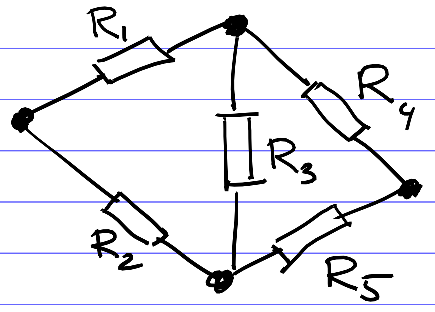
Electrical Networks

$G=(V,E)$ graph on vertex set $V=[n]$.
We'll view edges of G as resistors.



Edge weights = Conductivities = $\dfrac{1}{\text{resistances}}$

Let's fix directions of edges of G.
So, G is a digraph.

Remark.
We can pick edge directions in any way. (Like we did in the proof of undirected M.T.T. based on the Cauchy-Binet formula.)
We need edge directions to decide which directions of current should be regarded positive or negative.

Directed MTT, arborescences, etc depend on a directed graph in an essential way.

But here edge directions are needed just for notational purposes. The construction will not change if we reverse directions of some edges.

Any (directed) edge e in G has

- resistance $R_e \in \mathbb{R}_{>0}$.
- current $I_e \in \mathbb{R}$
- voltage (or potential difference) $V_e \in \mathbb{R}$.

Remark. If we reverse the direction of edge e, then resisdence remains the same, but the current & voltage reverse the sign:

$$R_{-e} = R_e$$

$$I_{-e} = -I_e$$

$$V_{-e} = -V_e$$

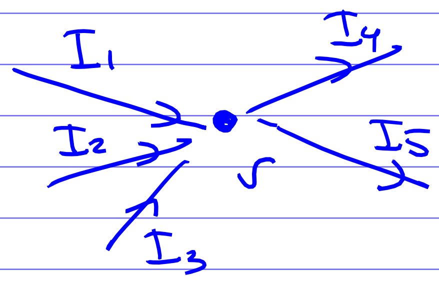
Typically, we'll assume that the graph G and resistances R_e are given. And we'll try to calculate the currents I_e and voltages V_e using the law of electricity.

Laws of Electricity

1st Kirchhoff's Law:

For any vertex $v \in V$,
the sum of in-currents =
= the sum of out-currents:

$$\sum_{e \to v} I_e = \sum_{v \xrightarrow{e'}} I_{e'}$$



$$I_1 + I_2 + I_3 = I_4 + I_5$$

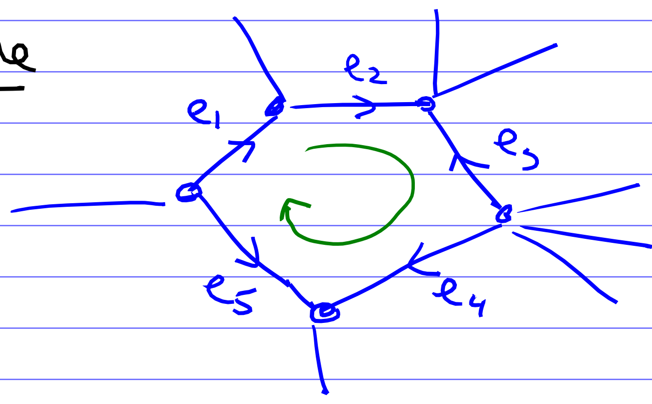
2nd Kirchhoff's Law:

For any cycle C in G with edges $e_1, e_2, \dots, e_m$ the signed sum of voltages of e_i's is zero:

$$\pm V_{e_1} \pm V_{e_2} \pm \dots \pm V_{e_m} = 0,$$

where the sign is "+" if the direction of edge e_i agrees with the orientation of C, and "−" if not.

