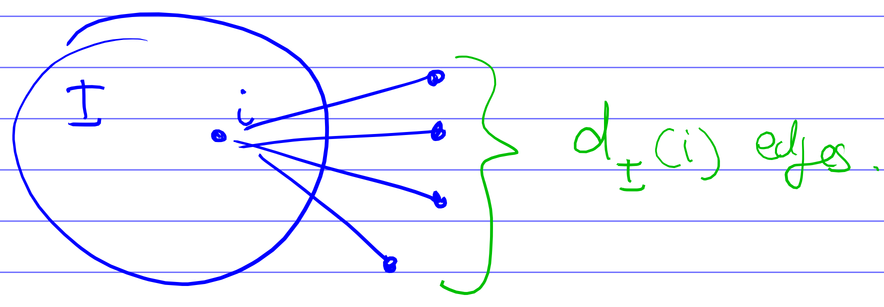
G-parking functions

Definition. For a connected graph G on vertex set $\{0,1,2,\dots,n\}$, a G-parking function is a positive integer vector $(a_1,\dots,a_n) \in (\mathbb{Z}_{>0})^n$ such that

$\forall$ nonempty subset $I \subseteq [n]$ $\exists\, i \in I$ s.t. $a_i \le d_I(i)$,

where $d_I(i) := \#$ edges (i,j) in G with $j \notin I$



Example. If $G = K_{n+1}$ then G-parking functions are the usual parking functions.

Indeed, $(a_1,\dots,a_n) \in \mathbb{Z}_{>0}^n$ is a K_{n+1}-parking function if $\forall$ nonempty $I \subseteq [n]$ $\exists\, i \in I$ s.t. $a_i \le d_I(i) = n+1-|I|$.

Equivalently, there is no nonempty $I \subseteq [n]$, $|I| = k$ s.t. $a_i > n-k+1$ $\forall\, i \in I$.

This is one of the definitions of parking functions.

Theorem. $\#$ G-parking functions $= \#$ spanning trees of G.

Theorem. G-parking functions $\vec{a} = (a_1,\dots,a_n)$ are in bijection with recurrent configurations $\vec{c} = (c_1,\dots,c_n)$ for the Abelian Sandpile Model.

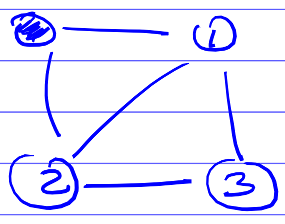
Namely, $\vec{c} = (d_1,\dots,d_n) - \vec{a}$, where d_i are degrees of vertices in G.

[Dhar 1990] proved one direction & conj the other direction

[Gabrielov 1993] complete proof

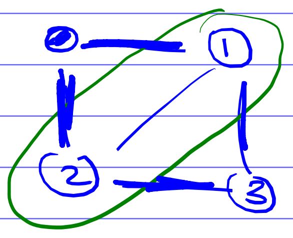
[Ivashkevich-Priezzhev 1998]
[Meester-Redig-Znamensky 2001]
[Cori-Rossin-Salvy 2002]
[P.-Shapiro 2004]
other proofs

etc.

Example. $G =$ 

G-parking functions (a_1, a_2, a_3)

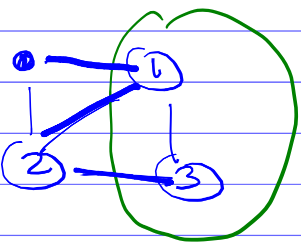
- $a_1, a_2, a_3 \geq 1$
- $a_1 \leq 3, \quad a_2 \leq 3, \quad a_3 \leq 2$
- $I = \{1,2\}$



$d_{\{1,2\}}(1) = 2, \quad d_{\{1,2\}}(2) = 2$

either $a_1 \leq 2$ or $a_2 \leq 2$

- $I = \{1,3\}$:



$d_{\{1,3\}}(1) = 2$

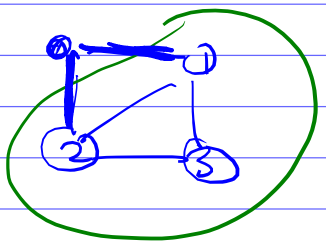
$d_{13}(3) = 1$

either $a_1 \leq 2$ or $a_3 \leq 1$

- $I = \{2,3\}$

either $a_2 \leq 2$ or $a_3 \leq 1$

- $I = \{1,2,3\}$



$d_{123}(1) = d_{123}(2) = 1$

$d_{123}(3) = 0$

either $d_1 \leq 1$ or $d_2 \leq 1$ or $a_3 \leq 0$.

