18.212 Lecture 9 03/09/2021

last time: • posets, chains, antichains.

• Any product of chains $[a] \times [b] \times \dots \times [c]$ (in particular, the Boolean lattice $B_n = [2] \times \dots \times [2]$) has a symmetric chain decomposition (SCD). Thus it is rank symmetric, unimodal, an Sperner.

$\Rightarrow$ Sperner's Theorem: Any collection of pairwise incomparable subsets of $[n]$ has $\leq \binom{n}{\lfloor n/2 \rfloor}$ elements.

Other results about chains and antichains.

Let P be <u>any</u> finite poset

Define

$MA(P) :=$ the maximal number of elements in an antichain in P.

$mc(P) :=$ the minimal number of chains needed to cover all elements of P

Theorem (Dilworth 1950)

$$MA(P) = mc(P).$$

Here is a dual version:

$MC(P) :=$ the maximal number of elements in a ~~anti~~chain in P.

$ma(P) :=$ the minimal number of antichains needed to cover all elements of P

Theorem (Minsky, 1971)

$$MC(P) = ma(P).$$

Both Dilworth's and Minsky's theorems are special cases of

Greene's Theorem

For a finite poset P, let $\ell_k :=$ the maximal size of a union of k chains;

$m_k :=$ the maximal size of a union of k antichain.

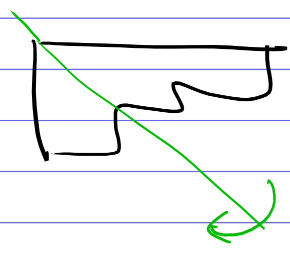
In particular, $\ell_1 = MC(P)$ and $m_1 = MA(P)$.

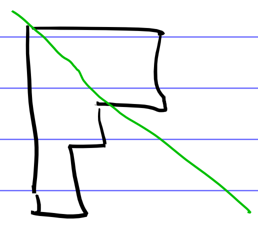
Theorem (Greene, 1976)

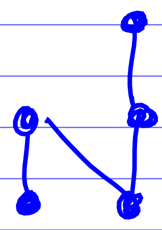
Let $\lambda(P) = (\lambda_1, \lambda_2, \dots) := (\ell_1, \ell_2 - \ell_1, \ell_3 - \ell_2, \dots)$

and $\mu(P) = (\mu_1, \mu_2, \dots) := (m_1, m_2 - m_1, m_3 - m_2, \dots)$

Then $\lambda(P)$ and $\mu(P)$ are partitions: $\lambda_1 \geq \lambda_2 \geq \dots$, $\mu_1 \geq \mu_2 \geq \dots$ and their Young diagrams are conjugate to each other (i.e. obtained by reflection w.r.t. the main diagonal)

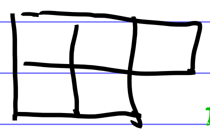
$\lambda(P) =$ 

$\mu(P) =$ 

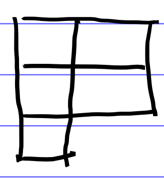
Example. $P =$ 

$(\ell_1, \ell_2, \ell_3, \dots) = (3, 5, 5, \dots)$

$(m_1, m_2, m_3, \dots) = (2, 4, 5, 5, \dots)$

$\lambda(P) = (3, 2) =$ 

Conjugate partitions

$\mu(P) = (2, 2, 1) =$ 

Remark. Dilworth's Thm: 1st column of $\lambda(P)$ = 1st row of $\mu(P)$.

Minky's Thm: 1st row of $\lambda(P)$ = 1st column of $\mu(P)$.

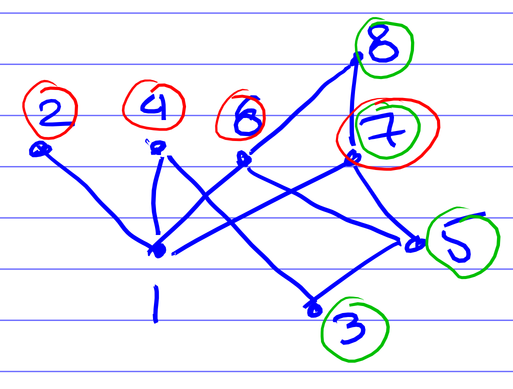
We will not prove Greene's Thm now. But we'll talk about it again later, when we will discuss the <u>Schensted correspondence</u> & RSK.