Example



$$V_{e_1} + V_{e_2} - V_{e_3} + V_{e_4} - V_{e_5} = 0$$

Lemma. 2nd Kirchhoff's Law is equivalent to existence of function $U : \{\text{vertices of } G\} \to \mathbb{R}$ such that, for any edge $e : u \to v$,

$$V_e = U_v - U_u.$$

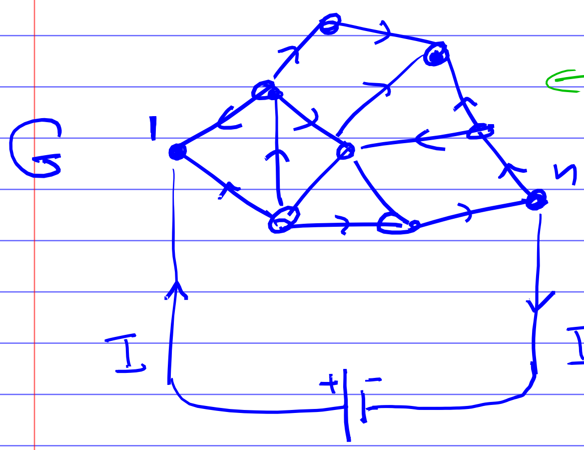
Such function U is called the **potential function**.

__Ohm's Law.__ For any edge e,

$$V_e = I_e \cdot R_e.$$

These 3 laws determine everything we need to know about an electrical network.

__A typical problem__



some graph

battery connected to the vertices 1 and n.

__Known:__

- graph G
- resistances R_e of all edges in G
- the voltage of the battery $U_1 - U_n$

__Find:__ All currents and voltages of edges in G.

Let's express electrical laws in matrix form.

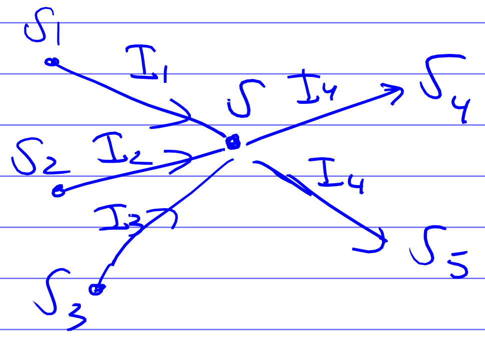
We'll express everything using the potential function $v \mapsto U_v$ (not the voltages

$$V_e = U_v - U_u \quad \text{for} \quad u \xrightarrow{e} v \text{)}$$

So 2nd Kirchhoff's law will hold automatically.

Ohm's law: $I_e = \dfrac{V_e}{R_e} = \dfrac{U_v - U_u}{R_e}$

Let's express the 1st Kirchhoff's law using the potential:



$$\underbrace{I_1 + I_2 + I_3}_{\text{in-current to } v} = \underbrace{I_4 + I_5}_{\text{out-current from } v}$$

The same equation in terms of potential:

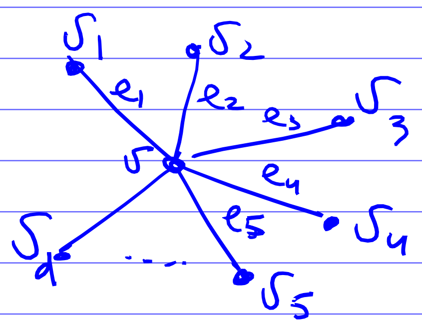
$$\underbrace{\frac{U_v - U_{v_1}}{R_1}}_{I_1} + \underbrace{\frac{U_v - U_{v_2}}{R_2}}_{I_2} + \underbrace{\frac{U_v - U_{v_3}}{R_3}}_{I_3} =$$

$$= \underbrace{\frac{U_{v_4} - U_v}{R_4}}_{I_4} + \underbrace{\frac{U_{v_5} - U_v}{R_5}}_{I_5},$$

or

$$\sum_i \frac{U_v - U_{v_i}}{R_i} = 0$$

In general, we have:

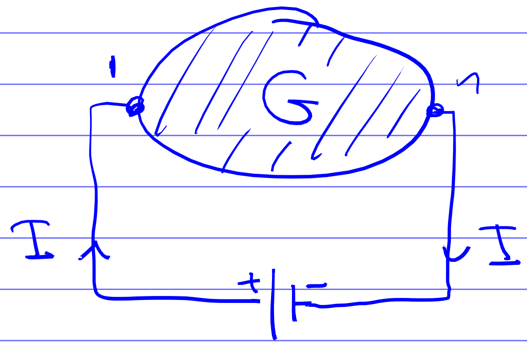


For any vertex $v \in V$ connected to vertices $v_1, \dots, v_d$ by edges $e_1, \dots, e_d$ with resistances $R_1, \dots, R_d$ we have

$$\left(\frac{1}{R_1} + \dots + \frac{1}{R_d}\right) U_v - \sum_{i=1}^{d} \frac{1}{R_i} U_{v_i} =$$

$$= \begin{cases} 0 & i \neq 1, n \\ -I & i = 1 \\ I & i = n \end{cases}$$

Here we assume that we connect a battery to vertices 1 and n, and $I = I_{battery}$ is the current through the battery:



So for vertex 1 we have additional in-current I from the battery, and for vertex n we have additional out-current I.

