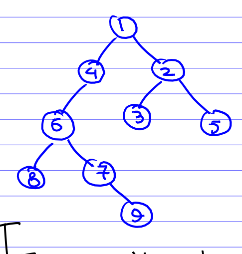
4. ④ 4th step is horizontal
   wt = 2·2.
   the "color" = j (for ④ with left child) or j' (for ④ with right child),
   where j is the position of the placeholder.

e.g. 

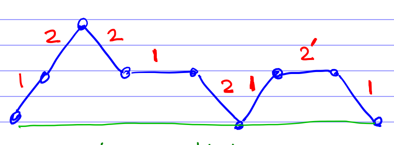
④ is attached to 1st placeholder, so the color = 1.

etc.

In our example, we get

T = 

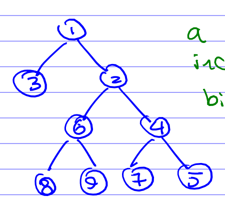
↓ Françon - Viennot



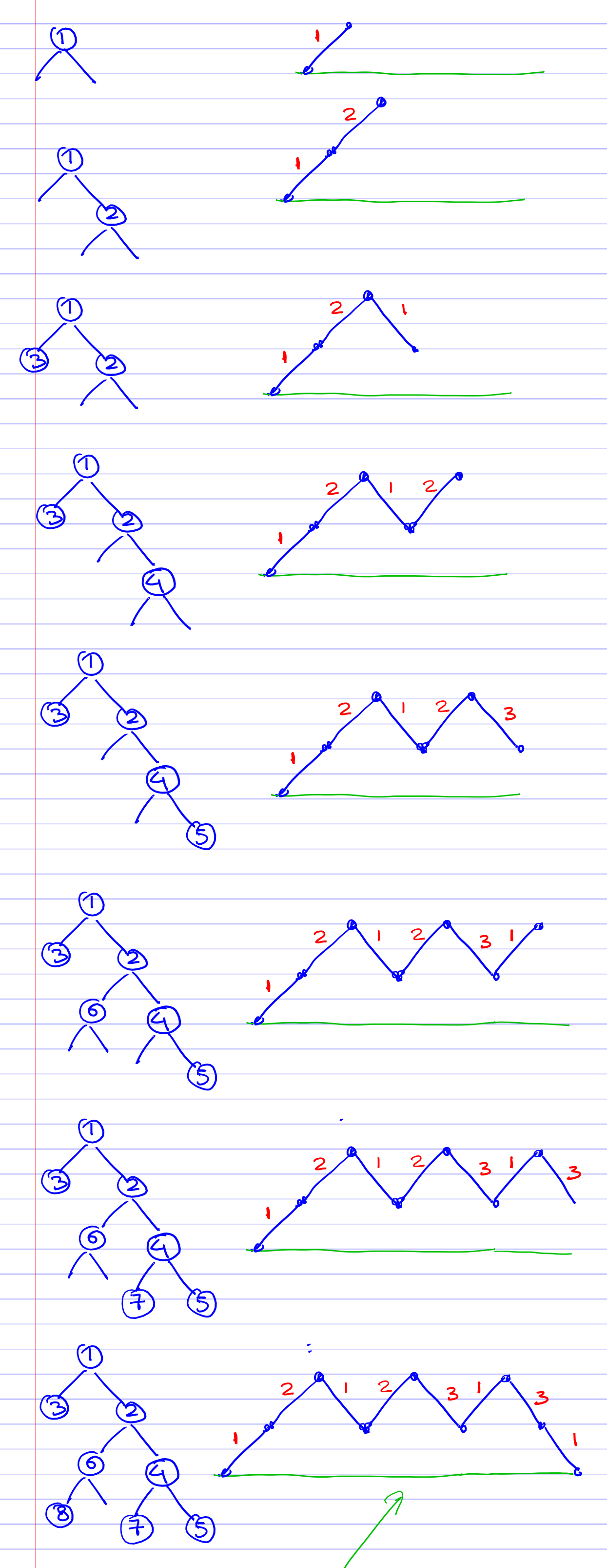
Colored Motzin path

This gives the needed bijection & proves the 1st claim.