Let $w = w_1 \ldots w_n$ be a permutation. Define the poset P_w of the permutation w, as follows:

- P_w is a poset on $[n] := \{1, 2, \ldots, n\}$
- $w_i <_{P_w} w_j$ iff $w_i < w_j$ & $i < j$ (in the usual order on $\mathbb{Z}$)

Example.

$w = 3, 5, 1, 7, 6, 8, 4, 2$

$P_w =$ 

chain 3, 5, 7, 8

antichain 2, 4, 6, 7

Then

- chains in $P_w \longleftrightarrow$ increasing subsequences of w
- antichains in $P_w \longleftrightarrow$ decreasing subsequences of w

not necessarily adjacent subseq.

Questions:
- Given a permutation $w \in S_n$, find the maximal size of an increasing / decreasing subsequences in w.
- What is the "typical" size of a maximal increasing decreasing subsequences in a random perm.?
- What is the meaning of the partition $\lambda(P_w)$ from Greene's theorem?

RSK will help us to answer these questions.

Here is another related result:

Erdős - Szekeres Theorem

For $m, n \geq 1$. Any permutation of size $\geq m \cdot n + 1$ either has an increasing subsequences of size $m+1$ or a decreasing subsequence of size $n+1$.

Greene's Theorem immediately implies Erdős-Szekeres Theorem.

<u>Proof</u> (of Greene $\Rightarrow$ Erdős-Szekeres)

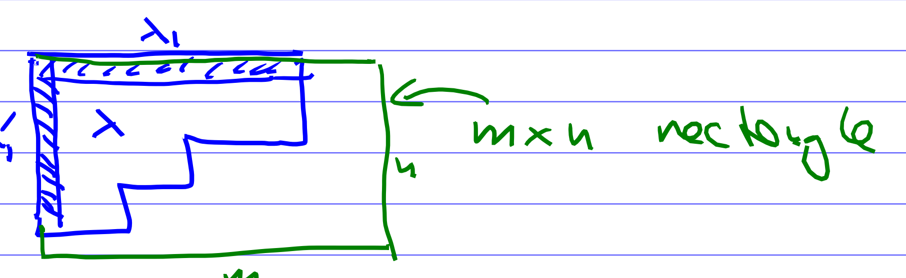
Let $\lambda = \lambda(P_w)$. By the definitions λ_1 (the size of the first row of λ) equals the max size of an increasing subsq. in w.

By Greene's Theorem, λ_1' (the size of the first column of λ) equals the max size of a decreasing subsequence in w.

Now Erdős-Szekeres Thm follows from the fact

<u>Lemma</u> Any Young diagram λ with $\geq m \cdot n + 1$ boxes either has 1^{st} row $\lambda_1 \geq m+1$ or has 1^{st} column $\lambda_1' \geq n+1$.

<u>Proof</u> (by contradiction). If $\lambda_1 \leq m$ and $\lambda_1' \leq n$, then λ fits inside the $m \times n$ rectangle and has $|\lambda| \leq m \cdot n$ boxes. $\square$



Actually, it is not very hard to prove Erdős-Szekeres Thm directly. For example, there is a nice argument based on the pigeonhole principle.

<u>Exercise.</u> Prove Erdős-Szekeres Thm without using Greene's thm.

