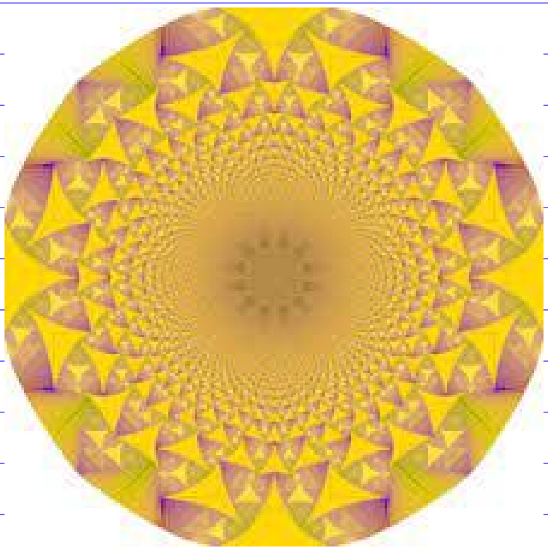
Abelian Sandpile Model (cont'd)

- Bak, Tang, Wisenfeld, 1987
- D. Dhar, 1990
- Björner, Lovász, Shor, 1991



- This is a game on graph with simple rules.
- It models complicated natural processes such as avalanches.
- It can have complicated "fractal-like" features.

- $G=(V,E)$ a finite connected graph on vertex set $V=\{0,1,\dots,n\}$ with an special vertex $q=0$, called the sink.

- $\tilde{L}=(\ell_{ij})_{1\le i,j\le n}$ the reduced Laplacian matrix of G (= the Laplacian matrix of G with 0^{th} row & column removed)

$$\ell_{ij} = \begin{cases} -\#\text{ edges between } i \ \& \ j, & i\neq j \\ d_i & \text{if } i=j \end{cases}$$

$d_i := \deg_G(i)$ - the degree of vertex i.

Recall, MTT: $\det(\tilde{L}) = \#\{\text{Spanning trees of } G\}$

- $\vec{L}_1, \vec{L}_2, \dots, \vec{L}_n \in \mathbb{Z}^n$ the row vectors of the reduced Laplacian matrix $\tilde{L}$.

- <u>Configurations</u>

$\vec{c} = (c_1, \dots, c_n) \in \mathbb{Z}^n_{\geq 0}$

- <u>Stable configurations</u>:

$\vec{c} = (c_1, \dots, c_n) \in \mathbb{Z}^n_{\geq 0}, \quad 0 \leq c_i < d_i \ \forall i$

- <u>Topplings</u> (or <u>firings</u>)

For a configuration $\vec{c}$,
a vertex i is <u>unstable</u> if $c_i \geq d_i$

We can <u>topple</u> an unstable vertex i

$\vec{c} \rightsquigarrow \vec{c} - \vec{L}_i$

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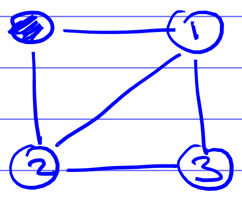
the sink

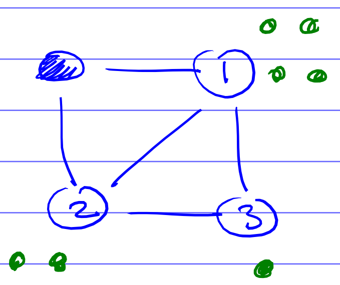
<u>Example</u>

$G =$ 

$$\tilde{L} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$ reduced Laplacian

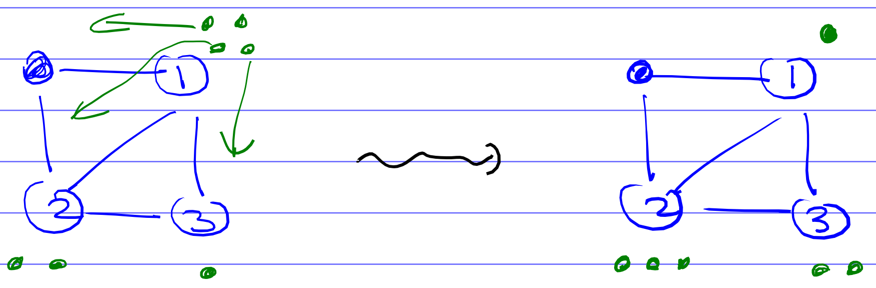
$\vec{L}_1 = (3, -1, -1)$
$\vec{L}_2 = (-1, 3, -1)$
$\vec{L}_3 = (-1, -1, 2)$

