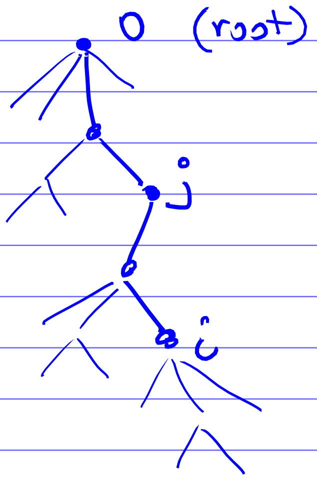
Inversions in Trees

Recall, that earlier in the semester we discussed statistics on permutations, such as # inversions in permutations. Let's do a similar thing for trees...

Let T be a (labelled) tree on the $n+1$ vertices $0, 1, 2, \ldots, n$. We'll think that the vertex 0 is the <u>root</u> of T.

<u>Definition.</u> A pair (i,j) $i,j \in \{1,\ldots,n\}$ is an <u>inversion</u> of T if

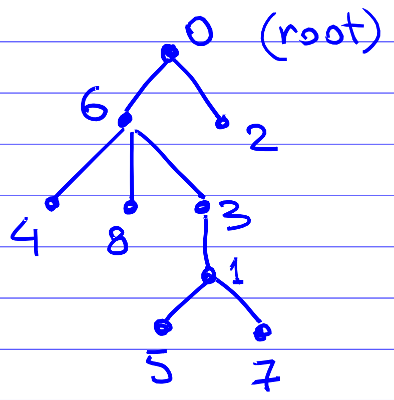
- $i<j$
- the vertex j belongs to the shortest path in T between the root 0 and the vertex i.



$i<j$ form an inversion

Let $\mathrm{inv}(T)$ be the number of inversions in T.

Example. $T =$



inversions:

(1,3), (1,6)

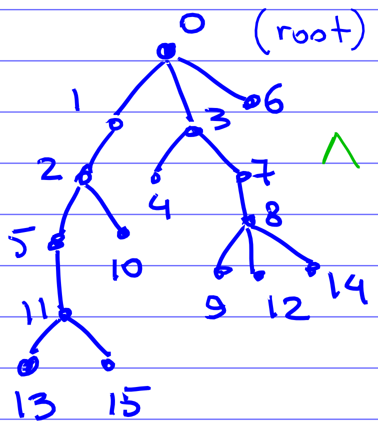
(3,6), (4,6), (5,6)

$$\mathrm{inv}(T) = 5.$$

Def. T is called an <u>increasing tree</u> if $\mathrm{inv}(T) = 0$.

Example.

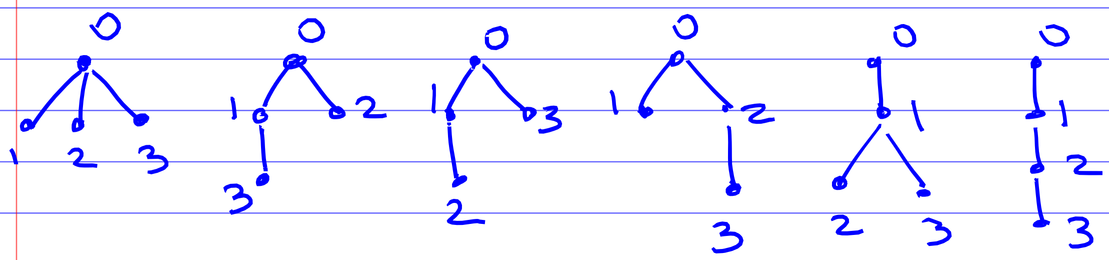
an increasing tree



the labels increase downwards

Proposition # increasing trees with $n+1$ vertices equals $n!$

Example $n=3$

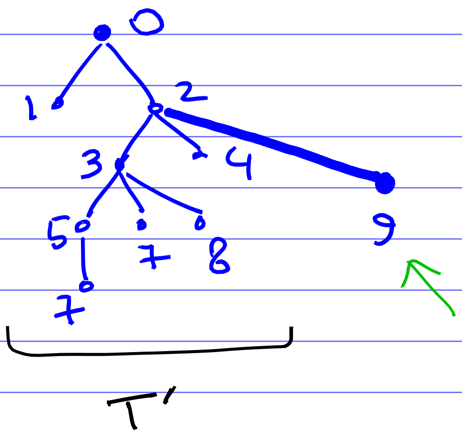


$3! = 6$ increasing trees on 4 vertices

Proof Induction on n.

