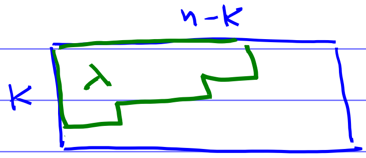
18.217 Lecture 17 03/29/2021

Unimodality of the Gaussian q-binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q} = \frac{(1-q)(1-q^2)\dots(1-q^n)}{((1-q)\dots(1-q^k))((1-q)\dots(1-q^{n-k}))}$$

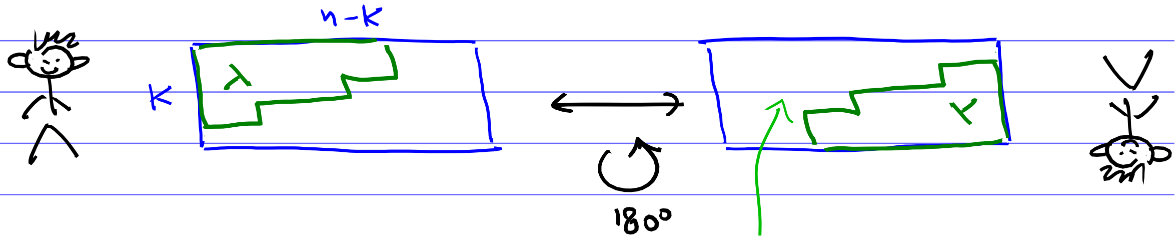
$$= a_0 + a_1 q + a_2 q^2 + \dots + a_{k(n-k)} q^{k(n-k)},$$

where $a_r = \#\{\lambda \subseteq k \times (n-k),\ |\lambda| = r\}$



Young diagram λ fits inside the $k \times (n-k)$ rectangle

Clearly, $a_r = a_{k(n-k)-r}$ (symmetry)



$\lambda^\vee := (n-k-\lambda_k,\ n-k-\lambda_{k-1}, \dots, n-k-\lambda_1)$

Example

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

Unimodality Theorem [Sylvester, 1878]

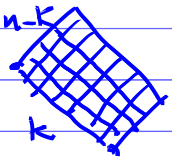
$a_0 \leq a_1 \leq \dots \leq a_{\lfloor \frac{N}{2} \rfloor} \geq \dots \geq a_N,$

$N := k(n-k).$

James Joseph Sylvester (1814 – 1897)

We'll give Sylvester's argument using weighted up & down operators.

Let P be a finite poset with a weight function $wt : P \to \mathbb{R}_{>0}$ on its elements.

(we will take $P = [k] \times [n-k] =$ 

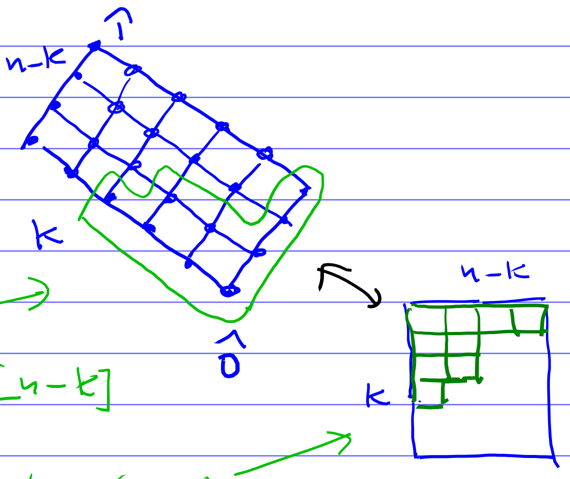
but right now we can assume that P is any poset.)

Let $J(P)$ be the lattice of order ideals in P, and

$J(P)_r :=$ the set of order ideals with r elements.

Recall that $J([k] \times [n-k])$ is Young's lattice $L(k, n-k)$ of Young diagrams $\lambda \subseteq k \times (n-k)$ (ordered by inclusion).

Example $P =$



an order ideal in $P = [k] \times [n-k]$ corresponds to a Young diagram $\lambda \subseteq k \times (n-k)$

$k = 6$
$n-k = 4$
$\lambda = (4,2,2,1)$

For $P = [k] \times [n-k]$, elements of $J(P)_r$ correspond to Young diagrams $\lambda \subseteq k \times (n-k)$ with $|\lambda| = r$.

So $|J(P)_r| = a_r$.