To get the second claim about Dyck paths & A_{2n+1} we just restrict this construction to complete increasing binary trees, which correspond to colored Dyck paths.

__Example__ 

a complete increasing binary tree



the colored Dyck path that corresponds to the tree T

There are many other special cases of the Françon-Viennot bijection.

Corollary

\# permutations in S_{n+1} without double ascents $w_i < w_{i+1} < w_{i+2}$ (assuming that $w_0 = 0$)

$=$ \# increasing binary trees on $n+1$ vertices s.t. $\nexists$ vertices that have only the right child.

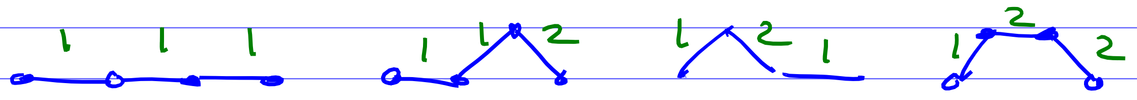
$=$ \# weighted Motzkin paths with n steps and the weights given by

$$wt(s) = ht(s) \text{ for any } s$$

(no factor 2 for horizontal steps.)

Example $n=3$

\# permutations in S_4 w/o double descent (& no initial descent) equals:



$$1 + 2 + 2 + 4 = 9.$$

Here are these 9 permutations:

2143, 3142,
3241, 3214
4132, 4213, 4231
4312, 4321

We expressed # alternating permutations A_{2n+1} (the tangent numbers) as # weighted Dyck paths.

We can reformulate this by putting the weights only on the "up" edges in a Dyck path.

Theorem $A_{2n+1} =$

$$= \sum_{\substack{P \text{ Dyck path} \\ \text{with } 2n \text{ steps}}} \prod_{\substack{s \text{ is an} \\ \text{"up" step} \\ \text{in } P}} ht(s) \cdot (ht(s)+1)$$

How about the secant numbers A_{2n} ?

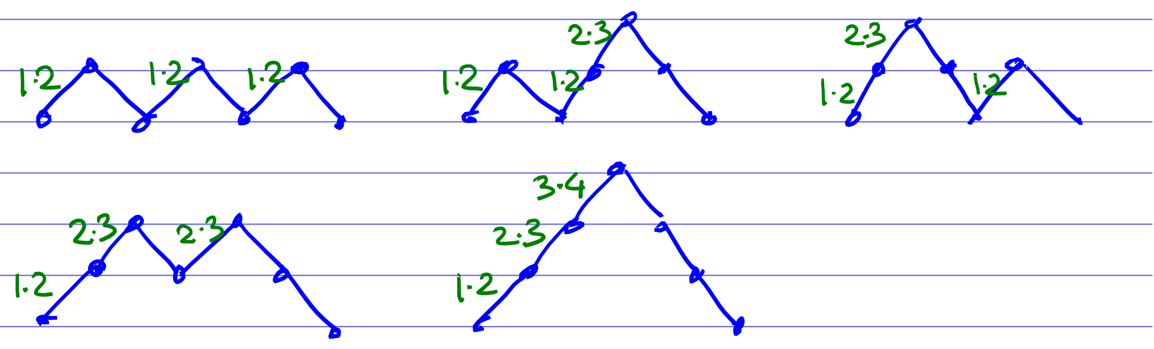
We just need to slightly modify the weights

Theorem $A_{2n+1} =$

$$= \sum_{\substack{P \text{ Dyck path} \\ \text{with } 2n \text{ steps}}} \prod_{\substack{s \text{ is an} \\ \text{"up" step} \\ \text{in } P}} ht(s)^2$$