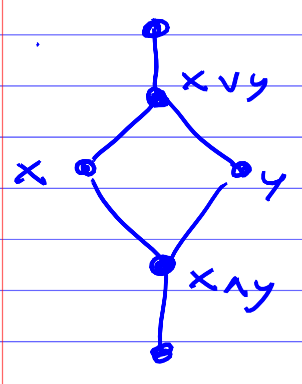
Lattices

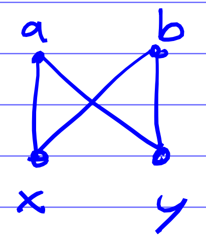
Definition. Let P be a poset, and $x, y \in P$. We say that:

- $z \in P$ is the join of x and y (denoted $z = x \vee y$) if z is the unique minimal element of P among all elements of P which are $\geq x$ and $\geq y$.
- t is the meet of x & y (denoted $t = x \wedge y$) if t is the unique maximal element of P among all elemenent which are $\leq x$ and $\leq y$.

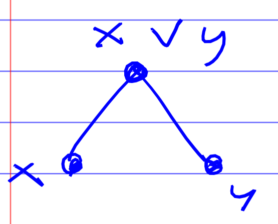
A poset P is a lattice if any two elements $x, y \in P$ have the join $x \vee y$ and the meet $x \wedge y$

Examples:





$\{u \in P \mid u \geq x$ & $u \geq y\} = \{a, b\}$. This set does not have a unique minimal element. x & y don't have a join. So this is not a lattice.

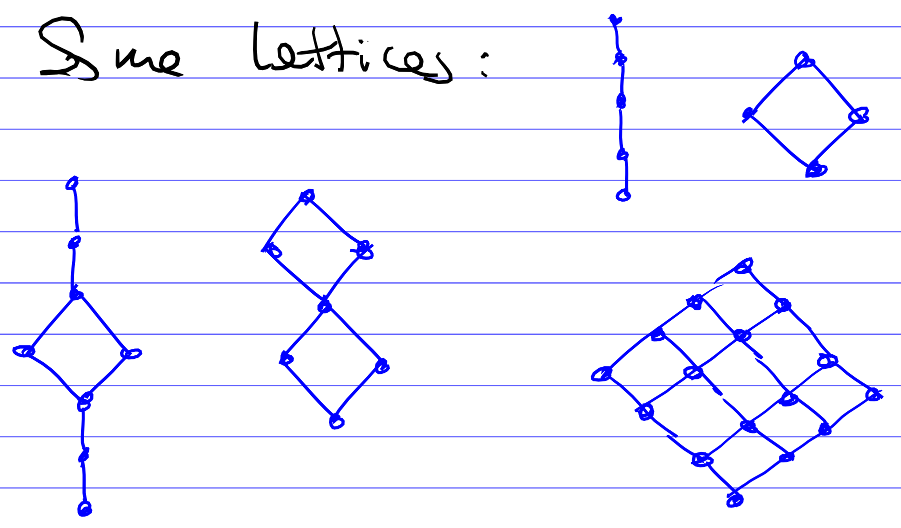


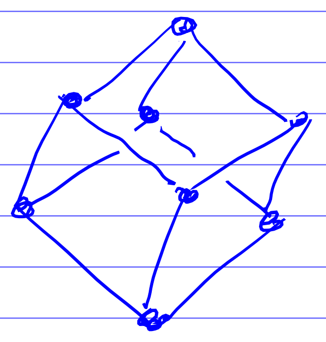
Is this a lattice? No. x & y have a join but they don't have a meet.

$$\{s \in P \mid s \leq x,\ s \leq y\} = \emptyset.$$

Remark Each finite lattice should have a unique minimal element and a unique maximal element.

Some lattices:



$B_3 =$ 

Axiomatic definition of lattices:

A set L with 2 binary operations $\vee$, $\wedge$ satisfying the axioms:

- Commutative laws:

$$x \wedge y = y \wedge x$$
$$x \vee y = x \vee y$$

- associative laws:

$$(x \wedge y) \wedge z = x \wedge (y \wedge z)$$
$$(x \vee y) \vee z = x \vee (y \vee z)$$

- absorption laws:

$$x \vee (x \wedge y) = x$$
$$x \wedge (x \vee y) = x$$

Exercise. Prove that the first definition of lattices (as a special kind of posets) is cryptomorphic (i.e. equivalent) to the above axiomatic def.