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8 G-parking functions:

(1 1 1) (1 1 2) (1 2 1) (2 1 1)

(1 2 2) (2 1 2) (1 3 1) (3 1 1)

G has 8 spanning trees.

$$\det(\tilde{L}) = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 8$$

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8 Recurrent configurations:

(2 2 1) (2 2 0) (2 1 1) (1 2 1)

(2 1 0) (1 2 0) (2 0 1) (0 2 1)

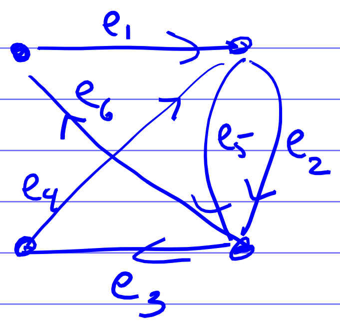
Eulerian Cycles

$G=(V,E)$ is a finite connected directed graph.

Definition. An Eulerian cycle (a.k.a Eulerian circuit, Eulerian tour) is an ordering $e_1, \dots, e_M$ of all directed edges $e \in E$ (without repetitions) s.t. $\forall i$ the end of e_i = the beginning of e_{i+1} & the end of e_M = the beginning of e_1.

In other words, it is a directed closed walk on G that uses all edges of G exactly once.

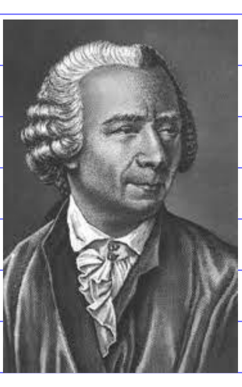
Example.



an Eulerian cycle $e_1, e_2, e_3, e_4, e_5, e_6$

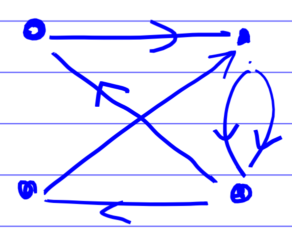
Does G have an Eulerian cycle?

Euler's Theorem (1736)



A connected digraph has a Eulerian cycle iff for any vertex v the indegree of v equals the outdegree of v.

Example

$G =$ 

$\text{indeg}(v) = \text{outdeg}(v)$
$\forall$ vertex v

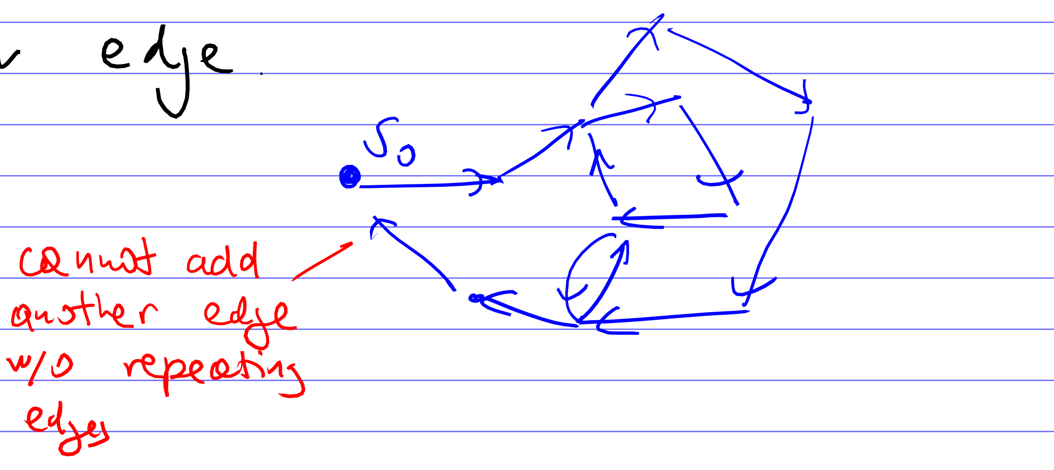
Proof Clearly, the condition

$(*)$ $\text{indeg}(v) = \text{outdeg}(v)$, $\forall$ vertex v

is necessary because an Eulerian cycle should leave a vertex v the same number of times it enters v.