An increasing tree T on vertices $0, 1, \dots, n$ can be constructed from an increasing tree T' on vertices $0, 1, \dots, n-1$ by connecting new vertex n with any of the vertices $0, 1, \dots, n-1$ by an edge.

$T =$



there are n ways to add new vertex n

$n = 9$

By induction, # increasing trees on $n+1$ vertices is $n \cdot (n-1)! = n!$

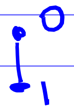
$\square$

Tree inversion polynomial

$$I_n(x) := \sum_{\substack{T \text{ labelled} \\ \text{tree on } n+1 \\ \text{vertices } 0,1,\dots,n}} x^{\mathrm{inv}(T)}$$

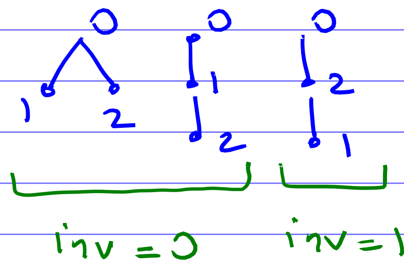
Examples. $n=0$: $\bullet 0$ $\qquad I_0 = 1$

$n=1$:



$I_1 = 1$

$n=2$:



$I_2 = x+2$

Parking functions again...

Theorem (G. Kreweras, 1980)

$$I_n(x) = \sum_{\substack{(f_1,\dots,f_n) \\ \text{parking function}}}' x^{\binom{n+1}{2} - (f_1+\dots+f_n)}$$

Recall $(f_1,\dots,f_n) \in \mathbb{Z}_{>0}^n$ is a parking function iff $\exists$ a permutation $w_1,\dots,w_n$ of $1,2,\dots,n$ such that $f_i \le w_i$ $\forall\, i=1,\dots,n$.

So

$$n \le f_1 + \dots + f_n \le 1+2+\dots+n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

Examples. n = 2

parking functions: $\underbrace{(1,2), (2,1)}_{0}, \underbrace{(1,1)}_{1}$

$$I_2(x) = 2 + x$$

n = 3. parking functions :

0 [(1,2,3) & all permutations ×6

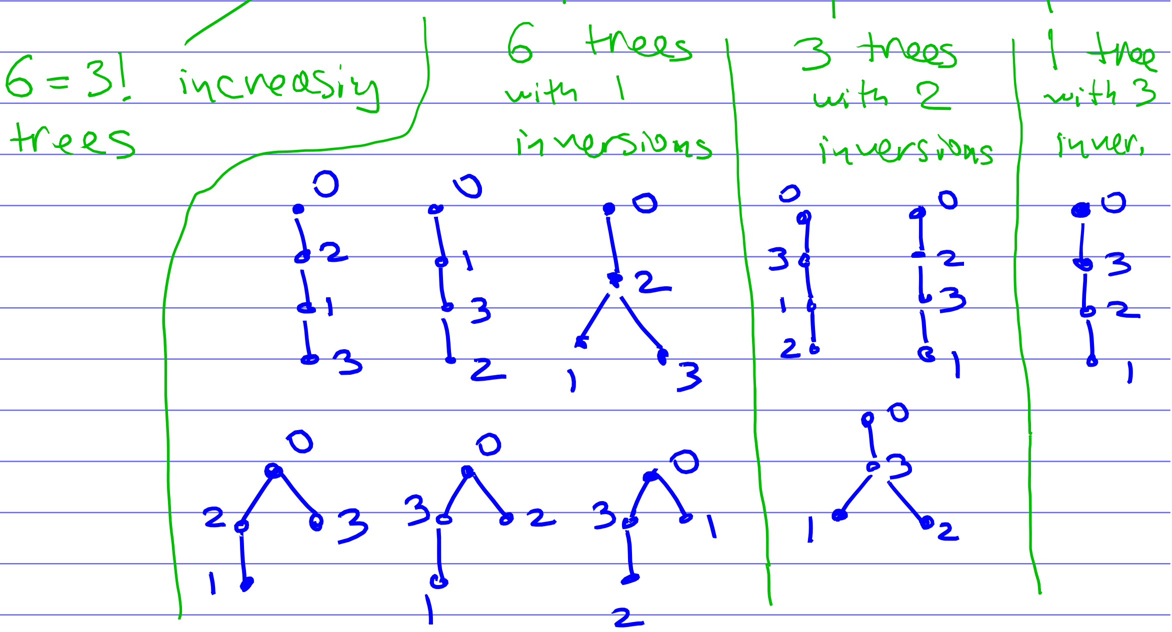
1 [(1,2,2) & permutations ×3
(1,1,3) & permutations ×3