Define the <u>Kirchhoff's matrix</u>

$K = (K_{ij})$ as the $n \times n$ matrix

$$K_{ij} = \begin{cases} \displaystyle\sum_{\substack{e \text{ edge} \\ \text{adjacent to } i}} c_e & \text{if } i = j \\ \displaystyle -\sum_{\substack{e:\text{ edge} \\ \text{between } i \,\&\, j}} c_e & \text{if } i \neq j \end{cases}$$

where $c_e := \dfrac{1}{R_e}$ are the <u>conductivities</u> of edges e.

<u>Remark</u>. The Kirchoff's matrix is exactly the weighted Laplacian matrix for the <u>undirected</u> graph G with edge weights c_e.

This is why we earlier said that the Laplacian matrix is the same thing as the Kirchoff's matrix.

Let $\vec{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix}$ be the

<u>vector of potentials</u>.

(Its entries are the potentials U_r of all vertices $r = 1, \ldots, n$.)

Now all 3 laws of electicity can be expressed by a single equation:

(*) $$\boxed{K \cdot \vec{U} = \begin{bmatrix} -I \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}}$$

This is a system of n linear equations for n unknowns (potentials) $U_1, \ldots, U_n$.

But we have a little problem: the matrix K is not invertible, i.e. $\det(K) = 0$.

How do we solve the above system of linear equations?

Actually, this is not really surpising because (*) does not have a unique solution.

The potential function is defined up to a constrant:

If $\begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix}$ is a solution of (*),

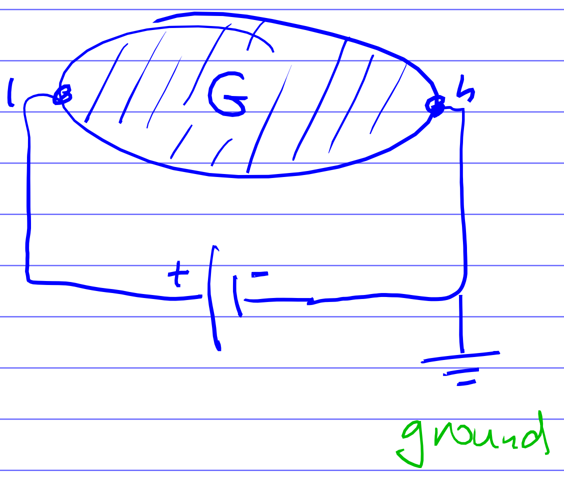
then $\begin{bmatrix} U_1+a \\ U_2+a \\ \vdots \\ U_n+a \end{bmatrix}$ is also a solution of (*).

Indeed, the law of electricity are expressed in terms of potential <u>differences</u> $U_v - U_u$, i.e. voltages.

(Only potential differences $U_v - U_u$ have physical meaning.)

We can fix this ambiguity by requiring that $U_n = 0$.

In other words we can "ground" vertex n.



Let $\tilde{K}$ be the <u>reduced</u> Kirchhoff's matrix obtained from K by removing the last row & the last column.

Then $(*) \Longleftrightarrow$

$$(**) \qquad \tilde{K} \cdot \begin{bmatrix} U_1 \\ \vdots \\ U_{n-1} \end{bmatrix} = \begin{bmatrix} -I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$(n-1)\times(n-1)$ matrix

Now $\tilde{K}$ is an invertible matrix (if G is a connected graph) and $(**)$ has a unique solution.

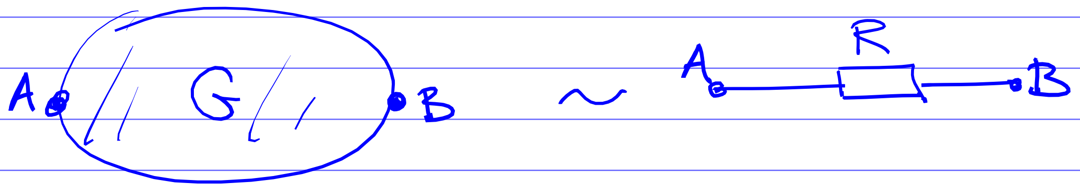
What is I?