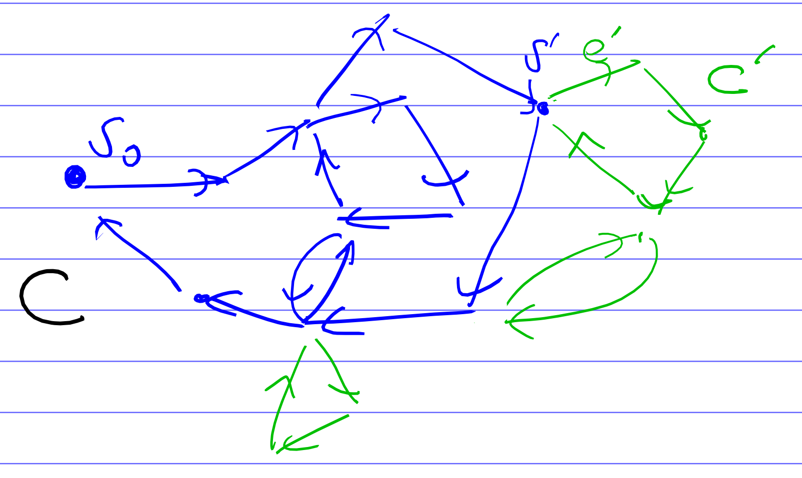
Let's show that the condition $(*)$ is sufficient.

Let's start at any vertex v_0 and start constructing a directed path from v_0, which is not allowed to use the same edge twice, until we cannot add an new edge.



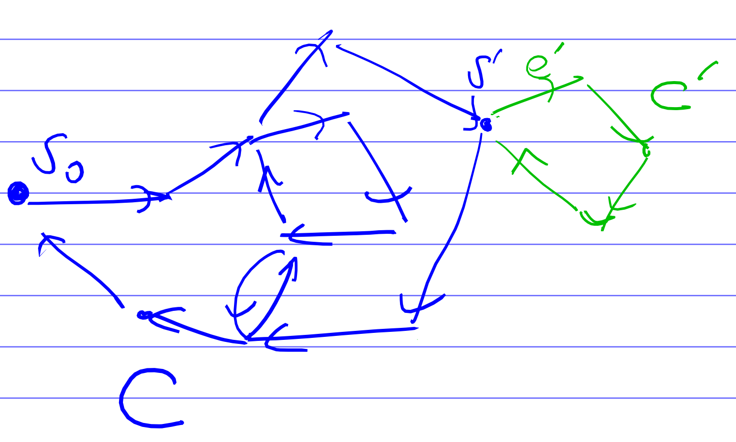
$(*)$ implies that we can get stuck only at the initial vertex v_0.

At this point, we have already constructed a cycle C w/o repeated edges. It is possible that C does not use all edges of G.



Pick any vertex v' on C s.t. that there is an edge $e' \notin C$ starting at v'.

Let's start constructing a directed path starting from e' (w/o repeating the edges that we have already used). Again by $(*)$ we should stop at the same vertex v'. At this point, we have constructed another cycle C'. Then we can combine C & C' into one cycle:



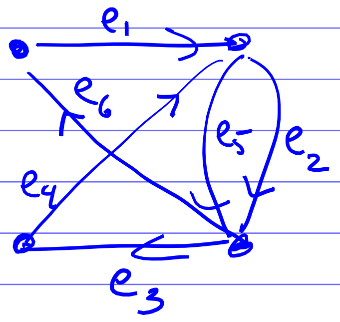
go from v_0 to v' along C
then follow the cycle C'
then go from v' to v_0 along C.

If there are other edges in G that we have not used, then repeat the same construction until we construct an Eulerian cycle. $\square$

Now assume that G is a connected digraph s.t. $\text{indeg}(v) = \text{outdeg}(v)$ $\forall$ vertex v.

What is the number of Eulerian cycles in G?

<u>Example</u>.

$G =$ 

Eulerian cycles:

$e_1, e_2, \dots, e_6$
$e_2, e_3, \dots, e_6, e_1$
$e_3, e_4, \dots, e_6, e_1, e_2$
$\vdots$
$e_6, e_1, e_2, \dots, e_5$

cyclic shifts of the first Eulerian cycle

$e_1, e_5, e_3, e_4, e_2, e_6$
$e_5, e_3, e_4, e_2, e_6, e_1$
$e_3, e_4, e_2, e_6, e_1, e_5$
$e_4, e_2, e_6, e_1, e_5, e_3$
$e_2, e_6, e_1, e_5, e_3, e_4$
$e_6, e_1, e_5, e_3, e_4, e_2$

This graph has 12 Eulerian that consist of 2 groups of 6 cycles obtained from each other by cyclic shifts.