2 [(1,1,2) & permutations ×3

3 [(1,1,1) ×1

So

$$I_3(x) = 6 + 6x + 3x^2 + x^3$$



How to prove this theorem?

- By induction, using recurrence relations.
- Bijective proof.

  Kreweras constructed a bijection: $\{\text{trees } T\} \longleftrightarrow \{\text{parking functions } f\}$

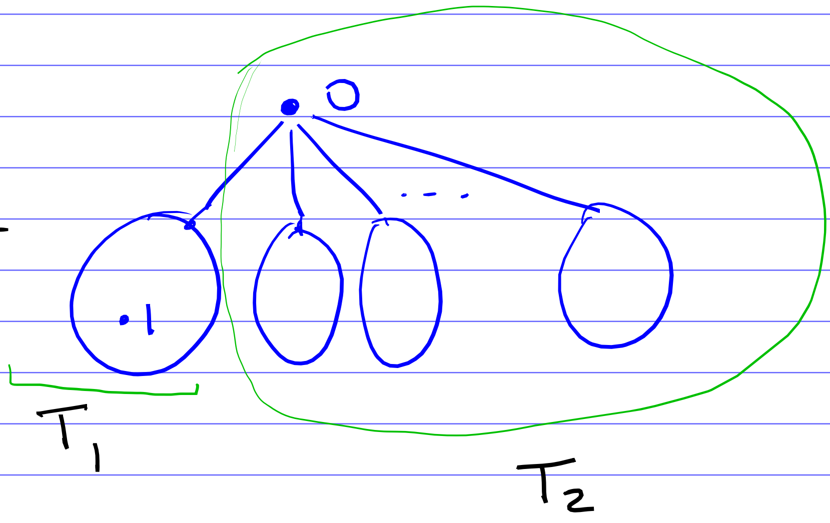
  such that $\mathrm{inv}(T) = \binom{n+1}{2} - (f_1 + \dots + f_n)$.

But the construction is a bit complicated. So we will not give it now.

<u>Recurrence Relation</u> for the inversion polynomial $I_n(x)$.

T a labelled tree on vertices $0, \dots, n$.

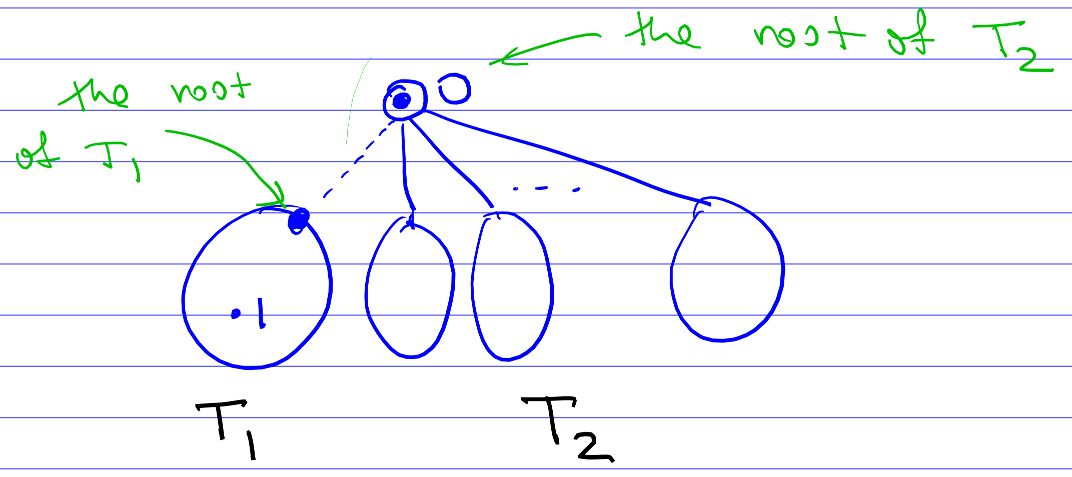
Let's subdivide it into 2 trees T_1 and T_2, as follows:

$T =$ 

T_1 is the connected component of the forest $T \setminus$ vertex 0 that contains the vertex 1.