Sylvester's Theorem $\Longleftrightarrow$

$$\# J(P)_r \leq \# J(P)_{r+1} \text{ for}$$

$$r < k(n-k)/2$$

For $\lambda \in J(P)_r$, let

$\mathrm{Add}(\lambda) := \{ x \in P$ s.t. $\lambda \cup \{x\}$ is an order ideal covering $\lambda \}$

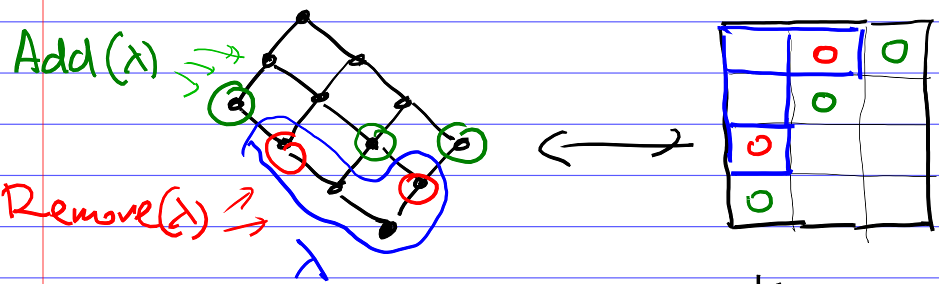
$\mathrm{Remove}(\lambda) := \{ y \in P$ s.t. $\lambda \setminus \{y\}$ is an order ideal covered by $\lambda \}$

For $P = [k] \times [n-k]$

$\mathrm{Add}(\lambda)$ correspond to all "addable" boxes to λ, i.e., outer corners of λ (located inside $k \times (n-k)$ rectangle).

$\mathrm{Remove}(\lambda)$ correspond to all removable boxes from λ.

Example $P = [4] \times [3]$



$k = 4, \quad n-k=3$

$\lambda = (2,1,1,0)$

Add(λ) consists of 3 green boxes

Remove(λ) consists of 2 red boxes

Theorem Fix poset P and rank r.

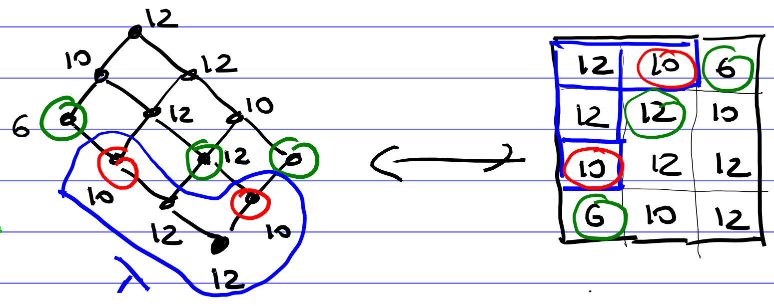
Assume that there exists a weight function $wt : P \to \mathbb{R}_{>0}$ such that

For any $\lambda \in J(P)_r$, we have

$$\sum_{x \in Add(\lambda)} wt(x) > \sum_{y \in Remove(\lambda)} wt(y).$$

Then $\# J(P)_r \leq \# J(P)_{r+1}$

Example $P = [4]\times[3]$, $r = 4$



Here is a weight function that works

⑥ + ⑫ + ⑥ > ⑩ + ⑩

If such inequalities are true for all Young diagrams $\lambda \subseteq 4\times 3$ with $|\lambda| = 4$ boxes,

then the above Theorem implies that

$$\begin{array}{ccc} J([3]\times[4])_4 & \le & J([3]\times[4])_5 \\ \parallel & & \parallel \\ a_4 & & a_5 \end{array}$$

How to find a weight function what works?

Could we just take $wt(\lambda) = 1$ for all λ?

Remember, we said that

addable boxes to λ

= # removable boxes from λ + 1

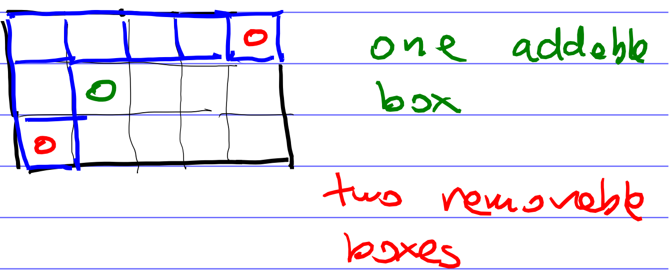
This is true if we consider all Young diagrams $\lambda \subseteq \mathbb{Y}$ without restrictions on their sizes. But for Young diagrams $\lambda \in L(k, n-k)$ this may not be true.

Example

$k = 3$, $n-k = 5$

$\lambda = (5,1,1)$

$|\lambda| = 7 < \frac{3\cdot 5}{2}$



The weight function $wt(\lambda) = 1$ works only for $r = |\lambda| < \min(k, n-k)$.