A configuration $\vec{c} = (4, 2, 1)$



vertex 1 is unstable

Toppling of vertex 1:



$\vec{c} = (4, 2, 1) \rightsquigarrow \vec{c} - \vec{L}_1 = (1, 3, 2)$

Lemma (Dhar)

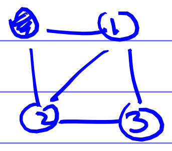
Finiteness: $\forall$ initial configuration $\vec{c}_{init} \in \mathbb{Z}^n_{\geq 0}$, after finitely many topplings, we obtain a stable configuration $\vec{c}_{stab}$.

Uniqueness: $\vec{c}_{stab}$ does not depend on a choice of order of topplings. (It depends only on $\vec{c}_{init}$)

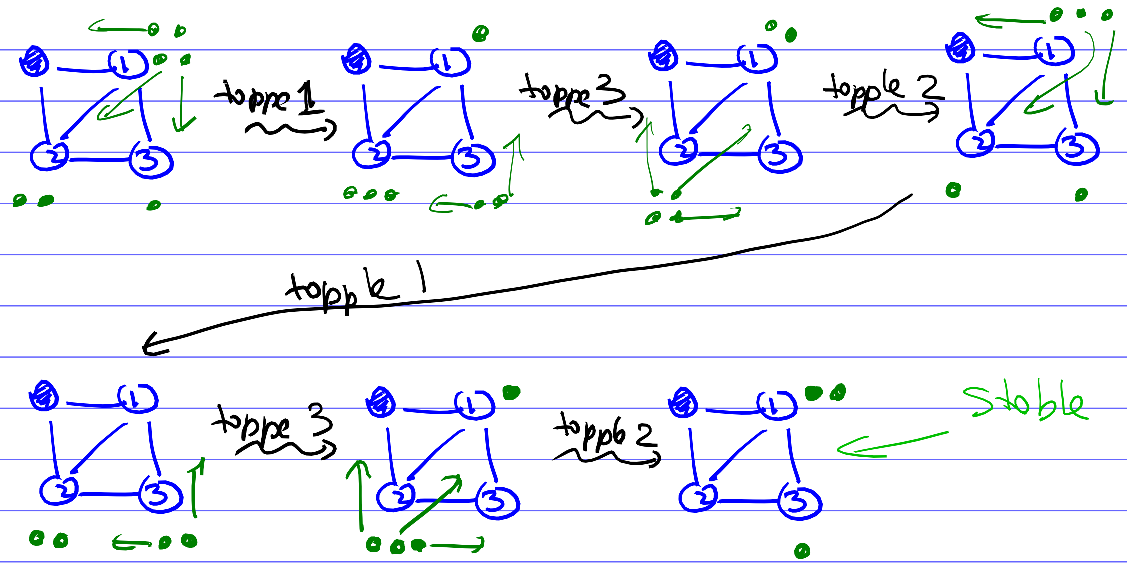
We proved this Lemma in the last lecture:

- Finitness: Each toppling strictly increases the "value" of a config.
- Uniqueness: "Diamond Lemma" argument.

- $\vec{c}_{stab}$ is called the <u>stabilization</u> of $\vec{c}_{init}$.

Example $G =$ 

$\vec{c}_{init} = (4,2,1)$



$\vec{c}_{stab} = (2,0,1)$

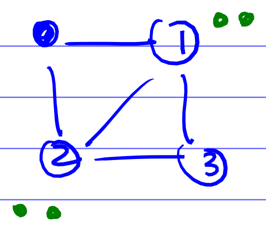
The stabilization of $\vec{c}_{init} = (4,2,1)$ is $\vec{c}_{stab} = (2,0,1)$

The <u>avalanche operators</u> $A_1, \dots, A_n$

- act on the set of stable configurations $\vec{c}$.
- $A_i(\vec{c})$ is the stabilization of $\vec{c} + \vec{e}_i$,

where $\vec{e}_i = (0, \dots 0, \overset{i}{1}, 0 \dots 0)$.