Assume that T_1 has k vertices.
Then T_2 has $n-k+1$ vertices
(including the root 0).

Both T_1 & T_2 have roots



The root of T_2 is 0
(the same as the root of T).

But the root of T_1 may not be the minimal vertex 1 of T_1.

We have

$$(*) \quad I_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \cdot \tilde{I}_{k-1}(x)\, I_{n-k}(x).$$

where

- $\binom{n-1}{k-1}$ counts # ways to distribute vertices between T_1 & T_2

  T_1 has vertex 1 and some other $k-1$ vertices from $\{2, \dots, n\}$

  T_2 has all remaining $n-k+1$ vertices

- $I_{n-k}(x)$ counts all choices for T_2 accoring to # inversions

- $\tilde{I}_{k-1}(x)$ counts all choices for T_1 according to # inversions.

But we need to be careful here, because the root r of T_1 may not be its minimal vertex.

The polynomial $\tilde{I}_{k-1}(x)$.

Let T_1 be a tree on k labelled vertices, say, $1, 2, \dots, k$ with one selected vertex r (the root of T_1).

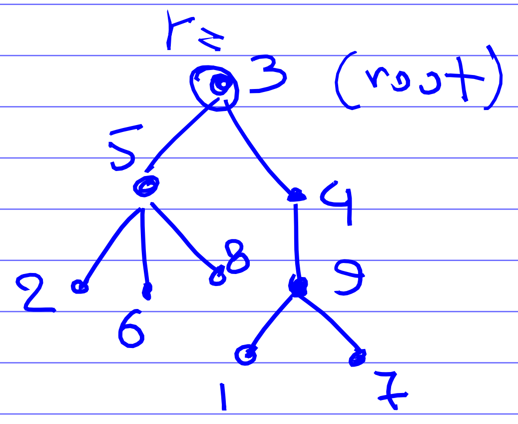
Define an inversion in T_1 as a pair (i,j) $i,j \in \{1,\dots,k\}$ such that

- $i < j$
- j belongs to the shortest path between the root r & the vertex i.

The difference between this definition and the previous definition of inversions is that we now allow j to be the root r. (This can happen if the root r is not the minimal vertex of T_1).

Example

$T_1 =$



inversions: $(1,3)$ $(1,4)$, $(1,9)$
$(2,3)$, $(2,5)$, $(7,9)$

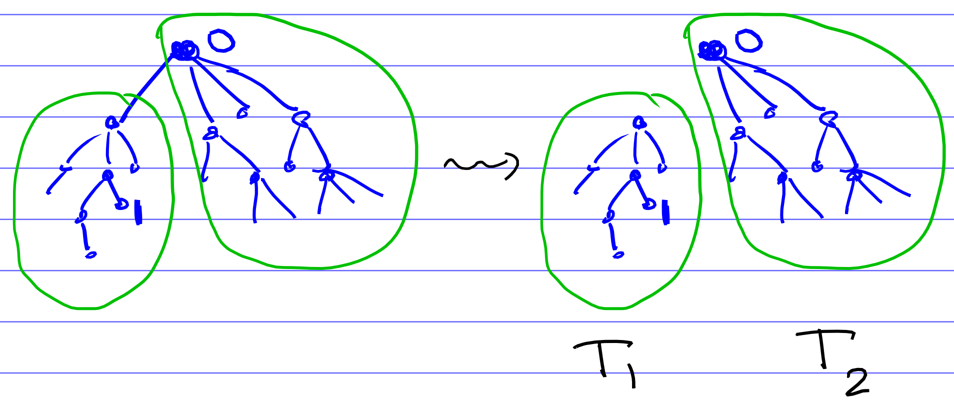
$\mathrm{inv}(T) = 6$.

Define

$$\tilde{I}_{k-1}(x) := \sum_{\substack{T_1 \text{ tree} \\ \text{on } k \text{ labelled} \\ \text{vertices } 1,\dots,k \\ \text{with selected root } r}} x^{\mathrm{inv}(T)}$$

Notice that, if

$T =$



the $\mathrm{inv}(T) = \mathrm{inv}(T_1) + \mathrm{inv}(T_2)$.