We can also say that it has 2 Eulerian cycles up to cyclic shifts.

B.E.S.T. Theorem

(de Bruijn, von Aardenne-Ehrenfest, Smith, Tutte, 1951)

$G=(V,E)$ connected digraph s.t. $\text{indeg}(v) = \text{outdeg}(v)$ $\forall\, v \in V$.

Then

\# Eulerian cycles in G is

$$\boxed{|E| \cdot \text{Arb}_r(G) \cdot \prod_{v\in V} (\text{outdeg}(v)-1)!}$$

where $|E|$ = \# edges in G,

$\text{Arb}_r(G)$ = \# arborescences in G rooted at $r \in V$.

Recall that, by the directed version of Matrix-Tree Theorem,

$\text{Arb}_r(G)$ = determinant of the reduced directed Laplacian matrix (with r^{th} row & column removed).

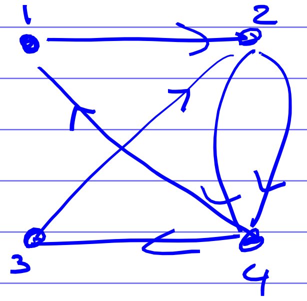
Remark LHS does not depend on r. So the theorem implies that $\text{Arb}_r(G)$ does not depend on r.

This is not true for an arbitrary digraph. By, if $\text{indeg}(v) = \text{outdeg}(v)$ $\forall\, v \in V$, then all cofactors of the directed Laplacian matrix are equal to each other, and

$\text{Arb}_r(G) = \text{Arb}_{r'}(G)$

$\forall\, r, r' \in V$.

Example.

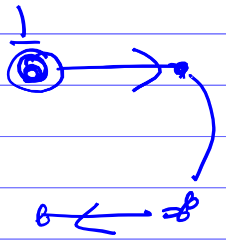
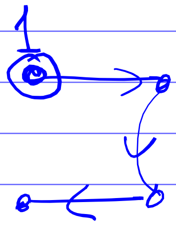
$G =$ 

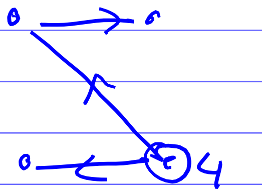
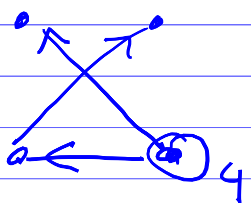
Laplacian $L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 2 \end{bmatrix}$

All cofactors of L are 2,

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \dots = \begin{vmatrix} 2 & 0 & -2 \\ -1 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} = 2$$

$\forall\, r \in V$, # arborescences rooted at r is 2.

$r = 1$  

$r = 4$  

Eulerian cycles is

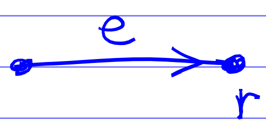
$$6 \cdot 2 \cdot 0!\, 1!\, 0!\, 1!$$

$$= 6 \cdot 2.$$

Proof of B.E.S.T. Theorem

The first factor $|E|$ accounts for cyclic shifts of Eulerian cycles.

Let's fix an edge e in G.
Let r be the end-point of e:



We'll show bijectively that # Eulerian cycles $e_1, \dots, e_M$ s.t. $e_1 = e$ equals
$$\mathrm{Arb}_r(G) \cdot \prod_{v \in V} (\mathrm{indeg}(v) - 1)!$$

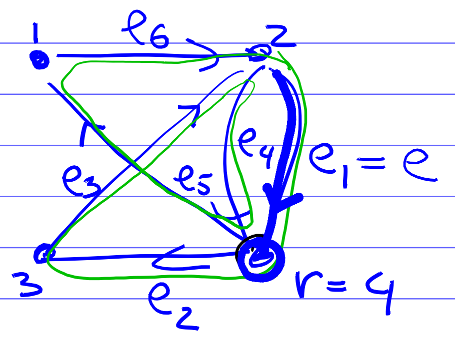
Pick an Eulerian cycle,
$C = (e_1, e_2, \dots, e_M)$ $e_1 = e$.

For any vertex $v \neq r$
mark the edge $\tilde{e}$ s.t.

- the end-point of $\tilde{e}$ is v
- $\tilde{e}$ appears first in $C = (e_1, e_2, \dots, e_M)$ among all edges whose end-point is v.