Axiomatic def $(L, \wedge, \vee) \rightsquigarrow$ the poset structure:

$$x \leq y \text{ iff } x \wedge y = x.$$

Lemma The Boolean lattice B_n is a lattice.

Proof B_n is the collection of all subsets of $\{1, 2, \dots, n\}$ ordered by containment:

$$A \leq B \iff A \subseteq B \quad (\text{as sets})$$

For any $A, B \in B_n$ we have
$$A \vee B = A \cup B \quad \text{and}$$
$$A \wedge B = A \cap B.$$

$\square$

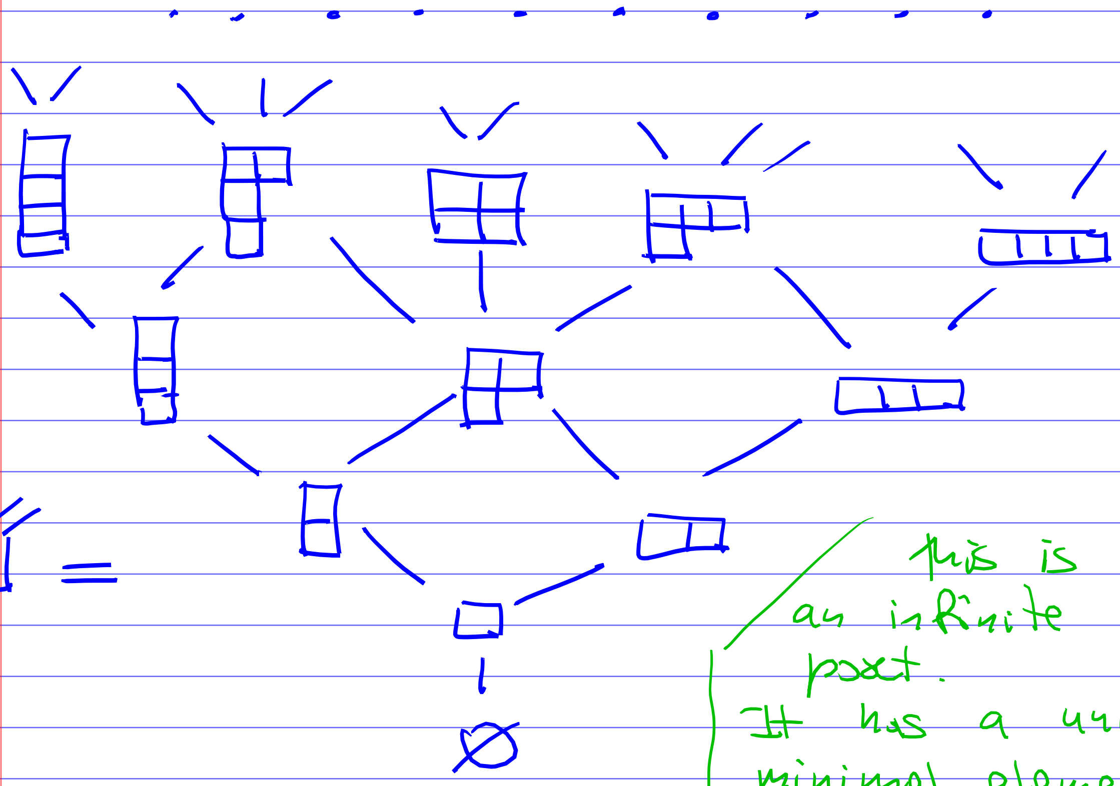
In fact, the join and meet operations can be thought of some kind of generalized union and intersection operations.