So the relation $(*)$ holds.

Let's relate the polynomials

(1) $$\tilde{I}_{k-1}(x) := \sum_{\substack{T_1 \text{ rooted tree} \\ \text{on } k \text{ vertices } 1,\dots,k \\ \text{with any possible} \\ \text{root } r \in \{1,\dots,k\}}} x^{\mathrm{inv}(T_1)}$$

and

(2) $$I_{k-1}(x) := \sum_{\substack{T_1 \text{ rooted tree} \\ \text{on } k \text{ vertices } 1,\dots,k \\ \text{such that} \\ \text{the root } r=1}} x^{\mathrm{inv}(T_1)}$$

Lemma

$$\tilde{I}_{k-1}(x) = \left(1+x+\dots+x^{k-1}\right) I_{k-1}(x).$$

This follows from

Lemma. $$\sum_{\substack{T_1 \text{ rooted tree on } 1,\dots,k \\ \text{with fixed root } r}} x^{\mathrm{inv}(T)}$$

$$= x^{r-1} I_{k-1}(x).$$

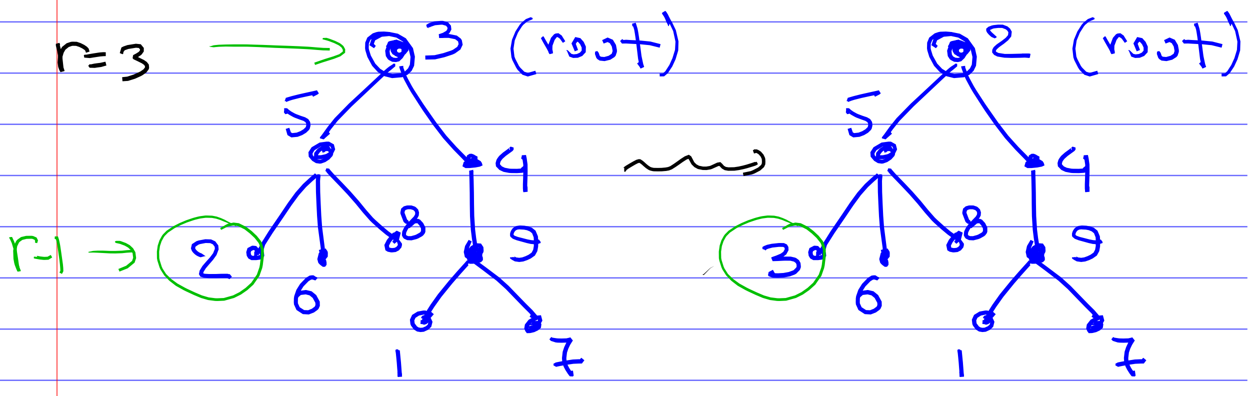
Proof Induction on r.

$r=1$: $\mathrm{LHS} = I_{r-1}(x)$ by def.

Induction step. $r>1$

Switch the root r with the vertex $r-1$. This decreases # inversions in T_1 by 1.

Example



inv = 6 $\qquad$ inv = 6 - 1 = 5

$(r-1, r)$ is no longer an inversion

So we get

$$\sum_{\substack{T_1 \text{ has} \\ \text{root } r}} x^{\mathrm{inv}(T_1)} = x \cdot \sum_{\substack{T_1 \text{ has} \\ \text{root } r-1}} x^{\mathrm{inv}(T_1)}$$

$$= \dots = x^{r-1} \sum_{T \text{ has root } 1} = x^{r-1} I_{k-1}(x).$$

$\square$

Now we obtain

Theorem The inversion polynomial $I_n(x)$ satisfies

$$I_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} (1+x+\dots+x^{k-1}) I_{k-1}(x) I_{n-k}(x),$$

for $n \geq 1$.

And $I_0(x) = 1$.

In order to show that

$$I_n(x) = \sum_{\substack{(f_1 \dots f_n) \\ \text{parking function}}}' x^{\binom{n+1}{2} - (f_1 + \dots + f_n)}$$

it is enough to show that the R.H.S. satisfies the same recurrence relation as $I_n(x)$.

Exercise. Show this.