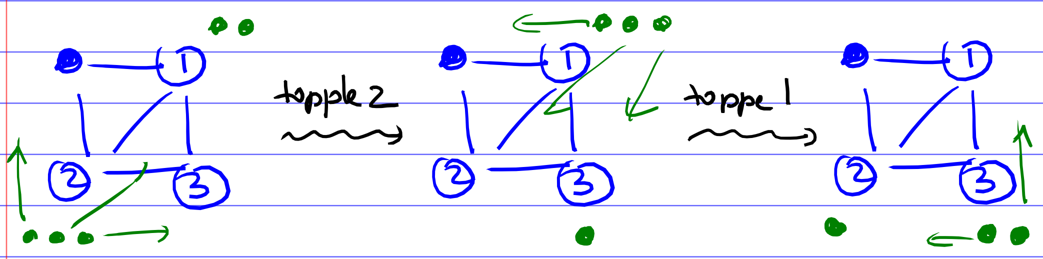
<u>Example</u>



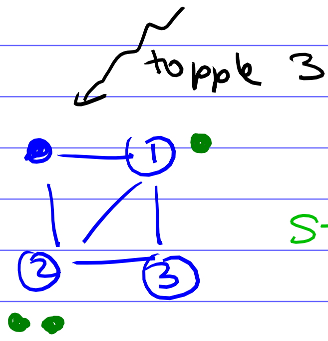
$\vec{c} = (2, 0, 2)$

a stable config.

$A_2(\vec{c})$: add 1 chip to vertex 2 and stabilize :



$\vec{c} + \vec{e}_2$



stable

$(1, 2, 0)$

$$A_2 : (2, 0, 2) \longmapsto (1, 2, 0)$$

<u>Lemma</u> (Dhar) The avalanche operators $A_1, \dots, A_n$ commute pairwise

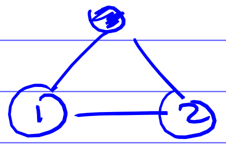
<u>Proof</u> $A_i A_j(\vec{c}) = A_j A_i(\vec{c})$

$=$ the stabilization of $\vec{c} + \vec{e}_i + \vec{e}_j$. $\square$

The <u>Abelian Sandpile Model</u> is the random walk on the set of stable configurations $\vec{c}$:

- randomly pick $i \in \{1, \dots, n\}$ with probability $1/n$.
- $\vec{c} \rightsquigarrow A_i \vec{c}$ (perform the i^{th} avalanche operator A_i)

<u>Example</u> (from last lecture)

$G =$ 

4 stable configurations:

$(0,0)$ $(0,1)$ $(1,0)$ $(1,1)$.

The avalanche operators A_1 & A_2:

$$\begin{tikzcd}
(0,0) \arrow[r, "A_2"] \arrow[d, "A_1"'] & (0,1) \arrow[d, "A_1"] \arrow[dl, bend right, "A_2"'] \\
(1,0) \arrow[r, "A_2"'] \arrow[ur, bend right, "A_1"'] & (1,1) \arrow[u, bend right, "A_2"'] \arrow[l, bend left, "A_1"]
\end{tikzcd}$$

The Abelian sandpile model is given by

$$\begin{tikzcd}
(0,0) \arrow[r, "1/2"] \arrow[d, "1/2"'] & (0,1) \arrow[d, "1/2"] \arrow[dl, bend right, "1/2"'] \\
(1,0) \arrow[r, "1/2"'] \arrow[ur, bend right, "1/2"'] & (1,1) \arrow[u, bend right, "1/2"'] \arrow[l, bend left, "1/2"]
\end{tikzcd}$$

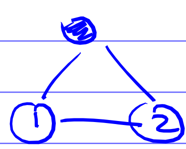
Definition. A stable configuration $\vec{c}$ is called **recurrent** if $\exists\, N \in \mathbb{Z}_{>0}$

$$(A_i)^N \vec{c} = \vec{c} \qquad \forall\, i = 1, \dots, n.$$

In other words, we can always come back to a recurrent configuration in this random walk.

Remark. Since G is a finite graph, there are finitely many stable configurations. So we can assume that N is the same for all recurrent configurations.

Example. (The previous example)

For $G =$ 

:

recurrent configurations : $(0,1)$ $(1,0)$ $(1,1)$

(we can take $N = 3$)

stable but not recurrent : $(0,0)$

Let's restrict the avalanche operators $A_1, \dots, A_n$ to the set R of all recurrent configurations.

