Abelian Sandpile Model (cont’d)

- Bak, Tang, Wiesenfeld, 1987
- D. Dhar, 1990
- Björner, Lovász, Shor, 1991

- This is a game on graph with simple rules.
- It models complicated natural processes such as avalanches.
- It can have complicated "fractal-like" features.

\[ G = (V, E) \] a finite connected graph on vertex set \( V = \{ 0, 1, \ldots, n \} \) with on special vertex \( q = 0 \), called the sink.

\( L = (L_{ij}) \) the reduced Laplacian matrix of \( G \) ( = the Laplacian matrix of \( G \) with 0\textsuperscript{th} row & column removed).

\[ L_{ij} = \begin{cases} -\# \text{edges between } i \text{ & } j, i \neq j & \text{if } i = j \end{cases} \]

\[ d_i = \text{deg}_G(i) - \text{ the degree of vertex } i. \]

Recall, MTT: \( \det(L) = \# \text{spanning trees in } G \)

\( \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n \in \mathbb{R}^n \) the row vectors of the reduced Laplacian matrix \( \hat{L} \).
Configurations
\( \bar{C} = (C_1, ..., C_n) \in \mathbb{Z}_{\geq 0}^n \)

Stable configurations:
\( \bar{C} = (C_1, ..., C_n) \in \mathbb{Z}_{\geq 0}^n \), \( 0 \leq C_i < d_i \), \( \forall i \)

Topplings (or flips)
For a configuration \( \bar{C} \), a vertex \( i \) is unstable if \( C_i = d_i \).
We can topple an unstable vertex \( i \):
\( \bar{C} \mapsto \bar{C} - \vec{L}_i \)

**Example**
\[ L = \begin{bmatrix} 3 & -1 & -1 \\ -3 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \]
Reduced Laplacian

\[ \vec{L}_1 = (3, -1, -1) \]
\[ \vec{L}_2 = (-1, 3, -1) \]
\[ \vec{L}_3 = (-1, -1, 2) \]

A configuration \( \bar{C} = (4, 2, 1) \)

Vertex 1 is unstable

Toppling of vertex 1:
\( \bar{C} = (4, 2, 1) \mapsto \bar{C} - \vec{L}_1 = (1, 3, 2) \)
Lemma (Dhar)

Finiteness: A initial configuration $C_{\text{init}} \in \mathbb{Z}^{\geq 0}$, after finitely many topplings, we obtain a stable configuration $C_{\text{stab}}$.

Uniqueness: $C_{\text{stab}}$ does not depend on a choice of order of topplings. (It depends only on $C_{\text{init}}$.)

We proved this Lemma in the last lecture:

- Finiteness: Each toppling strictly increases the "value" of a country.
- Uniqueness: "Diamond Lemma" argument.

- $C_{\text{stab}}$ is called the stabilization of $C_{\text{init}}$.

Example $G = \begin{array}{c}
\end{array}$

\[ C_{\text{init}} = (4, 2, 1) \]

The stabilization of $C_{\text{init}} = (4, 2, 1)$ is $C_{\text{stab}} = (2, 0, 1)$. 

\[ \text{topple 1} \]

\[ \text{topple 2} \]

\[ \text{topple 3} \]

\[ \text{Stable} \]

\[ C_{\text{stab}} = (2, 0, 1) \]
The avalanche operators $A_1, \ldots, A_n$ act on the set of stable configurations $\mathcal{C}$.

- $A_i(\mathcal{C})$ is the stabilization of $\mathcal{C} + \mathcal{e}_i$, where $\mathcal{e}_i = (0, \ldots, 0, i, 0, \ldots, 0)$.

**Example**

$\mathcal{C} = (2,0,2)$

A stable config

$A_2(\mathcal{C})$: add 1 chip to vertex 2 and stabilize:

$\mathcal{C} + \mathcal{e}_2$

- Topple 2

$\mathcal{C} + \mathcal{e}_2$

- Topple 3

Stable

$(1,2,0)$

$A_2: (2,0,2) \mapsto (1,2,0)$

**Lemma (Dhar)** The avalanche operators $A_1, \ldots, A_n$ commute pairwise.

**Proof** $A_i A_j(\mathcal{C}) = A_j A_i(\mathcal{C}) = \text{the stabilization of } \mathcal{C} + \mathcal{e}_i + \mathcal{e}_j$. □
The Abelian Sandpile Model is the random walk on the set of stable configurations $C$:

- randomly pick $i \in C_{\text{stable}}$ with probability $\gamma_i$.
- $C \rightarrow A_i C$ (perform the $i^{th}$ avalanche operator $A_i$).

Example (from last lecture)

$G = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
1 & 2 & 3
\end{array}$

4 stable configurations:

$(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.