Some special values of the inversion polynomial

- $I_n(1) = (n+1)^{n-1}$ — # labelled trees on $n+1$ vertices
- $I_n(0) = n!$ — # increasing trees on $n+1$ vertices

Theorem

- $I_n(-1) = A_n$,

where A_n is the number of <u>alternating permutations</u>

$$w = w_1 < w_2 > w_3 < \dots \quad w_n$$

of the numbers $1, 2, \dots, n$.

Example. $I_3(-1) =$

$= (-1)^3 + 3(-1)^2 + 6\cdot(-1) + 6 = 2$

There are 2 alternating permutations is S_3:

$1 < 3 > 2$ and $2 < 3 > 1$.

Remark The numbers A_n have many different names:

Euler numbers, André numbers, zigzag numbers, updown numbers, tangent & secant numbers, ...

They are related to the Bernoulli numbers.

Should not be confused with the Eulerian numbers

# of alternating permutations

n	0	1	2	3	4	5	6	7	8	...
A_n	1	1	1	2	5	16	61	271	1385	...

Proof Let's plug in $x=-1$ in the recurrence relation for $I_n(x)$.

Observe that

$$(1+x+\dots+x^{k-1})\Big|_{x=-1} = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

We obtain

$$I_n(-1) = \sum_{\substack{k\in[n] \\ k \text{ even}}} \binom{n-1}{k-1} I_{k-1}(-1)\, I_{n-k}(-1)$$

We can show that the same recurrence relation holds for the numbers A_n of alternating permutations:

$$(**) \quad A_n = \sum_{\substack{k\in[n] \\ k \text{ even}}} \binom{n-1}{k-1} A_{k-1}\, A_{n-k}$$

We can see this by subdividing an alternating permutation

$$w_1 < w_2 > w_3 < w_4 > w_5 < w_6 > \dots \; w_n$$

into 2 alternating permut., as follows:

$$\underbrace{w_1 < w_2 > \dots > w_{k-1}} < \overset{n}{\overset{\parallel}{w_k}} > \underbrace{w_{k+1} < \dots w_n}$$

($w_k = n$.
k should be even!)

$$w' = (w_1 < w_2 > \dots > w_{k-1})$$

$$w'' = (w_{k+1} < w_{k+2} > \dots \; w_n)$$

- There are $\binom{n-1}{k-1}$ ways to pick the subset $\{w_1,\dots,w_{k-1}\}$ in $\{1,2,\dots,n-1\}$
- There are A_{k-1} ways to order $w_1,\dots,w_{k-1}$ as an alternating permutation.
- There are A_{n-k} ways to order the remaining entries $\{w_{k+1},\dots,w_n\} = [n-1] - \{w_1,\dots,w_{k-1}\}$

So we get the recurrence (**)

Together with the initial condition $A_0 = I_0(-1) = 1$ this proves that

$$A_n = I_n(-1)$$

by induction on n. $\square$

Exp. Generating functions

Labelled trees & alternating permutations are <u>labelled</u> objects. So we need to use <u>exponential</u> generating functions.

Let

$$A_n(x) := \sum_{n\geq 0} A_n \frac{x^n}{n!}$$

Let's express the recur. rel. (**) in terms of $A(x)$.

(**) $\Leftarrow$

$$A_n \cdot \frac{x^{n-1}}{(n-1)!} = \sum_{\substack{k\in[n] \\ k \text{ even}}} A_{k-1} \frac{x^{k-1}}{(k-1)!} \cdot A_{n-k} \frac{x^{n-k}}{(n-k)!}$$

Sum this over all $n\geq 1$.

$$\underbrace{\sum_{n\geq 1} A_n \frac{x^{n-1}}{(n-1)!}}_{A'(x)} = \underbrace{\left(\sum_{\substack{k\geq 1 \\ \text{even}}} A_{k-1} \frac{x^{k-1}}{(k-1)!}\right)}_{\text{the odd part of } A(x)} \underbrace{\left(\sum_{m\geq 0} A_m \frac{x^m}{m!}\right)}_{A(x)}$$

Let's consider the even and odd parts of $A(x)$

$$A(x) = A^{even}(x) + A^{odd}(x),$$

where

$$A^{even}(x) := \sum_{\substack{n \geq 0 \\ n \text{ even}}} A_n \frac{x^n}{n!}$$

$$A^{odd}(x) := \sum_{\substack{n \geq 1 \\ n \text{ odd}}} A_n \frac{x^n}{n!}$$

$$(**) \iff A'(x) = A^{odd}(x) \cdot A(x)$$

Equivalently,

<u>Proposition</u>.