The operators $A_1, \dots, A_n$ are <u>invertible</u> on R. Indeed,

$$A_i^{-1} \vec{c} = A^{N-1} \vec{c}$$ for any recurrent configuration $\vec{c}$.

<u>Definition</u> The <u>sandpile group</u> SG (a.k.a. the critical group) is the finite abelian group of generated by the avalanche operators $A_1, \dots, A_n$ acting on the set R of recurrent configurations.

(So SG is a certain abelian subgroup of the symmetric group $S_{|R|}$.)

Theorem [Dhar]

- $SG \cong \mathbb{Z}^n / \langle \vec{L}_1, \dots, \vec{L}_n \rangle$

the subgroup of $\mathbb{Z}^n$ generated by rows $\vec{L}_i$ of the reduced Laplacian

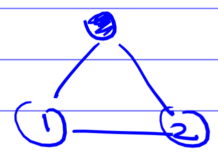
- $|R| = |SG| = \det \tilde{L}$

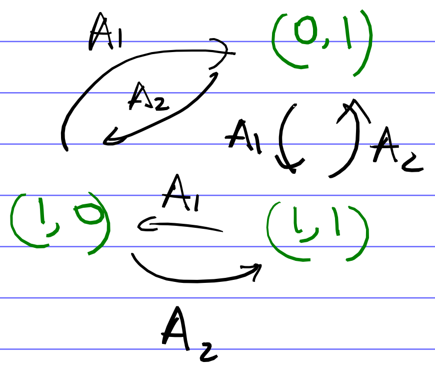
\# recurrent configurations

the order of the sandpile group

the det of reduced Laplacian.

- In particular (by the MTT), \# recurrent configurations $=$ \# spanning trees in G.

Example $G =$ 

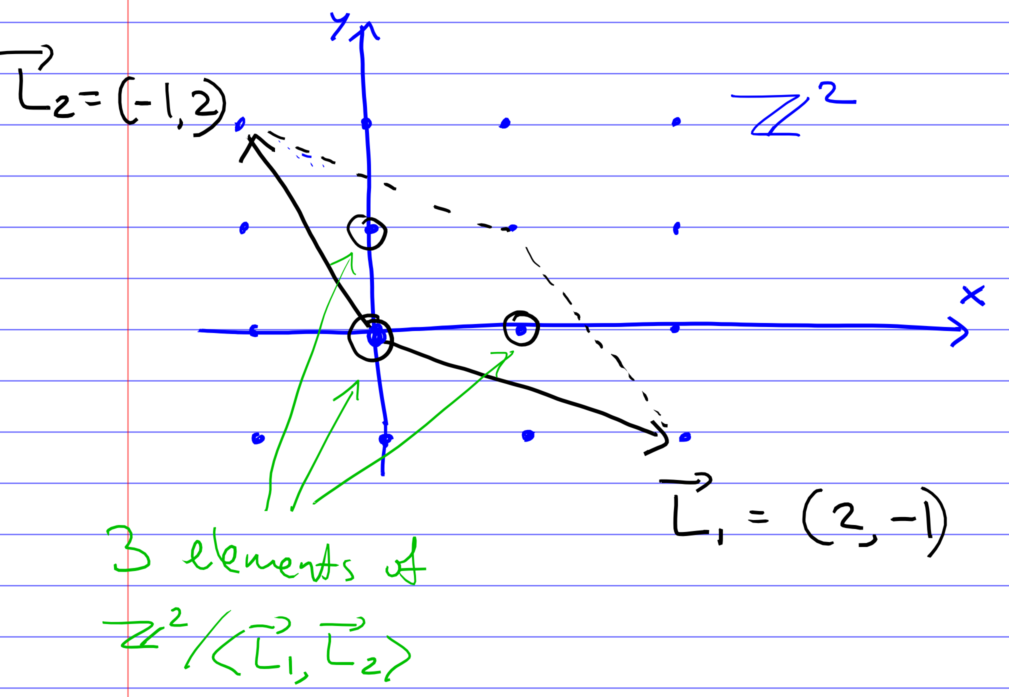


$$\tilde{L} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

reduced Laplacian

$$SG \cong \mathbb{Z}^2 / \langle (2,-1), (-1,2) \rangle$$

$$\cong \mathbb{Z}/3\mathbb{Z}$$ the cyclic group of order 3.