Young's lattice $\mathbb{Y}$

$\mathbb{Y}$ is the set of all Young diagram (of all sizes) ordered by containment:

$\lambda \leq \mu$ if $\lambda \subseteq \mu$ (as a collection of boxes)

$\mathbb{Y} =$



this is an infinite poset. It has a unique minimal element $\varnothing$ (the empty Young diagram), but no maximal element

Covering relations in $\mathbb{Y}$

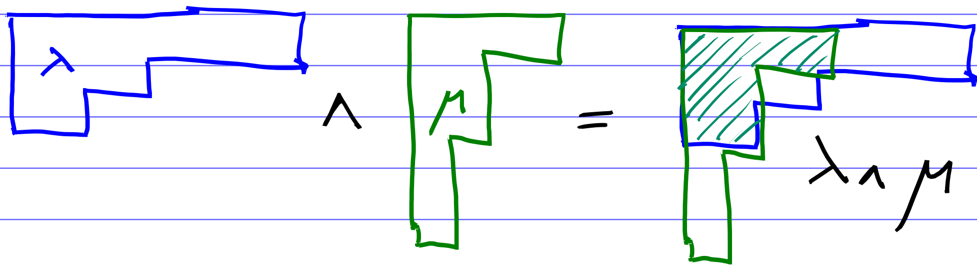
$\lambda \lessdot \mu$ if μ is obtained from λ by adding a single box.

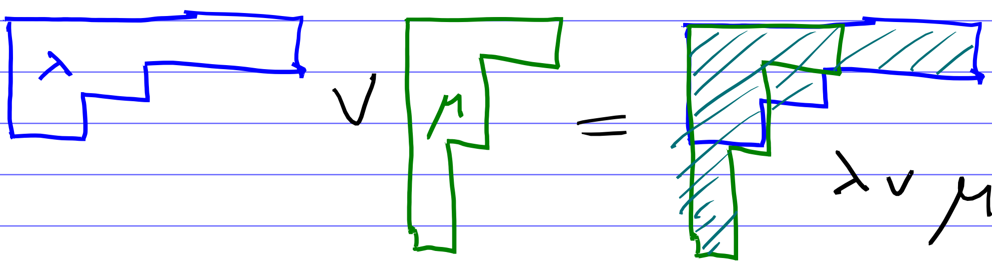
Lemma: Young's lattice $\mathbb{Y}$ is a lattice.

Proof

$$\lambda \vee \mu = \lambda \cup \mu$$
$$\lambda \wedge \mu = \lambda \cap \mu$$

the union & intersection of λ & μ, viewed as a collection of boxes





The partition lattice Π_n

Π_n is the collection of all <u>set partitions</u> ordered by <u>refinement</u>:

π & σ are set partitions

$\pi \le \sigma$ iff any block of π is contained in a block of σ.

(equiv., any block of σ is a union of some block of π).

In this case, we'll say that π <u>refines</u> σ and σ <u>coarsens</u> π.

<u>Example</u>. $n = 9$

$$\pi = (13 \mid 245 \mid 689 \mid 7)$$
$$\sigma = (137 \mid 245689)$$

$$\Pi_3 = \begin{tikzcd}[column sep=small] & (123) & \\ (12|3) \arrow[ur, dash] & (13|2) \arrow[u, dash] & (23|1) \arrow[ul, dash] \\ & (1|2|3) \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

<u>Covering relations</u> in Π_n

$\pi \lessdot \sigma$ iff σ is obtaine from π by combining <u>exactly</u> 2 blocks of π.

<u>Proposition</u>. Π_n is a lattice.

<u>Proof</u>. Let $\pi, \sigma \in \Pi_n$.

The meet $\pi \wedge \sigma$ is easy to construct: blocks of $\pi \wedge \sigma$ should by all possible intersections of blocks of π & σ.

<u>Example</u>

$$\pi = (1357 \mid 2468)$$
$$\sigma = (12 \mid 345 \mid 678)$$
$$\sigma \wedge \pi = (1 \mid 2 \mid 35 \mid 4 \mid 68 \mid 7)$$

<u>Exercise</u> Describe the join $\pi \vee \sigma$ of two set partitions, and finish the proof...