$$\frac{d\, A^{even}(x)}{dx} = A^{odd}(x) \cdot A^{even}(x)$$

$$\frac{d\, A^{odd}(x)}{dx} = A^{odd}(x) \cdot A^{odd}(x)$$

Initial conditions:

$$A^{even}(0) = 1, \quad A^{odd}(0) = 0.$$

Secant & Tangent Numbers

Theorem

$$A^{even}(x) = \sec(x) := \frac{1}{\cos(x)}$$

$$A^{odd}(x) = \tan(x).$$

Proof. Check that $\sec(x)$ & $\tan(x)$ satisfy the same differential equations:

$$\sec(x)' = \tan(x) \cdot \sec(x)$$

$$\tan(x)' = \tan(x) \cdot \tan(x)$$

and $\sec(0) = 1$, $\tan(0) = 0$. $\square$

Remark. This is why the numbers A_{2n-1} are also called the tangent numbers and A_{2n} are called the secant numbers.

- The secant numbers A_{2n} are also called the Euler numbers.
- The tangent numbers A_{2n-1} are related to the Bernoulli numbers B_m.

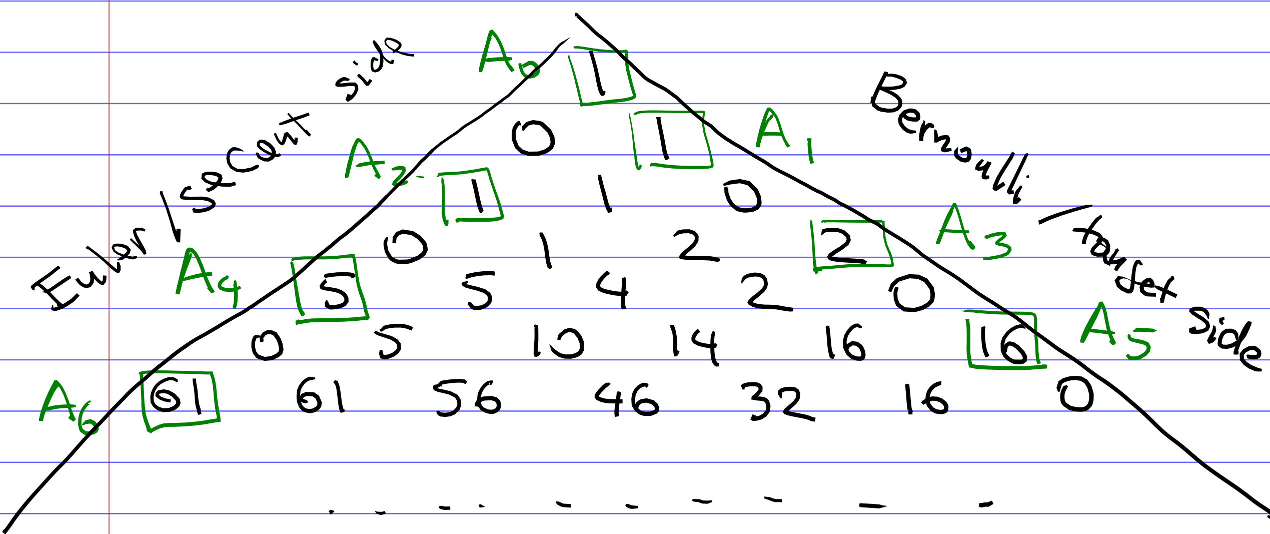
The Bernoulli numbers are defined by the Taylor series

$$\frac{x}{1-e^{-x}} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}.$$

Then $B_{2n} = (-1)^{n-1} \frac{2n}{4^{2n} - 2^{2n}} A_{2n-1}$

and $B_{2n-1} = 0$, for all $n \geq 2$ except $B_1 = \frac{1}{2}$.

The Euler–Bernoulli triangle



<u>Rule</u>: To get the i^{th} row add the entries of the $(i-1)^{st}$ row <u>from left to right</u> if i is <u>even</u>, or <u>from right to left</u> if i is <u>odd</u>.

<u>Exercise</u> Show that this triangle contains the numbers A_n of alternating permutations on its sides, as shown above