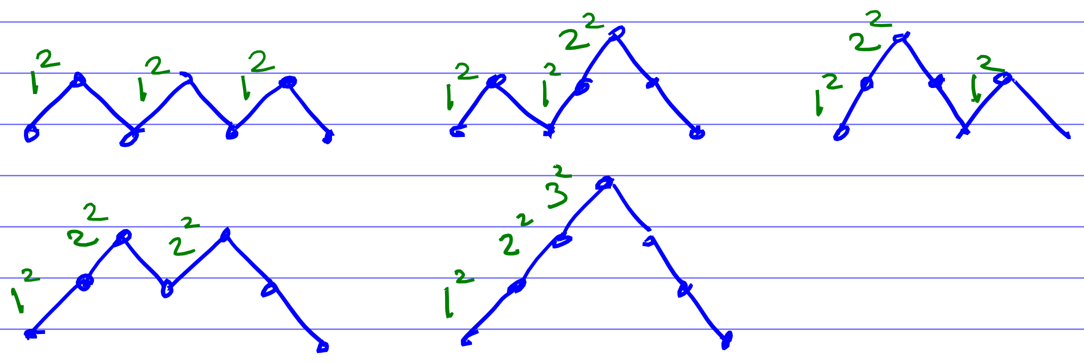
Exercise. Prove this.

Example $n=3$



$$A_7 = (1\cdot2)(1\cdot2)(1\cdot2) + (1\cdot2)(1\cdot2)(2\cdot3)$$
$$+(1\cdot2)(2\cdot3)(1\cdot2) + (1\cdot2)(2\cdot3)(2\cdot3)$$
$$+ (1\cdot2)(2\cdot3)\cdot(3\cdot4)$$

$$= 272$$



$$A_6 = 1^2\cdot1^2\cdot1^2 + 1^2\cdot1^2\cdot2^2 +$$
$$+ 1^2\cdot2^2\cdot1^2 + 1^2\cdot2^2\cdot2^2$$
$$+ 1^2\cdot2^2\cdot3^2$$

$$= 61.$$

Continued Fractions and weighted Dyck (& Motzkin) paths.

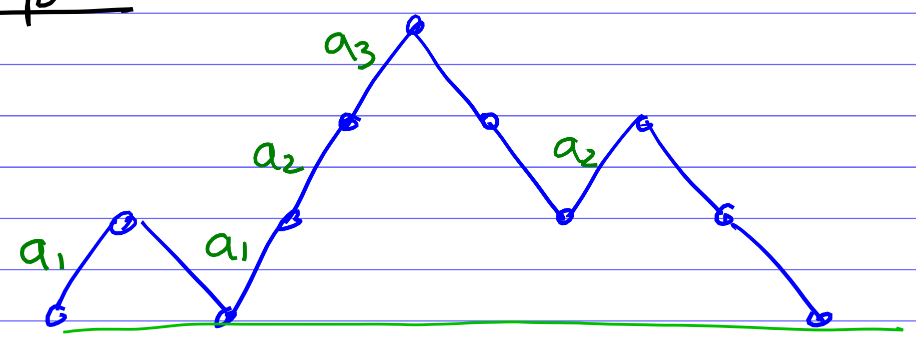
Let $a_1, a_2, a_3, \ldots$ be some "weights". (Let's treat them as variables.)

Weighted Dyck paths

For a Dyck path P, define

$$\mathrm{wt}(P) := \prod_{\substack{s \text{ is an} \\ \text{"up" step in } P}} a_{ht(s)}$$

Example

$P =$ 

$$\mathrm{wt}(P) = a_1 \cdot a_1 \cdot a_2 \cdot a_3 \cdot a_2$$

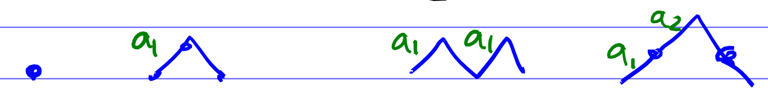
Flajolet's "Fundamental Lemma"

Theorem (P. Flajolet 1980)