$|R| = |SG| = 3 =$ \# spanning trees.

Proof Since the avalanche operators $A_1, \dots, A_n$ are invertible (and pairwise commuting) operators on the set R of recurrent configurations $\vec{c}$, we can define

$$A^{\vec{b}} : \vec{c} \mapsto A_1^{b_1} A_2^{b_2} \dots A_n^{b_n}(\vec{c})$$

for any integer vector $\vec{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$.

By definition,

$$SG := \{ A^{\vec{b}} \mid b \in \mathbb{Z}^n \}.$$

For any recurrent config. $\vec{c} \in R$ and any $i \in \{1, 2, \dots, n\}$:

$A^{\vec{L_i}}(\vec{c})$ = the stabilization of $\vec{c} + \vec{L_i} + (N, \dots, N)$

= the stabilization of $\vec{c} + (N, \dots, N)$ — we can start the process of stabilization by toppling vertex i.

$= A^{(N, \dots N)}(\vec{c}) = \vec{c}$.

So $A^{\vec{b}} = Id$, — the identity operator action on recurrent configuration

if $\vec{b}$ is a linear combination of vectors $\vec{L_1}, \dots, \vec{L_n}$.

On the other hand, $A^{\vec{b}} \neq Id$ if $\vec{b}$ is not a linear comb. of $\vec{L_1}, \dots, \vec{L_n}$, because stabilization only involves subtracting $\vec{L_i}$'s.

Thus $SG \simeq \mathbb{Z}^n / \langle L_1, \dots L_n \rangle$

Explicitly,

$$\begin{array}{ccc} A^{\vec{b}} & \xleftrightarrow{\text{bij.}} & \vec{b} \bmod (L_1 \dots L_n) \\ \cap & & \cap \\ SG & & \mathbb{Z}^n / \langle L_1, \dots, L_n \rangle \end{array}$$

Lemma. For any two recurrent configurations $\vec{c}, \vec{c}' \in R$
$$\vec{c}' = A^{\vec{c}' - \vec{c}}(\vec{c}).$$

Proof of Lemma.

$A^{\vec{c}' - \vec{c}}(\vec{c}) :=$ the stabilization of
$$\vec{c} + (\vec{c}' - \vec{c} + (N, \dots, N))$$

we need to add this vector to make sure that all coord. become nonnegative

$=$ the stabilization of
$$\vec{c}' + (N, \dots, N)$$
$$= A^{(N,\dots,N)}(\vec{c}') = \vec{c}'. \square$$

This implies that the sandpile group acts simply transitively on the set R of recurrent configurations. In particular,
$$|SG| = |R|.$$

Here is an explicit bijection between SG and R:

Fix one reference recurrent configuration $\vec{c}_{ref} \in R$.
$$\begin{array}{ccc} A^{\vec{b}} & \overset{\text{bij.}}{\longleftrightarrow} & \vec{c} = A^{\vec{b}}(\vec{c}_{ref}) \\ \cap & & \cap \\ SG & & R \end{array}$$

Finally, $|SG| = \left| \mathbb{Z}^n / \langle L_1, \dots, L_n \rangle \right|$
$$= |\det(\tilde{L})| = \# \text{ spanning trees of } G$$

this equality holds for any collection of integer vectors that form a basis of $\mathbb{R}^n$.

This finishes the proof of theorem. $\square$

How to describe recurrent configurations?

Definition. A configuration $\vec{c} = (c_1, \dots, c_n)$ is <u>allowed</u> if

$\forall$ nonempty subset $I \subseteq \{1, \dots, n\}$
$\exists\, i \in I$ such that

$$c_i \geq \deg_{G_I}(i),$$

where G_I is the induced graph on vertex set I, i.e.,

$$\deg_{G_I}(i) = \sum_{j \in I \setminus \{i\}} \#\{\text{edges between } i \,\&\, j\}$$

Theorem. A configuration is recurrent iff it is stable and allowed.

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relatively easy   harder.

[Dhar 1990] proved $\Rightarrow$ & conjectured $\Leftarrow$.

Thm was proved in

[Gabrielov 1993]

[Ivashkevich-Priezzhev 1998]

[Meester-Redig-Znamensky 2001]

[Cori-Rossin-Salvy 2002]

[P.-Shapiro 2004]

etc.