The avalanche operators $A_1$ and $A_2$:

$\begin{array}{ccc}
(0,0) & \xrightarrow{A_2} & (0,1) \\
\downarrow A_1 & & \downarrow A_1 \\
(1,0) & \xrightarrow{A_2} & (1,1)
\end{array}
$

$\begin{array}{ccc}
(0,0) & \xrightarrow{\gamma_2} & (0,1) \\
\downarrow \gamma_2 & & \downarrow \gamma_2 \\
(1,0) & \xrightarrow{\gamma_2} & (1,1)
\end{array}
$

The Abelian sandpile model is given by...
Definition. A stable configuration \( \mathcal{C} \) is called \underline{recurrent} if \( \exists N \in \mathbb{Z}_{\geq 0} \) such that \( (A_i)^N \mathcal{C} = \mathcal{C} \) for all \( i = 1, \ldots, n \).

In other words, we can always come back to a recurrent configuration in this random walk.

Remark. Since \( G \) is a finite graph, there are finitely many stable configurations. So we can assume that \( N \) is the same for all recurrent configurations.

Example. (The previous example)

For \( G = \begin{array}{c} 3 \\ 1 \rightarrow 2 \end{array} \):

- recurrent configurations: \((0,1)\), \((1,0)\), \((1,1)\)
  
- (we can take \( N = 3 \))

Stable but \underline{not} recurrent: \((0,0)\)
Let's restrict the avalanche operators $A_1, \ldots, A_n$ to the set $R$ of all recurrent configurations.

The operators $A_1, \ldots, A_n$ are invertible on $R$. Indeed, $A_i^{-1} \vec{c} = A^{n-1} \vec{c}$ for any recurrent configuration $\vec{c}$.

**Definition.** The *sandpile group* $SG$ (a.k.a. the critical group) is the finite abelian group generated by the avalanche operators $A_1, \ldots, A_n$ acting on the set $R$ of recurrent configurations.

(So $SG$ is a certain abelian subgroup of the symmetric group $S_{181}$.)
Theorem \([\text{Dhar}]\)

- \(SG \cong \mathbb{Z}^n / \langle L_1, \ldots, L_m \rangle\)

  the subgroup of \(\mathbb{Z}^n\) generated by rows \(L_i\)
  of the reduced Laplacian

- \(|R| = |SG| = \det L\)

  \# recurrent configurations the order of \(L\)
  \# spanning trees in \(G\)

In particular (by the MTT),

\(\#\) recurrent configurations

\(\cong\) \# spanning trees in \(G\).

Example \(G = \begin{array}{c}
\begin{array}{c}
A_1 \\
A_2 \\
A_3
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
(0,1) \\
(1,0) \\
(1,1)
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
L = \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}
\end{array}
\end{array}
\)

\(SG \cong \mathbb{Z}^2 / \langle (2,-1), (-1,2) \rangle \cong \mathbb{Z}/3\mathbb{Z}\) the cyclic group of order 3

\(L_1 = (2,-1)\) \(L_2 = (-1,2)\)

\(5\) elements of \(\mathbb{Z}^2 / \langle L_1, L_2 \rangle\)

\(|R| = |SG| = 3 = \#\) spanning trees.
Proof. Since the available operators $A_1, \ldots, A_n$ are invertible (and pairwise commuting) operators on the set $R$ of recurrent configurations $\mathcal{C}$, we can define

$$A^B : \mathcal{C} \rightarrow A_1^b \cdot A_2^b \cdot \ldots \cdot A_n^b (\mathcal{C})$$

for any integer vector $B = (b_1, \ldots, b_n) \in \mathbb{Z}^n$.

By definition,

$$SG := \{ A^B \mid b \in \mathbb{Z}^n \}.$$

For any recurrent config. $\mathcal{C} \in R$ and any $i \in \{1, \ldots, n\}$,

$$A_1^{b_1} (\mathcal{C}) = \text{the stabilization of } \mathcal{C} + t_1 (N_1, \ldots, N_n)$$

$$= \text{the stabilization of } \mathcal{C} + (N_1, \ldots, N_n)$$

$$= A_1^{(b_1, \ldots, N_n)} (\mathcal{C}) = \mathcal{C}.$$

So $A^B = Id$, the identity operator acting on recurrent configurations.

If $B$ is a linear combination of vectors $t_1, \ldots, t_n$,

On the other hand, if $B$ is not a linear comb. of $t_1, \ldots, t_n$, we can stabilize $\mathcal{C}$ by adding, say, $-t_1$.

Thus $SG = \mathbb{Z}^n \setminus \langle t_1, \ldots, t_n \rangle$.