But it does not work in general.

We need to:

(1) Prove the above thm.

(2) For $P = [k] \times [n-k]$ and $r < k \cdot (n-k)/2$, construct a weight function that satisfies the conditions of the theorem.

For (1), we'll use weighted up & down operators and some linear algebra.

For (2) we'll use combinatorics.

Let $V_r = \mathbb{R}[J(P)_r]$

vector space of formal linear combinations of elements $\lambda \in J(P)_r$

(For $P = [k] \times [n-k]$, V_r is the space of formal lin. comb. of Young diagrams $\lambda \subseteq k \times (n-k)$ with $|\lambda| = r$.)

"up" operator:

$$U_r : V_r \to V_{r+1}$$

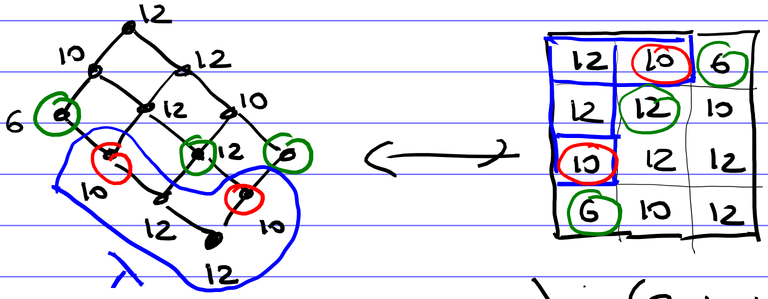
$$\lambda \mapsto \sum_{x \in Add(\lambda)} \sqrt{wt(x)} \cdot \lambda \cup \{x\}$$

"down" operator

$$D_r : V_r \to V_{r-1}$$

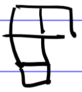
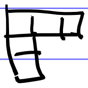
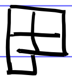
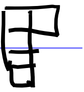
$$\lambda \mapsto \sum_{y \in Remove(\lambda)} \sqrt{wt(y)} \quad \lambda \setminus \{y\}$$

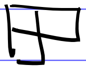
Example $P = [4] \times [3]$, $r = 4$



12	10	6
12	12	10
10	12	12
6	10	12

$\lambda = (2,1,1,0)$

U_4 :  $\mapsto \sqrt{6} \cdot$  $+ \sqrt{12}$  $+ \sqrt{6}$ 

D_4 :  $\mapsto \sqrt{10}$  $+ \sqrt{10}$ 

In terms of matrices

$U_r : V_r \to V_{r+1}$ is given by a matrix of size $|J(P)_{r+1}| \times |J(P)_r|$

and $D_{r+1} : V_{r+1} \to V_r$ is given by the transpose matrix

$$D_{r+1} = U_r^T$$

<u>Lemma</u> Let

$$H_r := D_{r+1} U_r - U_{r-1} D_r$$

$$\begin{tikzcd}
 & V_{r+1} \arrow[dr, "D_{r+1}"] & \\
V_r \arrow[ur, "U_r"] \arrow[rr, "H_r"] \arrow[dr, "D_r"'] & & V_r \\
 & V_{r-1} \arrow[ur, "U_{r-1}"'] &
\end{tikzcd}$$

Then H_r is given by a <u>diagonal</u> matrix with <u>positive</u> entries on the diagonal.

Explicitly,

$$H_r : \lambda \longmapsto \Big(\sum_{x \in Add(\lambda)} wt(x) - \sum_{y \in Remove(\lambda)} wt(y) \Big) \lambda$$

<u>Proof.</u> (This is basically the same argument that we gave when we proved $DU - UD = I$ for Young's lattice $\mathbb{Y}$.)

$$H_r = D_{r+1} U_r - U_{r-1} \cdot D_r$$

has all zero off diagonal entries. Indeed, if we add a box x to λ and then remove a different box y, we can reverse these operation (first remove y & then add x). We get

$$\sqrt{wt(x)} \cdot \sqrt{wt(y)} - \sqrt{wt(y)} \cdot \sqrt{wt(x)} = 0.$$

The diagonal entries of H_r:
Add a box x to λ & then remove the same box x. we get

$$\sum_{x \in Add(\lambda)} \sqrt{wt(x)} \sqrt{wt(x)} \lambda$$

or remove box y from λ and then add the same box y

we get

$$\sum_{y \in Remove(\lambda)} \sqrt{wt(y)} \sqrt{wt(y)} \cdot \lambda.$$

Combining this, we get

$$H_r : \lambda \mapsto \underbrace{\left(\sum_{x \in Add(\lambda)} wt(x) - \sum_{y \in Remove(\lambda)} wt(y) \right)}_{>0 \text{ by the assumption of the theorem.}} \lambda$$