G-parking functions

Definition. [P.- Shapiro]

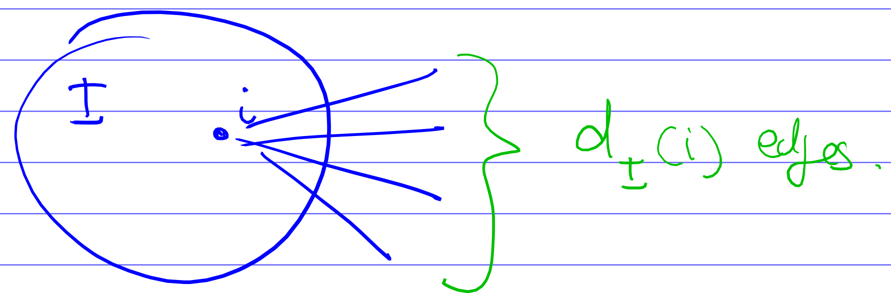
For a connected graph G on vertex set $\{0,1,2,\dots,n\}$ a **G-parking function** if

a positive integer vector $(a_1,\dots,a_n) \in (\mathbb{Z}_{>0})^n$ such that

$\forall$ nonempty subset $I \subseteq [n]$
$\exists\ i \in I$ s.t.

$$a_i \leq d_I(i),$$

where $d_I(i) :=$ # edges (i,j) in G with $j \notin I$



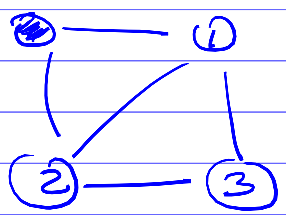
Example. If $G = K_{n+1}$ then G-parking functions are the usual parking functions.

Theorem. # G-parking functions = # spanning trees of G.

Theorem. G-parking functions $\vec{a} = (a_1,\dots,a_n)$ are in bijection with recurrent configurations $\vec{c} = (c_1,\dots,c_n)$.

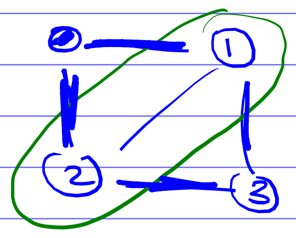
Namely, $\vec{c} = (d_1,\dots,d_n) - \vec{a}$.

where d_i are degrees of vertices in G.

Example. $G =$ 

G-parking functions (a_1, a_2, a_3)

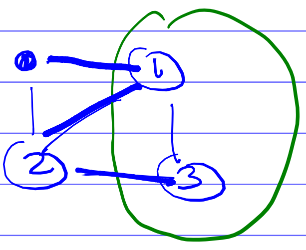
- $a_1, a_2, a_3 \geq 1$
- $a_1 \leq 3, \quad a_2 \leq 3, \quad a_3 \leq 2$
- $I = \{1, 2\}$



$d_{\{1,2\}}(1) = 2, \quad d_{\{1,2\}}(2) = 2$

either $a_1 \leq 2$ or $a_2 \leq 2$

- $I = \{1, 3\}$:



$d_{\{1,3\}}(1) = 2$

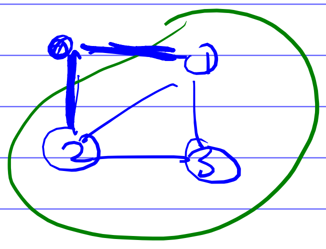
$d_{13}(3) = 1$

either $a_1 \leq 2$ or $a_3 \leq 1$

- $I = \{2, 3\}$

either $a_2 \leq 2$ or $a_3 \leq 1$

- $I = \{1, 2, 3\}$



$d_{123}(1) = d_{123}(2) = 1$

$d_{123}(3) = 0$

either $a_1 \leq 1$ or $a_2 \leq 1$ or $a_3 \leq 0$.

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8 G-parking functions:

(1 1 1) (1 1 2) (1 2 1) (2 1 1)

(1 2 2) (2 1 2) (1 3 1) (3 1 1)

G has 8 spanning trees.

$$\det(\tilde{L}) = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 8$$

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8 Recurrent configurations:

(2 2 1) (2 2 0) (2 1 1) (1 2 1)

(2 1 0) (1 2 0) (2 0 1) (0 2 1)