Explicitly,

$$A^B \leftrightarrow B \mod (t_1, \ldots, t_n)$$

$$SG \approx \mathbb{Z}^n \setminus \langle t_1, \ldots, t_n \rangle.$$
Lemma. For any two recurrent configurations \( \mathbf{c}, \mathbf{c}' \in \mathcal{R} \)
\[ \mathbf{c}' = A^{\mathbf{c} - \mathbf{c}'}(\mathbf{c}). \]

Proof of Lemma:
\[ A^{\mathbf{c} - \mathbf{c}'}(\mathbf{c}) = \text{the stabilization of} \]
\[ \mathbf{c} + (\mathbf{c}' - \mathbf{c} + (N, \ldots, N)) \]
\[ = \text{the stabilization of} \]
\[ \mathbf{c}' + (N, \ldots, N) \]
\[ = A^{\mathbf{c}'}(\mathbf{c}) = \mathbf{c}'. \]

This implies that the sandpile group acts simply transitively on the set \( \mathcal{R} \) of recurrent configurations. In particular,
\[ |SG| = |\mathcal{R}|. \]

Here is an explicit bijection between \( SG \) and \( \mathcal{R} \):
Fix one reference recurrent configuration \( \mathbf{c}_{\text{ref}} \in \mathcal{R} \).
\[ A^{\mathbf{c}_{\text{ref}}} \leftrightarrow_{bij} \mathbf{c} = A^{\mathbf{c}_{\text{ref}}} \]
\[ \overline{\mathcal{R}} \leftrightarrow SG \]

Finally,
\[ |SG| = |\mathcal{R}| = \left| \mathbf{e}^{\mathbf{c}_{\text{ref}}} \right| \]
\[ = |\det (\mathbf{L_i})| = \pm \text{Spanning tree of } \mathbf{c}_{\text{ref}}. \]

This equality holds for any collection of integer vectors that form a basis of \( R^n \).

This finishes the proof of the theorem. \( \square \)
How to describe recurrent configurations?

**Definition.** A configuration \( C = (c_1, \ldots, c_n) \) is allowed if

A nonempty subset \( I \subseteq \{1, \ldots, n\} \)

\( \exists i \in I \) such that

\[ c_i > \deg_{G_I}(i), \]

where \( G_I \) is the induced graph on vertex set \( I \), i.e.,

\[ \deg_{G_I}(i) = \sum_{j \in I \setminus \{i\}} 1 \]

**Theorem.** A configuration is recurrent iff it is stable and allowed.

[Shaw 1990] proved \( \Rightarrow \) & conjectured \( \Leftarrow \).

This was proved in

[Gabrielov 1993]

[Inashkevich-Prikozev 1998]

[Meester-Redig-Zumzensky 2001]

[Cori-Rossin-Salvy 2002]

[Pi-Skapino 2004]

etc.
**Definition.** \([P\text{-}\text{Parking}]\)

For a connected graph \(G\) on vertex set \(\{0,1,2,\ldots,n\}\)

a **\(G\)-parking function** if

a positive integer vector \((a_0, a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n\) such that

For every nonempty subset \(I \subseteq \{0,1,2,\ldots,n\}\)

\(\exists \ i \in I \) s.t.

\[ a_i < d_i(i), \]

where \(d_i(i) : = \# \text{ edges } (i,j) \in E \quad \text{with } \ j \not\in I \)

**Example.** If \(G = K_{n+1}\) then

\(G\)-parking functions are the usual parking functions.

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**Theorem.** \# \(G\)-parking functions

= \# spanning trees of \(G\).

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**Theorem.** \(G\)-parking functions

\(\overrightarrow{a} = (a_0, a_1, \ldots, a_n)\) are

in bijection with recurrent configurations \(\overrightarrow{c} = (c_0, c_1, \ldots, c_n)\).

Namely,

\(\overrightarrow{c} = (d_0, d_1, \ldots, d_n) - \overrightarrow{a}\)

where \(d_i\) are degrees of vertices in \(G\).
Example. $G = \begin{array}{c}
\begin{array}{c}
\textbf{Example. } G = \\
\textbf{G-parking functions} \ (a_1, a_2, a_3)
\end{array}
\end{array}$

- $a_1, a_2, a_3 \geq 1$
- $a_1 \leq 3$, $a_2 \leq 3$, $a_3 \leq 2$
- $I = \{1, 2, 3\}$

\[
d_{i,j,3} (1) = 2, \ d_{i,j,3} (2) = 1
\]

either $a_1 \leq 2$ or $a_2 \leq 2$

- $I = \{2, 3\}$

\[
d_{i,3} (1) = 2, \ d_{i,3} (2) = 1
\]

either $a_1 \leq 2$ or $a_3 \leq 1$

- $I = \{2, 3, 5\}$

\[
d_{i,3} (1) = d_{i,3} (2) = 1
\]

either $a_1 \leq 1$ or $a_2 \leq 1$ or $a_3 \leq 0$

8 G-parking functions:

\[
(111) \ (112) \ (121) \ (211) \\
(122) \ (212) \ (131) \ (311)
\]

$G$ has 8 spanning trees.

\[
\text{det}(L) = \begin{vmatrix}
3 & -1 & 1 \\
-1 & 3 & -1 \\
-1 & -1 & 2
\end{vmatrix} = 8
\]

8 Recurrent configurations:

\[
(221) \ (220) \ (211) \ (121) \\
(210) \ (120) \ (201) \ (021)
\]