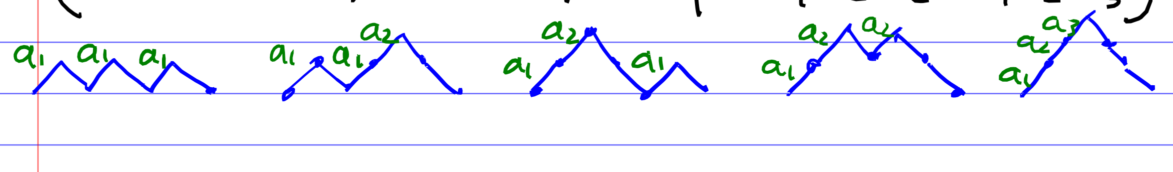
$$\sum_{n \geq 0} \Big(\sum_{\substack{P \text{ Dyck} \\ \text{path with} \\ 2n \text{ steps}}} \mathrm{wt}(P) \Big) x^n = \cfrac{1}{1 - \cfrac{a_1 x}{1 - \cfrac{a_2 x}{1 - \cfrac{a_3 x}{1 - \ldots}}}}$$

(This is an identity of formal power series.)

$$\text{L.H.S.} = 1 + a_1 x + (a_1 a_1 + a_1 a_2) x^2$$



$$+ (a_1^3 + a_1 a_1 a_2 + a_1 a_2 a_1 + a_1 a_2 a_2 + a_1 a_2 a_3) x^3$$



$+ \ldots$

$$\text{R.H.S.} = \cfrac{1}{1 - \cfrac{a_1 x}{1 - \cfrac{a_2 x}{\ldots}}} =$$

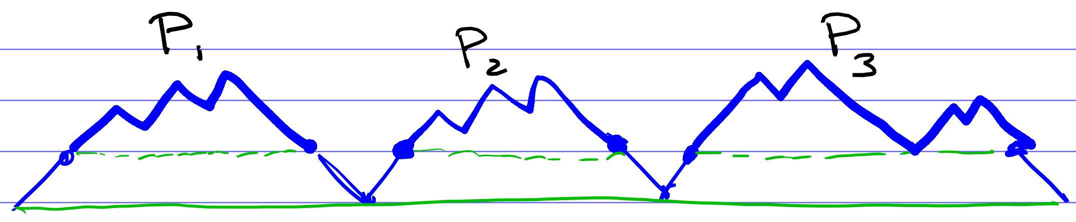
$$1 + \left(\cfrac{a_1 x}{1 - \cfrac{a_2 x}{\ldots}} \right) + \left(\cfrac{a_1 x}{1 - \cfrac{a_2 x}{\ldots}} \right)^2 + \ldots$$

$$= 1 + a_1 x + a_1 a_2 x^2 + a_1^2 x^2 + \ldots$$

Proof. Let's think of the L.H.S as a function

$F(x, a_1, a_2, \dots)$ in the infinitely many variables $x, a_1, a_2, a_3, \dots$

We can decompose a Dyck path P into several smaller Dyck paths $P_1, \dots, P_k$:

$P =$ 

Clearly, all heights in $P_1 \dots P_k$ are increased by 1.

So we get

$$F(x, a_1, a_2, \dots) =$$

$$\sum_{k \geq 0} \left(a_1 x \, F(x, a_2, a_3, \dots)\right)^k$$

$$= \frac{1}{1 - a_1 x \cdot F(x, a_2, a_3, \dots)}$$

Repeating this we get the needed continued fraction:

$$F(x, a_1, a_2, \dots) =$$

$$= \frac{1}{1 - a_1 x \cdot \frac{1}{1 - a_2 x \, F(x, a_3, a_4, \dots)}}$$

$$= \cfrac{1}{1 - \cfrac{a_1 x}{1 - \cfrac{a_2 x}{1 - \cfrac{a_3 x}{1 - \dots}}}}$$

$\square$

Examples

1. $a_1 = a_2 = a_3 = \dots = 1$

$$\sum_{n\geq 0} C_n x^n = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-x\cdots}}}$$

Clearly, the R.H.S satisfies

$$f = \frac{1}{1 - x\cdot f},$$

which is equivalent to the quadratic equation

$$x f^2 - f + 1 = 0$$

for the generating function $f(x)$ for the Catalan numbers.

