Unimodality of the Gaussian $q$-binomial coefficients:

\[ \binom{n}{k}_q = \frac{[n]!}{[k]_q! [n-k]_q} \frac{(1-q)(1-q^2) \ldots (1-q^n)}{(1-q)(1-q^k)(1-q^2) \ldots (1-q^k)} \]

\[ = a_0 + a_1 q + a_2 q^2 + \ldots + a_{k(n-k)} q^{k(n-k)} \]

where $a_r = \# \{ \lambda \subseteq k \times (n-k), |\lambda| = r \}$

Clearly, $a_r = a_{k(n-k)-r}$ (symmetry)

Example: $\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$
Unimodality Theorem [Sylvester, 1878]

$a_0 \leq a_1 \leq \ldots \leq a_{\lfloor \frac{n}{2} \rfloor} > \ldots > a_n$,

$N := k(n-k)$.

We'll give Sylvester's argument using weighted up & down operators.

Let $P$ be a finite poset with a weight function $w: P \to \mathbb{R}_{>0}$ on its elements.

(we will take $P = [k] \times [n-k]$ but right now we can assume that $P$ is any poset.)

Let $J(P)$ be the lattice of order ideals in $P$, and $J(P)_r :=$ the set of order ideals with $r$ elements.

Recall that $J([k] \times [n-k])$ is Young's lattice $L(k,n-k)$ of Young diagrams $\lambda \leq k \times (n-k)$ (ordered by inclusion).
Example \( P = \) 

an order ideal in \( P = [k] \times [n-k] \)

Corresponds to a Young diagram \( \lambda \subseteq k \times (n-k) \)

For \( P = [k] \times [n-k] \),
elements of \( J(P)_r \) correspond to Young diagrams \( \lambda \subseteq k \times (n-k) \)

with \( |\lambda| = r \).

So \( |J(P)_r| = a_r \).

Sylvester's Theorem \( \Rightarrow \)

\[ \# J(P)_r \leq \# J(P)_{r+1} \text{ for } r < k(n-k)/2 \]

For \( \lambda \in J(P)_r \), let

Add(\( \lambda \)) := \{ y \in P \text{ s.t. } \lambda \cup \{y\} \text{ is an order ideal covering } \lambda \} \]

Remove(\( \lambda \)) := \{ y \in P \text{ s.t. } \lambda \setminus \{y\} \text{ is an order ideal covered by } \lambda \} \]

For \( P = [k] \times [n-k] \)

Add(\( \lambda \)) correspond to all addable boxes to \( \lambda \), i.e., outer corners of \( \lambda \)

(located inside \( k \times (n-k) \)

rectangle)

Remove(\( \lambda \)) correspond to all removable boxes from \( \lambda \).

\[ \lambda = (2, 1, 1, 0) \]

Add (\( \lambda \)) consists of 3 green boxes

Remove (\( \lambda \)) consists of 2 red boxes

Theorem.

Fix poset \( P \) and rank \( r \).

Assume that there exists a weight function \( w^+ : P \to \mathbb{R}_{\geq 0} \) such that

For any \( \lambda \in J(P)_r \), we have

\[ \sum_{x \in \text{Add}(\lambda)} w^+(x) > \sum_{y \in \text{Remove}(\lambda)} w^+(y). \]

Then \( \# J(P)_r < \# J(P)_{r+1} \).
Example \( P = [4] \times [3], \, r = 4 \)

Here is a weight function that works

\[
6 \cdot (12) + 6 > 15 + 10
\]

If such inequalities are true for all Young diagrams \( \lambda \leq 4 \times 3 \) with \( |\lambda| = 9 \) boxes,

Then the above Theorem implies that

\[
J ([3] \times [4])_4 \preceq J ([3] \times [4])_5
\]

\[
\alpha_4 \preceq \alpha_5
\]

How do we find a weight function what works?
Could we just take \( w_t(\lambda) = 1 \) for all \( \lambda \)?

Remember, we said that

\[
\# \text{addable boxes to } \lambda \ni \lambda
\]

\[
= \# \text{removable boxes from } \lambda + 1
\]

This is true if we consider all Young diagrams \( \lambda \leq \lambda \) without restrictions on their sizes. But for Young diagrams \( \lambda \leq L (k, n-k) \) this way not be true.