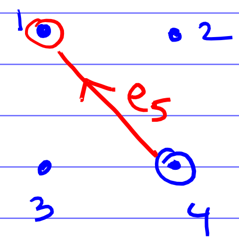
Example

$r = 4$

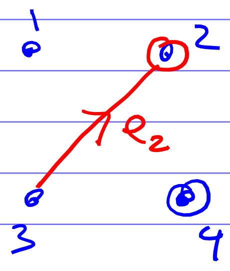


$C = (e_1, \dots, e_6)$

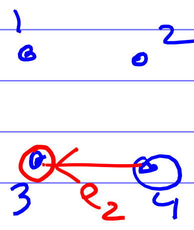
$v = 1$:



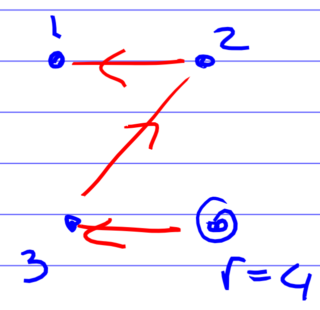
$v = 2$



$v = 3$:



All marked edges:

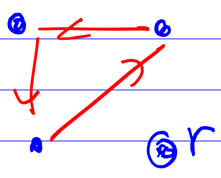


In general, the marked edges satisfy:

- there are $|V| - 1$ marked edges
- $\forall\, v \neq r\ \exists$ exactly one marked edge entering v,
- There are no marked edges entering r.
- The marked edges cannot form a directed cycle.

Proof of the last claim:

Assume that $\exists$ a directed cycle

$\tilde{C} =$ 

$\tilde{C}$ cannot contain the root r ($\nexists$ marked edges entering r)

In the Eulerian cycle C, as we go from the root r, as some point we should visit one of the vertices v of $\tilde{C}$. The edge that enters v for the first time should be marked.

But the marked edge entering v is in $\tilde{C}$.

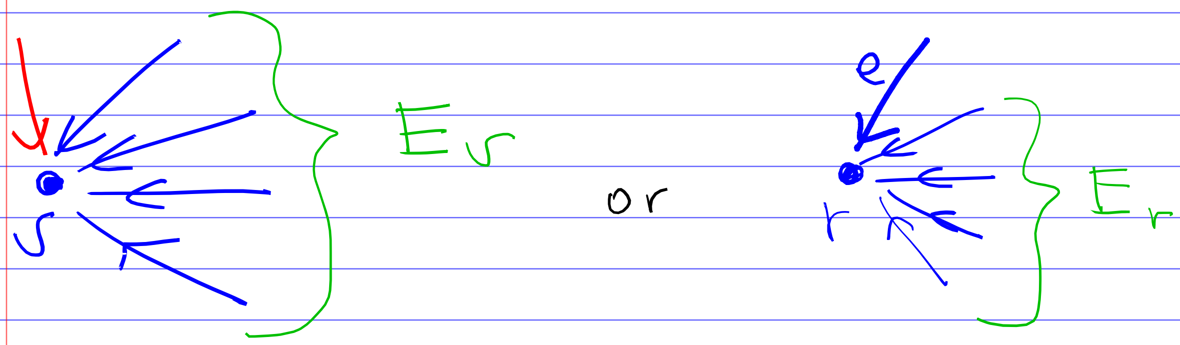
We get a contradiction. $\square$

The above claims $\Rightarrow$

The marked edges form an arborescence T rooted at r.

For any vertex $v \in V$ (now we allow $v = r$), let E_v be the set of all edges f in G s.t.

- f enters vertex v
- f is <u>not</u> marked
- $f \neq e$ (the first edge in C)



Clearly, $\# E_v = \operatorname{indeg}(v) - 1$, because, for $v \neq r$, f cannot be the marked edge, and, for $v = r$, f cannot be the edge e.

Let π_v be the permutation of the edges in E_v given by their order of appearance in $C = (e_1, \dots, e_m)$.

We've got the map

$$C \longmapsto (T, (\pi_v)_{v \in V})$$

an arborescence rooted at r

a permutation of $\operatorname{indeg}(v) - 1$ elements

We claim that this map is a bijection between all Eulerian cycles C that start with e and the set $\{(T, (\pi_v)_{v \in V})\}$.

Clearly, the R.H.S has the cardinality

$$\mathrm{Arb}_r(G) \cdot \prod_{v \in V} (\mathrm{indeg}(v)-1)!$$

In order to show that this is a bijection we need to construct the inverse map $(T, (\pi_v)) \mapsto C$.