We know the voltage of the battery $V_{bat} = U_1 - U_n$ $= U_1 - 0$. But we don't know the current I <u>through</u> the battery.

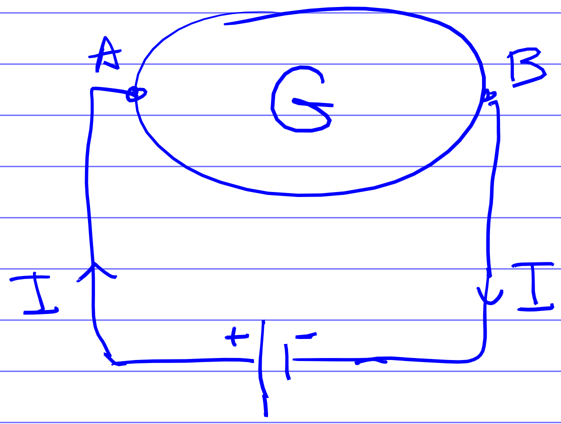
Here is how to fix this:

- Solve $(**)$ for $I = 1$ (or for any I).
- Rescale the solution $(U_1, \dots, U_{n-1})^T$ so that $U_1 - 0$ becomes the given voltage $V_{battery}$ of the battery.

The whole graph G acts as a single resistor between two marked vertices A & B.
(In above construction $A=1$ & $B=n$)



How to find the resistance $R_{AB}(G)$ if we know resistances of all edges in G?



By Ohm's Law

$$R_{AB}(G) = \frac{U_A - U_B}{I}$$

Solving (**) by Cramer's rule we deduce

$$R_{AB}(G) = \frac{\det \tilde{\tilde{K}}}{\det \tilde{K}}$$

matrix obtained from K by removing 2 rows & two columns with labels A & B

Both numerator & denominator can be interpreted using (weighted) matrix tree theorem

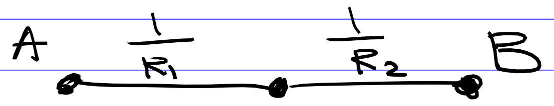
Theorem Let G be a weighted graph with edge weights $C_e = \frac{1}{R_e}$ and two marked vertices A & B.

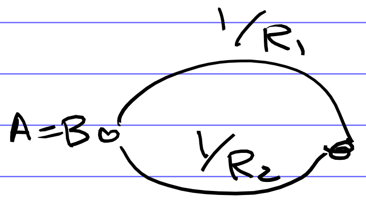
Let G' be the graph obtained from G by gluing vertices A & B into a single vertex. Then

$$R_{AB}(G) = \frac{\displaystyle\sum'_{\substack{T' \text{ spanning} \\ \text{tree of } G'}} \text{weight}(T')}{\displaystyle\sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \text{weight}(T)}$$

Examples.

Series connection

$G =$ 

$G' =$ 

2 spanning trees of G'

$$R_{AB}(G) = \frac{\frac{1}{R_1} + \frac{1}{R_2}}{\frac{1}{R_1} \cdot \frac{1}{R_2}}$$

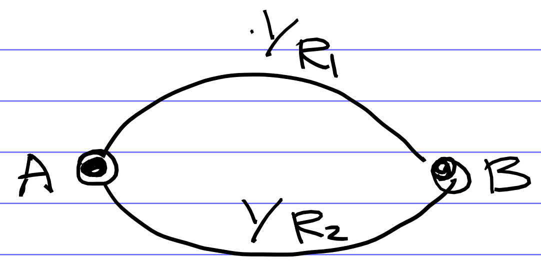
1 spanning tree of G

$$= R_1 + R_2 .$$

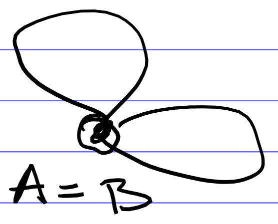
For a series connection, we need to add resistances

Parallel connection

$G =$



$G' =$



$$R_{AB}(G) = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

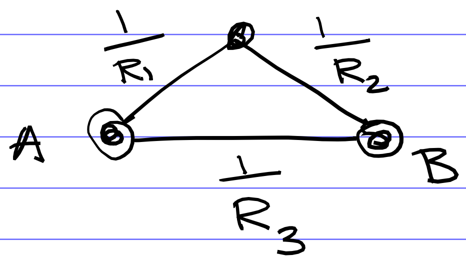
1 empty spanning tree of G'

2 spanning trees of G.