By the way,

$$\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-\dots}}} = \varphi := \frac{1+\sqrt{5}}{2}$$

(the golden ratio)

Indeed, we get the quadratic eqn. $\varphi = \frac{1}{1-\varphi}$

$\iff \varphi^2 - \varphi + 1 = 1.$

So the "sum of all Catalan numbers"

$$f(1) = C_0 + C_1 + C_2 + \dots$$

equals the golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

---

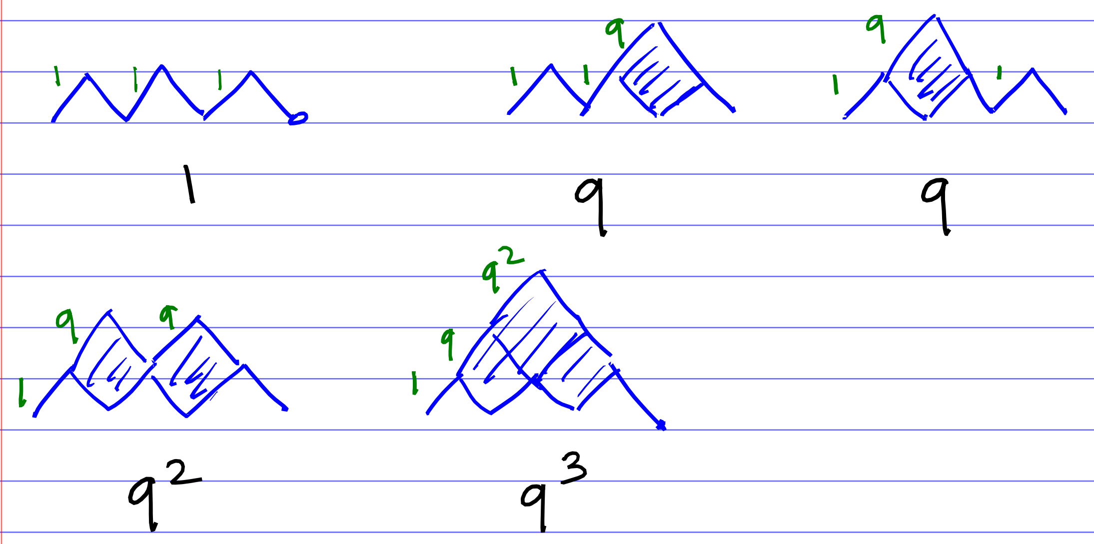
2. $a_i = q^{i-1}$

$$\sum_{n\geq 0} C_n(q)\, x^n = \cfrac{1}{1-\cfrac{x}{1-\cfrac{qx}{1-q^2x\cdots}}}$$

this is the Ramanujan's continued fraction

$C_n(q)$ counts Dyck paths by $q^{(\text{Area below the path})}$

Example $n=3$



$$C_3(q) = 1 + 2q + q^2 + q^3$$

3. $a_i = i\cdot(i+1)$

(*) $$\sum_{n\geq 0} A_{2n+1}\, x^n = $$

$$= \cfrac{1}{1-\cfrac{1\cdot 2\cdot x}{1-\cfrac{2\cdot 3\cdot x}{1-\cfrac{3\cdot 4\, x}{1-\cfrac{4\cdot 5\cdot x}{\cdots}}}}}$$

4. $a_i = i^2$

(**) $$\sum_{n\geq 0} A_{2n}\, x^n = $$

$$= \cfrac{1}{1-\cfrac{1^2 x}{1-\cfrac{2^2 x}{1-\cfrac{3^2 x}{1-\cdots}}}}$$

<u>Remark</u>. Examples 3 & 4 (formulas (*) and (**)) are (rare) exceptions to the rule that we should use <u>exponential</u> generating functions for <u>labelled</u> combinatorial objects.

Recall that in the last lecture we discussed exp. generating functions

$$\sum_{n\geq 0} A_{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \tan(x)$$

$$\sum_{n\geq 0} A_{2n} \frac{x^{2n}}{(2n)!} = \sec(x).$$

In (*) and (**) we have <u>ordinary</u> generating functions for A_{2n+1} and A_{2n}. These two power series <u>diverge</u> for <u>all</u> nonzero values of x.