The lattice of order ideals

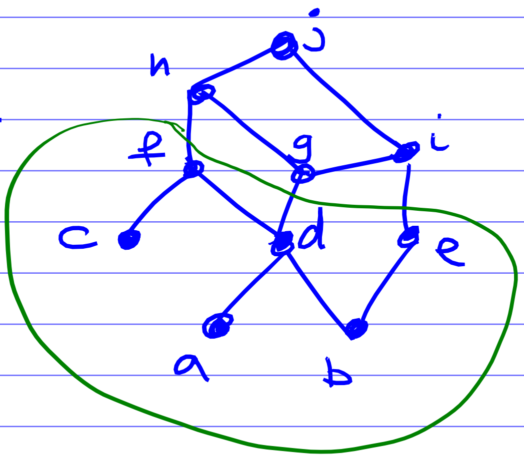
Let P be any poset (not necessarily a lattice).
An <u>order ideal</u> in P is a subset $I \subseteq P$ of elemets of P such that

$$x \in I,\ y \leq_P x \Rightarrow y \in I.$$

Example

$P =$

$I =$



$I = \{a, b, c, d, e, f\}$ is an order ideal

Let $J(P)$ be the poset of all order ideals in P ordered by containment:

$$I \leq J \text{ iff } I \subseteq J$$
(as sets)

Lemma $J(P)$ is a lattice.

Proof:

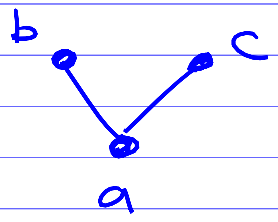
$$I \wedge J = I \cap J$$
$$I \vee J = I \cup J$$

intersection/union of sets

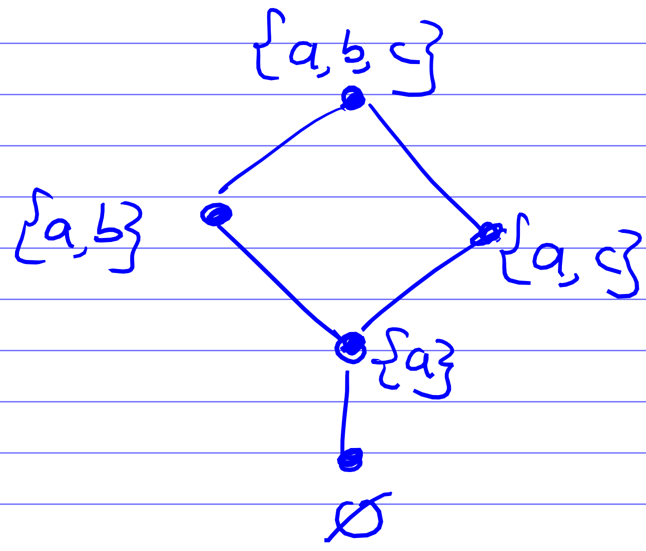
It is easy to see that the set theoretic intersection/union of two order ideals is an order ideal. $\square$

Example.

$P =$



$J(P) =$



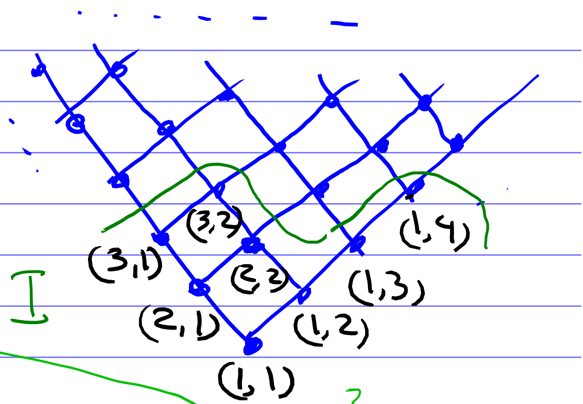
Example. Let $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ be the poset of positive integers with the usual order $1 < 2 < 3 < \cdots$ (an infinite chain)

Then $J(\mathbb{Z}_{>0}) \simeq \mathbb{Z}_{>0}$

$$J(\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}) \simeq \mathbb{Y}$$

Young's lattice

$\mathbb{Z}_{>0} \times \mathbb{Z}_{>0} =$



Identify an element $(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ with a box in position (i,j). Then an order ideal in $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ corresponds to a collection of boxes in a Young diagram.