Now we can finish the proof of theorem:

We know

- $H_r = D_{r+1} U_r - U_{r-1} D_r$

  is a diagonal matrix with <u>positive</u> eigenvalues

- $U_r = D_{r+1}^T$, $U_{r-1} = D_r^T$

So $D_{r+1} U_r = D_r^T D_r + H_r$

positive semi-definite matrix

positive definite

Thus $D_{r+1} U_r$ is a positive definite $|J(P)_r| \times |J(P)_r|$ matrix

$\Longrightarrow$ Its det $\neq 0 \Longleftrightarrow$ its rank $= |J(P)_r|$.

But U_r is a $|J(P)_{r+1}| \times |J(P)_r|$ matrix.

$\text{rank}(U_r) \geq \text{rank}(\underset{\overset{\shortparallel}{U_1^T U_r}}{H_r}) = |J(P)_r|$.

$\Longrightarrow |J(P)_{r+1}| \geq |J(P)_r|$,

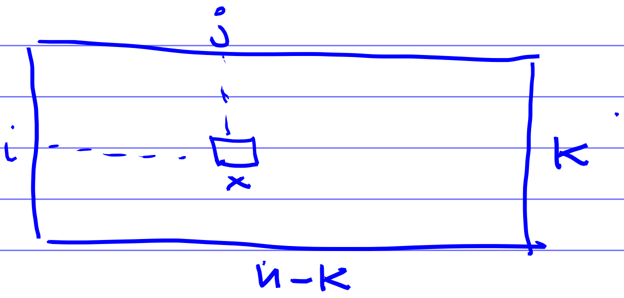
as needed. $\square$

We still need to construct a weight function for $P = [k] \times [n-k]$ $r < k(n-k)/2$

with required properties.

Lemma. Consider the weight function wt on boxes of the $k \times (n-k)$ rectangle given by

$$wt(x) := (n-k-c(x))\,(k+c(x))$$



$c(x) := j-i$ the content of x

Then this weight function satisfies the needed condition

$$\sum_{x \in Add(\lambda)} wt(x) - \sum_{y \in Remove(\lambda)} wt(y) > 0$$

for any $\lambda \subseteq k \times (n-k)$ with $r = |\lambda| < k(n-k)/2$ boxes.

Example $k=4, \quad n-k=3$

3·4	2·5	1·6
4·3	3·4	2·5
5·2	4·3	3·4
6·1	5·2	4·3

Lemma For this weight function wt, and any $\lambda \subseteq k \times (n-k)$ we have

$$\sum_{x \in Add(\lambda)} wt(x) - \sum_{y \in Remove(\lambda)} wt(y)$$

$$= k \cdot (n-k) - 2|\lambda|.$$

(Clearly, this > 0 if $|\lambda| < \frac{k(n-k)}{2}$)

Example

$k = 4$

$n-k = 3$

$\lambda = (2,1)$

n−k

k

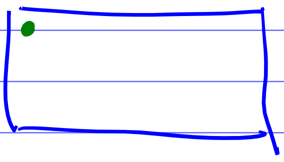
	2·5	1·6
	3·4	
5·2		
6·1		

$$1\cdot6 + 3\cdot4 + 6\cdot1 + 2\cdot5 + 5\cdot2$$

$$= \underbrace{4\cdot3}_{k\cdot(n-k)} - \underbrace{2\cdot4}_{2\cdot|\lambda|}$$

Idea of proof. Induction on $|\lambda|$.

Base $\lambda = \emptyset$



Add(λ) = a single box in the upper left corner whose wt is $k.(n-k)$

So we get $k.(n-k) = k.(n-k) - 0$ ✓

Induction Step

λ with $|\lambda| > 0$ and

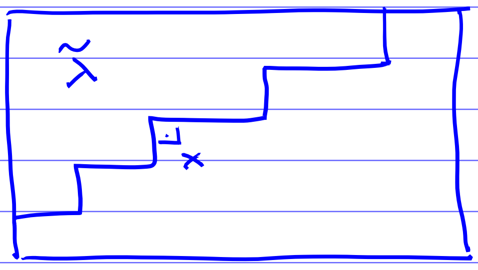
Let $\tilde{\lambda}$ is obtained from λ by removing a box x.

By induction the equality

$$\sum wt(x) - \sum wt(y) = k(n-k) - 2|\tilde{\lambda}|$$

holds for $\tilde{\lambda}$.