But idenities (*) & (**) are still true for <u>formal</u> power series in x.

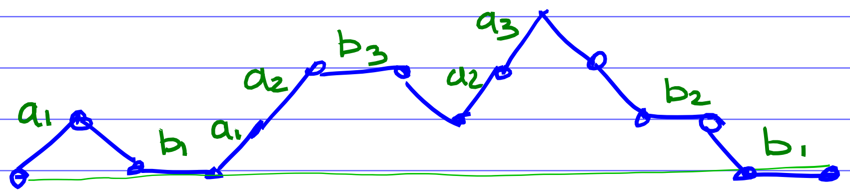
More generally we get a similar formula for weighted Motzkin paths.

$a_1, a_2, a_3, \ldots ; b_1, b_2, b_3, \ldots$
(formal variables)

For a Motzkin path P

$$wt(P) := \prod_{\substack{s \text{ up} \\ \text{step}}} a_{ht(s)} \prod_{\substack{\tilde{s} \text{ horizontal} \\ \text{step}}} b_{ht(\tilde{s})}$$

Example

$P =$ 

$$wt(P) = a_1 b_1 a_1 a_2 b_3 a_2 a_3 b_2 b_1$$

Flajolet's Fund. Lemma

$$\sum_{n \geq 0} \sum_{\substack{P \text{ Motzkin} \\ \text{path with} \\ n \text{ steps}}} wt(P) =$$

$$= \cfrac{1}{1 - b_1 x - \cfrac{a_1 x^2}{1 - b_2 x - \cfrac{a_2 x^2}{1 - b_3 x - \cfrac{a_3 x^2}{1 - \ldots}}}}$$

The proof is similar to the proof for the case of weighted Dyck paths. So we'll skip it.

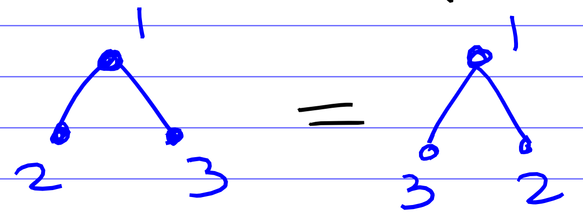
Increasing 012 - trees

Definition. A labelled tree T on the vertex set $1, 2, \ldots, n$ is an increasing 012 - tree if

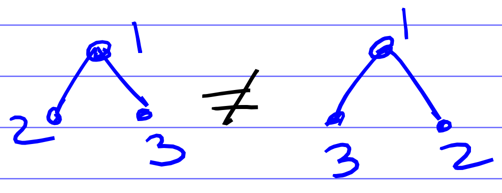
1. T is an increasing tree, i.e. $\operatorname{inv}(T) = 0$.
2. Each vertex has 0, 1, or 2 children.

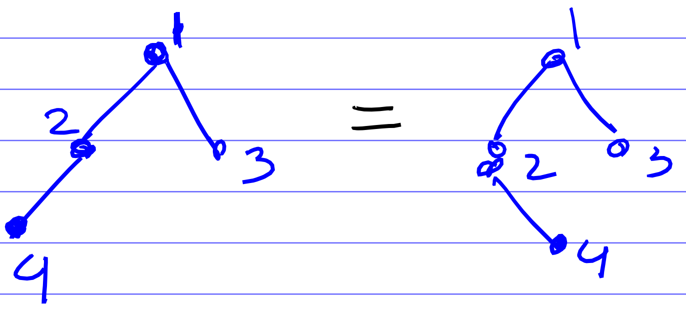
(Here we assume that the vertex 1 is the root of T.)