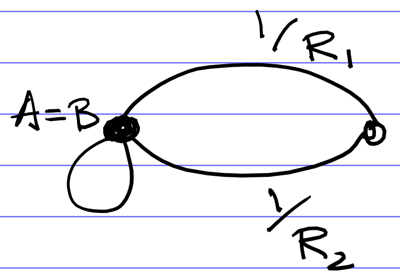
For a series connection inverses of resistances add.

Another example

$G =$



$G' =$



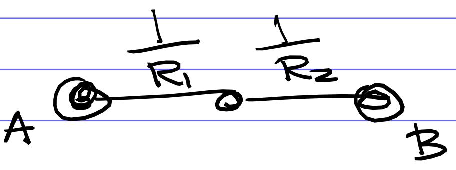
2 spanning trees of G'

$$R_{AB} = \frac{\frac{1}{R_1} + \frac{1}{R_2}}{\frac{1}{R_1}\cdot\frac{1}{R_2} + \frac{1}{R_1}\cdot\frac{1}{R_3} + \frac{1}{R_2}\cdot\frac{1}{R_3}}$$

3 spanning trees of G.

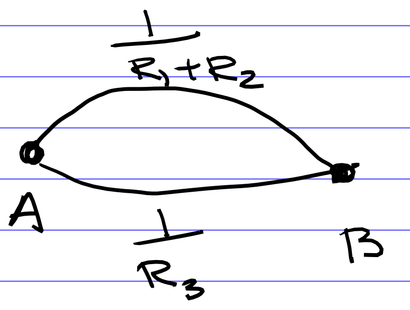
Let's try to get the same result by first doing a series connection & then a parallel connection:

series:



$R_{AB} = R_1 + R_2$

parallel:



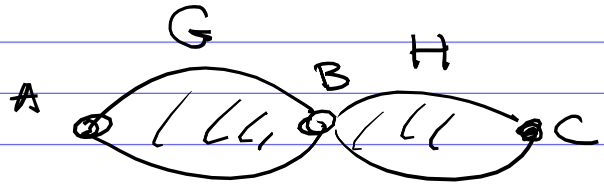
$$R_{AB}(G) = \frac{1}{\frac{1}{R_1+R_2} + \frac{1}{R_3}}$$

$$= \frac{R_1+R_2}{1 + \frac{R_1}{R_3} + \frac{R_2}{R_3}}$$

= same expression as above.

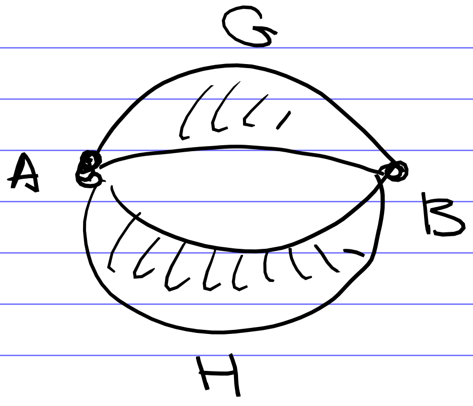
In general, for any two connected graphs G & H with edge weights $= \frac{1}{R_e}$ and 2 marked vertices in each graph, we have

series connection:



$$R_{AC}(G\text{-}H) = R_{AB}(G) + R_{BC}(H)$$

parallel connection.