1,1	1,2	1,3	1,4
2,1	2,2		
3,1	3,2		

Example. $B_n = J$ (the "empty" poset on n elements)

the Boolean lattice

the poset where any two elts are incomparable

$$B_n \simeq J\left(\underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \underset{3}{\bullet} \quad \cdots \quad \underset{n}{\bullet} \right)$$

Distributive Lattices

Actually, $J(P)$ is not only just a lattice, but also it belongs to an especially nice class of lattices.

<u>Definition.</u> A lattice L is called a <u>distributive</u> lattice if it satisfies the <u>distributive laws</u>:

- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y).$

<u>Remark.</u> For numbers (say, in $\mathbb{R}$) and the usual operations of addition "$+$" and multiplication "$\cdot$", we have the distributive law:

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z),$$

but we don't have the law:

$$x + (y \cdot z) = (x + y) \cdot (x + z)$$

this is usually not correct

So $(\mathbb{R}, +, \cdot)$ is <u>not</u> a distributive lattice, but it shares some similarities with distributive lattices.

Lemma. $J(P)$ is a distributive lattice.

Proof. It is easy to check that the both distributive laws hold for the unions & intersection of sets. $\square$

Fundemental Theorem on Finite Distributive Lattices (a.k.a **Birkhoff's Representation Theorem**), 1937.

For a finite poset P, $J(P)$ is a finite distributive lattice, and any finite distributive lattice L is isomorphic to $J(P)$ for some finite poset P.

Remark. Basically, this theorem says:

$$P \longleftrightarrow J(P)$$

is a "one-to-one correspondence" between finite posets and finite distributive lattices.

We need to be a little careful here: In order to talk about 1-1 correspondences we need to have some sets. But there is no such thing as the "set of all finite posets".

Strictly speaking, the above statement does not make sense mathematically.

In order to rigorously formulate it, we need to talk about the categories of finite posets & finite distributive lattices. But I want to avoid talking about category theory in this class.

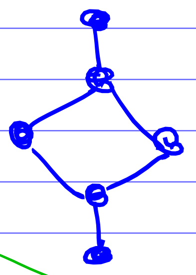
Idea of proof. For any finite distributive lattice L, we need to find a finite poset P such that $L \cong J(P)$.

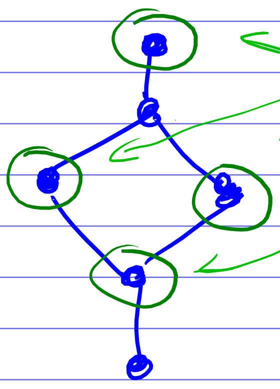
An element z of L is called *join-irreducible* if z is not a minimal element of L, and we cannot write it as $z = x \vee y$ for some $x, y \lneq z$.

Let P be the poset of all join-indecomposable elements.

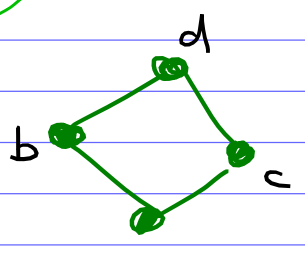
Then one can deduce, using the axioms of distributive lattices, that $L \cong J(P)$.

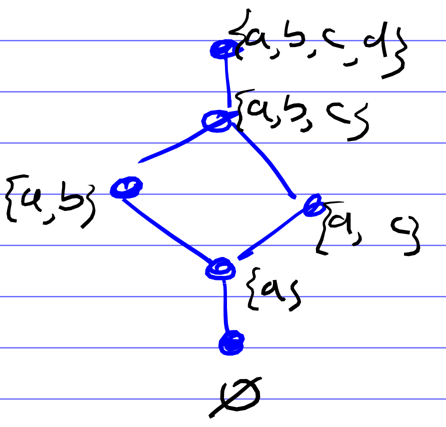
Exercise. Prove this.

Example $L =$ 



join-irreducible elements of L

$\rightsquigarrow P =$ 

$J(P) =$ 

$\cong L.$

this is the lattice of all Young diagrams that fit inside the 2×2 square