Example

\[
k = 3, \quad n-k = 5
\]

\[
\lambda = (5, 1)
\]

\[
|\lambda| = 6 < \frac{3 \times 5}{2}
\]

The weight function \( w_t(\lambda) = 1 \) works only for \( r = |\lambda| < \min(k, n-k) \).
But it does not work in general.
We need to:

1. Prove the above theorem.
2. For $P = [K] \times [L_{n-k}]$ and $r \leq k \cdot (n-k)/2$, construct a weight function that satisfies the conditions of the theorem.

For (1), we'll use weighted up & down operators and some linear algebra.

For (2) we'll use combinatorics.

Let $V_r = \mathbb{R}[J(P)_r]$ vector space of formal linear combinations of elements $\lambda \in J(P)_r$

(For $P = [K] \times [L_{n-k}]$, $V_r$ is the space of formal linear comb. of Young diagrams $\lambda \in K \times (n-k)$ with $|\lambda| = r$)
"Up" operator:

\[ U_r : V_r \to V_{r+1} \]

\[ \lambda \mapsto \sum_{x \in \text{Add}(\lambda)} \sqrt{w^+(x)} \cdot x \in \mathfrak{S}_3 \]

"Down" operator:

\[ D_r : V_r \to V_{r-1} \]

\[ \lambda \mapsto \sum_{y \in \text{Remove}(\lambda)} \sqrt{w^+(y)} \cdot y \in \mathfrak{S}_3 \]

Example: \( P = [4] \times [3] \), \( r = 4 \)

\[ \lambda = (2,1,1,0) \]

\[ U_q : \begin{pmatrix} 10 & 12 \\ 12 & 10 \end{pmatrix} \mapsto 12 \begin{pmatrix} 6 \\ 10 \end{pmatrix} + \sqrt{12} \begin{pmatrix} 12 \\ 12 \end{pmatrix} + \sqrt{6} \begin{pmatrix} 12 \\ 12 \end{pmatrix} \]

\[ D_q : \begin{pmatrix} 12 & 16 \\ 12 & 12 \end{pmatrix} \mapsto \sqrt{10} \begin{pmatrix} 12 \\ 12 \end{pmatrix} + \sqrt{10} \begin{pmatrix} 12 \\ 12 \end{pmatrix} \]
In terms of matrices $U_r : V_r \rightarrow V_{r+1}$ is given by a matrix of size $|J(r)| \times |J(r)|$ and $D_{r+1} : V_{r+1} \rightarrow V_r$ is given by the transpose matrix $D_{r+1} = U_r^T$.

Lemma: Let

\[ H_r := D_{r+1} U_r - U_{r-1} D_r \]

Then $H_r$ is given by a diagonal matrix with positive entries on the diagonal.

Explicitly,

\[ H_r : x \mapsto \left( \sum_{x \in \text{Add}(x)} - \sum_{x \in \text{Remove}(x)} \right) x \]

Proof. (This is basically the same argument that we used when we proved $UUU - U = I$ for Young's lattice $Y_r$.)
\[ H_r = D \cdot r - U \cdot r \cdot D \]

has all zero off diagonal entries. Indeed, if we add a box \( x \) to \( \lambda \) and then remove a different box \( y \), we can reverse these operations (first remove \( y \) & then add \( x \)) we get

\[
\sqrt{w^+(x)} \cdot \sqrt{w^+(y)} - \sqrt{w^+(y)} \cdot \sqrt{w^+(x)} = 0.
\]

The diagonal entries of \( H_r \):

Add a box \( x \) to \( \lambda \) &
then remove the same box \( x \),
we get \[ \sum_{x \in \text{Add}(\lambda)} \sqrt{w^+(x)} \cdot \lambda \]
or remove box \( y \) from \( \lambda \)
and then add the same box \( y \)
we get \[ \sum_{y \in \text{Remove}(\lambda)} \sqrt{w^+(y)} \cdot \lambda. \]

Combining this, we get

\[ H_r : \lambda \rightarrow (\sum_{x \in \text{Add}(\lambda)} \sqrt{w^+(x)} - \sum_{y \in \text{Remove}(\lambda)} \sqrt{w^+(y)}) \cdot \lambda \]

\[ > 0 \text{ by the assumption of the theorem.} \]
Now we can finish the proof of Theorem:

We know

- $H_r = D_{r+1} U_r - U_{r-1} D_r$
  
  is a diagonal matrix with positive eigenvalues

- $U_r = D_{r+1}^T$, $U_{r-1} = D_r^T$

So $D_{r+1} U_r = D_r^T D_r + H_r$

Thus $D_{r+1} U_r$ is a positive definite $|J(P)_r| \times |J(P)_r|$ matrix

$\implies$ Its $\det \neq 0 \iff$

its rank $= |J(P)_r|$

But $U_r$ is a $|J(P)_{r+1}| \times |J(P)_r|$ matrix,

$\text{rank } (U_r) \geq \text{rank } (H_r) = |J(P)_r|$

$U_r^T U_r$

$\implies |J(P)_{r+1}| \geq |J(P)_r|$, as needed. $\square$
We still need to construct a weight function for
\[ P = \left[k, j \right] \times \left[k, a-k \right] \quad r < k \frac{(n-k)}{2} \]
with required properties.

Lemma. Consider the weight function \( w_t \) on boxes of the \( k \times (n-k) \) rectangle given by

\[
 w_t(x) := (n-k-c(x))(k+c(x))
\]

Then this weight function satisfies the needed condition

\[
 \sum_{x \in \text{Add}(\lambda)} w_t(x) - \sum_{x \in \text{Remove}(\lambda)} w_t(x) > 0
\]

for any \( \lambda \subset k \times (n-k) \) with \( r = |\lambda| < k \frac{(n-k)}{2} \) boxes.
**Example**

\[ K = 4, \quad n-K = 3 \]

<table>
<thead>
<tr>
<th>3.4</th>
<th>2.5</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3</td>
<td>3.4</td>
<td>2.5</td>
</tr>
<tr>
<td>5.2</td>
<td>4.3</td>
<td>3.4</td>
</tr>
<tr>
<td>6.1</td>
<td>5.2</td>
<td>4.3</td>
</tr>
</tbody>
</table>

**Lemma**

For this weight function \( w^+ \) and any \( \lambda \in \mathcal{K} \times (n-K) \) we have

\[
\sum_{x \in \text{Add}(\lambda)} w^+(x) - \sum_{y \in \text{Remove}(\lambda)} w^+(y) = K(n-K) - 2|\lambda|,
\]

(Clearly, this > 0 if \( |\lambda| < \frac{K(n-K)}{2} \)).

**Example**

\[
\begin{array}{|c|c|}
\hline
K = 4 & n-K = 3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
3.4 & 2.5 & 1.6 \\
4.3 & 3.4 & 2.5 \\
5.2 & 4.3 & 3.4 \\
6.1 & 5.2 & 4.3 \\
\hline
\end{array}
\]

\[
1.6 + 3.4 + 6.1 + 2.5 + 5.2 = 4.3 - 2.4
\]

\[
K(n-K) = 2|\lambda|
\]
Idea of proof. Induction on $\lambda$.

**Base** $\lambda = \emptyset$

Add $(x)$ is a single box in the upper left corner where $wt$ is $k (n-k)$

So we get $k (n-k) = k (n-k) - 0$

**Induction Step**

$\lambda$ with $|\lambda| > 0$ and let $\lambda'$ is obtained from $\lambda$ by removing a box $x$.

By induction the equality

$$\sum wt(x) - \sum wt(y) = k (n-k) - 2 |x|$$

holds for $\lambda'$.

For $\lambda$, the RHS decreases by 2.

One can check that the LHS also decreases by 2.

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**Exercise:** Prove the lemma.

There is also a more interesting proof of the lemma that does not use induction.
How about shifted Young diagrams?

Fix $n$ and let $r$ be the number of shifted Young diagrams with $r$ boxes that fit inside the "staircase of size $n$".

Example $n = 4$

$\begin{array}{cccc}
b_0 &=& 1 & \times \\
b_1 &=& 1 & \\
b_2 &=& 1 & \\
b_3 &=& 2 & \\
b_4 &=& 2 & \\
b_5 &=& 2 & \\
b_6 &=& 2 & \\
b_7 &=& 2 & \\
b_8 &=& 1 & \\
b_9 &=& 1 & \\
b_{10} &=& 1 & \\
\end{array}$

We've got a unimodal sequence

$1, 1, 1, 2, 2, 2, 2, 1, 1$
Can you extend Sylvester's proof to shifted Young diagrams?

**Exercise.** Prove the unimodality
\[ b_0 < b_1 < \ldots < b_{M/2} \geq \ldots \geq b_M \]

\[ M = \frac{n(n+1)}{2} \]

or present a counter example.