For λ, the R.H.S decreases by 2. One can check that the L.H.S also decreases by 2...



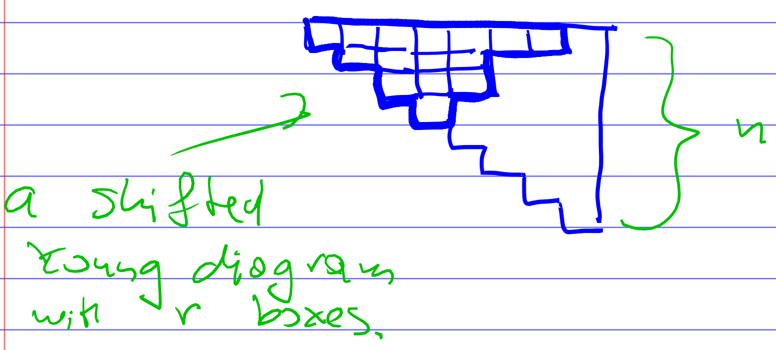
when we add a box, the sets Add$(\tilde{\lambda})$ and Remove$(\tilde{\lambda})$ slightly changes...

Exercise. Prove the lemma.

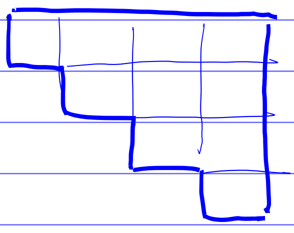
There is also a more interesting proof of the Lemma that does not use induction.

How about shifted Young diagrams?

Fix n. Let b_r be the number of shifted Young diagrams with r boxes that fit inside the "staircase of size n"

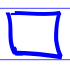


Example $n = 4$

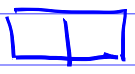


$b_0 = 1$ ∅

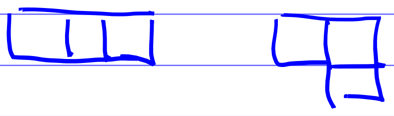
$b_1 = 1$



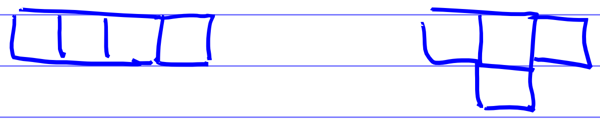
$b_2 = 1$



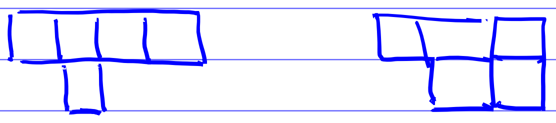
$b_3 = 2$



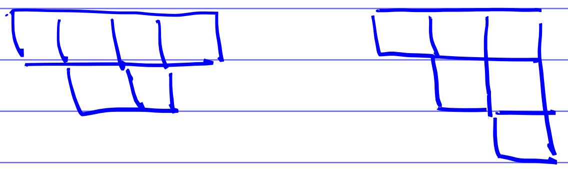
$b_4 = 2$



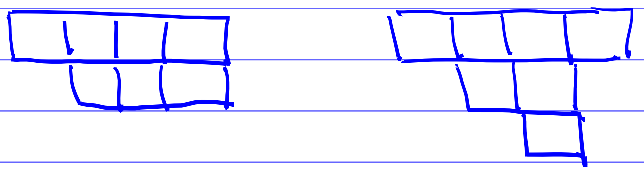
$b_5 = 2$



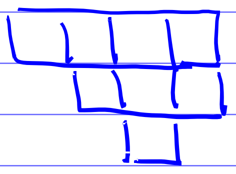
$b_6 = 2$



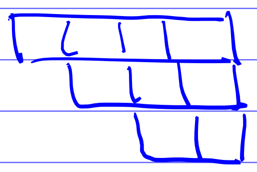
$b_7 = 2$



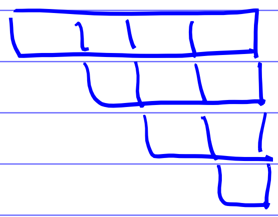
$b_8 = 1$



$b_9 = 1$



$b_{10} = 1$



We've got a unimodal sequence

1, 1, 1, 2, 2, 2, 2, 2, 1, 1, 1

Can you extend Sylvesters proof to shifted Young diagrams?

Exercise. Prove the unimodality

$$b_0 \le b_1 \le \dots \le b_{\lfloor M/2 \rfloor} \ge \dots \ge b_M$$

$$M = n(n+1)/2$$

or present a counter example.