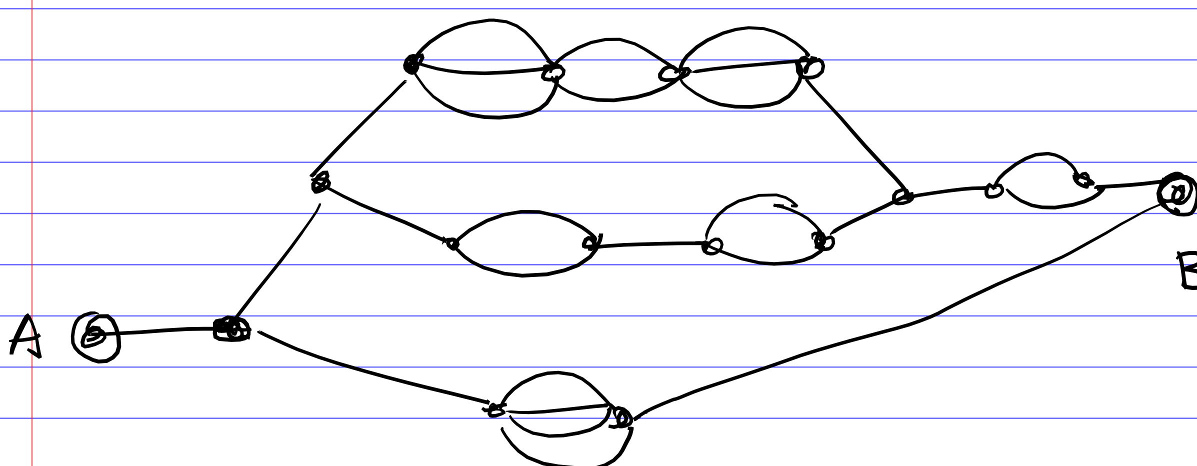


$$R_{AB}(G \,\|\, H) = \frac{1}{\frac{1}{R_{AB}(G)} + \frac{1}{R_{AB}(H)}}$$

<u>Exercise</u>. Interpret & prove both formulas using spanning trees.

Weighted graphs that can be constructed from single edges by a sequence of series & parallel connections are called series-parallel networks:

Example



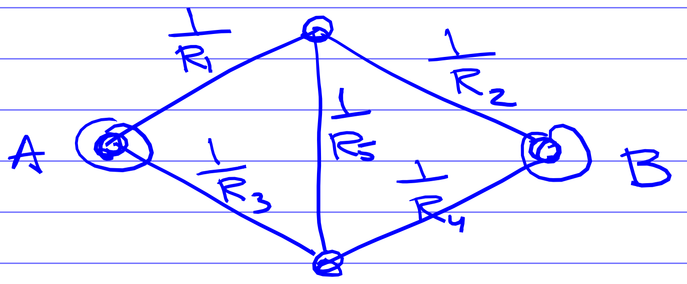
a series-parallel network

It is easy to calculate resistances of such networks using above formulas:

series: $R_1 + R_2$

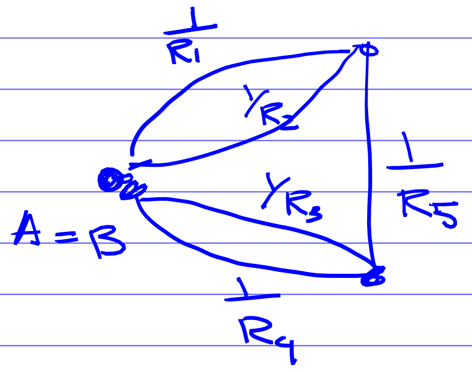
parallel: $\left((R_1)^{-1} + (R_2)^{-1}\right)^{-1}$

Here is the smallest example of a graph that cannot be obtained by series-parallel connections:

$G =$ 

Wheatstone bridge

We can calculate the resistance $R_{AB}(G)$ using spanning trees

$G' =$ 

$R_{AB}(G) =$

8 spanning trees of G'

$$= \frac{\left(\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)\cdot\frac{1}{R_5} + \left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left(\frac{1}{R_3}+\frac{1}{R_4}\right)\right)}{\left(\frac{1}{R_1}\cdot\frac{1}{R_2}\cdot\frac{1}{R_3} + \frac{1}{R_1}\cdot\frac{1}{R_2}\cdot\frac{1}{R_4} + \frac{1}{R_1}\cdot\frac{1}{R_3}\cdot\frac{1}{R_4} + \frac{1}{R_2}\cdot\frac{1}{R_3}\cdot\frac{1}{R_4} + \left(\frac{1}{R_1}+\frac{1}{R_3}\right)\cdot\left(\frac{1}{R_2}+\frac{1}{R_4}\right)\cdot\frac{1}{R_5}\right)}$$

8 spanning trees of G.