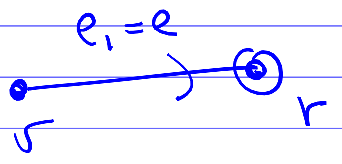
In other words, we need to reconstruct an Eulerian cycle from a given arborescence T rooted at r and a collection of permutations π_v.

Lemma $\forall$ arborescence T & a collection of permutations π_v, as above, there is a unique Eulerian cycle C corresponding to T & π_v.

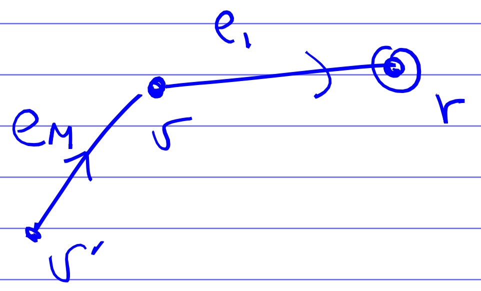
Let's construct

$$C = (e_1, e_2, \dots, e_M).$$

- start from $e_1 = e$

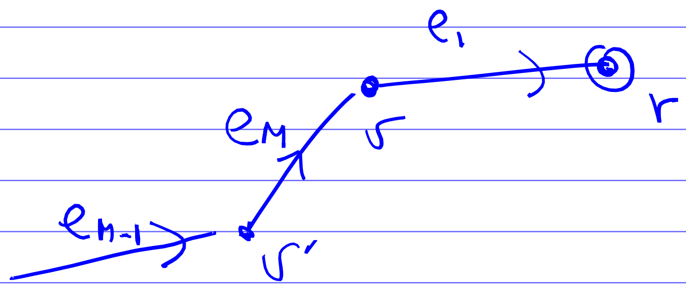


- go backward: find e_M



e_M should be the edge entering vertex v that appears last in the permutation π_v

- then find e_{M-1}:



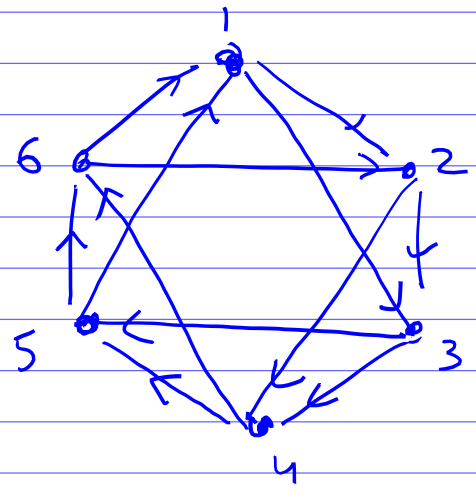
e_{M-1} should be the edge entering v' that appears last in the permutation $\pi_{v'}$.

- etc.

If, at some point we come to a vertex v that we already visited, we use the second to the last edge in π_v, etc.

After we use all edges in π_v, we should take the edge from the arborescence that enters v (one of the marked edges). $\square$

Example

$G =$ 

$\#$ Eulerian cycles $= ?$

$|E| = 12$

$\text{indeg}(v) = 2 \quad \forall\, v \in G$

Need to calculate $\text{Arb}_r(G)$.

$$\text{Laplacian } L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ -1 & 0 & 0 & 0 & 2 & -1 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\text{Arb}_6(G) = \begin{vmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & 2 \end{vmatrix} =$$

$$= 2 \cdot \begin{vmatrix} 2 & & & \\ & 2 & & \\ & O & 2 & \\ & & & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 2 & -1 \end{vmatrix}$$

$$= 2^5 - \big((-1)^4 + 2\cdot(-1)^2 + (-1)\cdot 2(-1) + (-1)^2\cdot 2 + 2\cdot 2\big)$$

$$= 32 - 11 = 21$$

G has 21 Eulerian cycles up to cyclic shifts, or $12 \cdot 21$ Eulerian cycles in total.

Let's summerize:

<u>Laplacian Matrix</u> L
(a.k.a the Kirchhoff matrix)

appears in

- <u>Matrix-Tree Theorem</u>
  \# spanning trees
  (or arborescences in directed case) are given by cofactors of L

- <u>Electical Networks</u>

- <u>Abelian Sandpile Model</u>
  (# recurrent configurations
  = a cofactor of L)

- <u>BEST Theorem</u>:
  \# directed Eulerian cycles is expressed in terms of a cofactor of L.