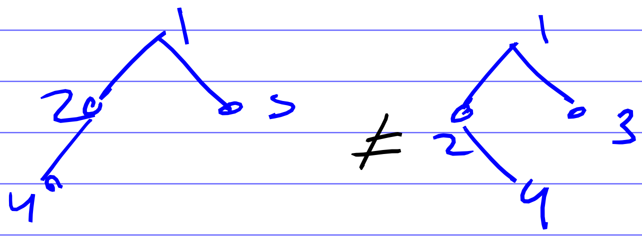
Remark These trees are labelled trees. They should not be confused with increasing binary trees. (These binary trees are not just labelled trees. They have additional information about left & right children.)



as increasing labelled trees

But  as binary trees

Also  as labelled trees, but



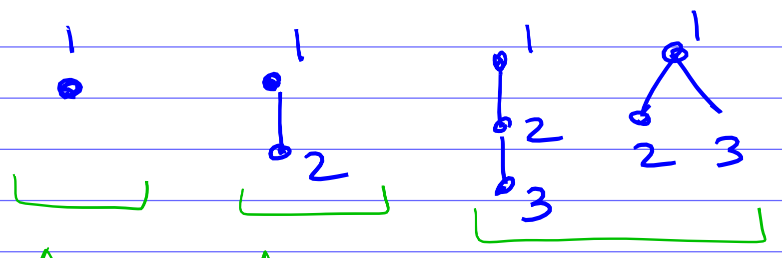
as binary trees.

Recall, # increasing trees on n vertices $= (n-1)!$

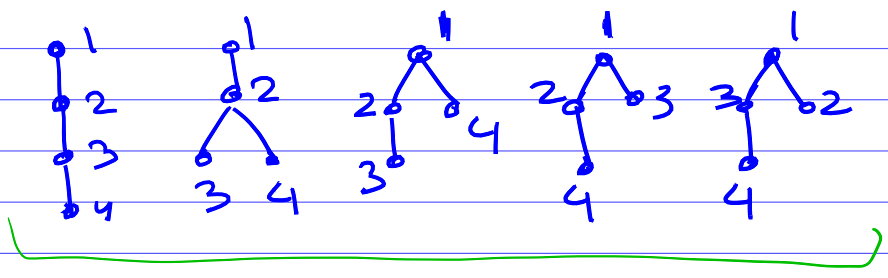
Theorem. # 012-trees on n vertices $= A_n$ (# alternating permutations)

Exercise. Prove this.

Examples



$A_1 = 1$ $A_2 = 1$ $A_3 = 2$



$A_4 = 5$

Compare:

- # all increasing binary trees on n vertices is $n!$

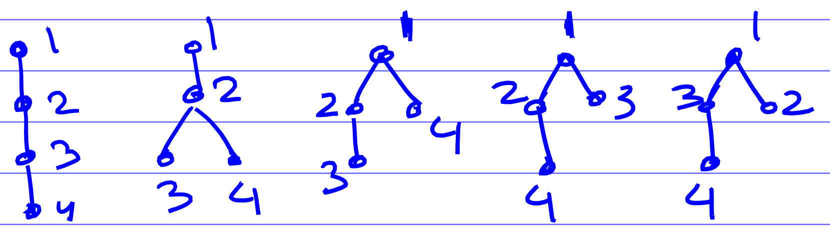
Among them, # complete or almost complete trees is A_n

- # increasing labelled trees on n vertices is $(n-1)!$

Among them, # 012-trees is A_n.

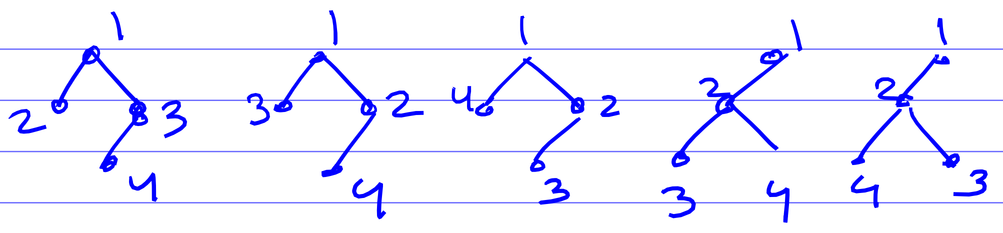
In particular, we have not just $A_n \leq n!$, but also $A_n \leq (n-1)!$

For $n=4$



(increasing 012-trees)

vs.



(almost complete increasing binary trees)