Sperner's Theorem 1928

Let $S_1, \dots, S_N$ be different subsets of $[n] := \{1, 2, \dots, n\}$ such that $\forall\, i \neq j$

$$S_i \not\subseteq S_j \quad \text{and} \quad S_j \not\subseteq S_i$$

In other words, all $S_1, \dots, S_N$ are pairwise incomparable with each other.

Then $N \le \binom{n}{\lfloor n/2 \rfloor}$.

One can easily see that the bound is sharp. Indeed, let us take all $\lfloor \frac{n}{2} \rfloor$-element subsets of $[n]$. They cannot contain each other, because they have the same cardinality. There are exactly $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ such subsets.

But how to prove that we cannot find a larger family of pairwise incomparable subsets?

Let us first generalize the setup of this theorem.

Posets

Definition. A poset P (abbreviation for "partially ordered set") is a set together with a binary relation "$\leq$" satisfying the axioms:

- reflexivity: $a \leq a \quad \forall\, a \in P$.
- symmetry: For $a, b \in P$, if $a \leq b$ and $b \leq a$, then $a = b$.
- transitivity: For $a, b, c \in P$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

The notation "$a < b$" means that $a \leq b$ and $a \neq b$.

"$a \geq b$" means that $b \leq a$

$a > b$ means that $b < a$.

Covering relations in P:

$a \lessdot b$ ("b covers a" or "a is covered by b") means
- $a < b$, and
- $\nexists\, c \in P$ such that $a < c < b$.

Two elements $a, b \in P$ are called incomparable if

$a \not\leq b$ and $b \not\leq a$.

Remark. If we want to distinguish the order relation in P from the usual order relation $\leq$ of numbers, we will use the notations "$\leq_P$", "$<_P$", "$\geq_P$", "$>_P$", etc for the order relation in P.

Examples: ① $P = [n] := \{1, \dots, n\}$ with the usual order relation $a \leq_P b$ iff $a \leq b$ (as numbers)

② $P = \mathbb{R}$ also with the usual order relation $a \leq_P b$ iff $a \leq b$ in $\mathbb{R}$.

Such posets are called totally ordered. They don't have incomperable elements.

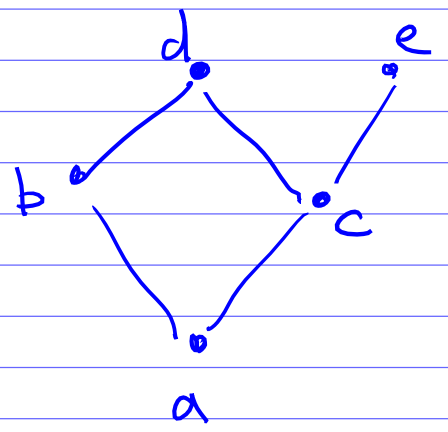
Definition. The Hasse diagram of a poset P is the directed graph on the set of elements of P with directed edges $a \to b$ for all covering relations $a \lessdot b$.

(Typically, Hasse diagrams are drawn on the plane without arrows on the edges but with b above a for any covering relation $a \lessdot b$.)

It is not hard to see that any finite poset is uniquely determined by its Hasse diagram.

But this is not true for (some) infinite posets. For example $(\mathbb{R}, \leq)$ has no covering relations, so its Hasse diagram has no edges.

Example



the Hasse diagram of the poset on the set $\{a,b,c,d,e\}$ such that $a \lessdot b$, $a \lessdot c$, $b \lessdot d$, $c \lessdot d$, $c \lessdot e$.

By transitivity, we also have $a < d$ and $a < c$.

We have the following pairs of incomparable elements: b and c, b and e, d and e.

Definitions. A chain in P is a sequence of elements $a_1 < a_2 < \cdots < a_k$ in P.

A saturated chain is a sequence $a_1 \lessdot a_2 \lessdot \cdots \lessdot a_k$

An antichain is a subset A of elements in P such that any two elements of A are incomparable.

For example, the poset shown above has (saturated) chain $a \lessdot b \lessdot d$ and an antichain $\{b, c\}$.

Some questions that we can ask about a poset P:

- What is the maximal possible size of a chain in P?
- What is the maximal possible size of an antichain in P?

For the above poset, the answers are 3 and 2.

Let $2^{[n]}$ denotes the set of all subsets of $[n]$.

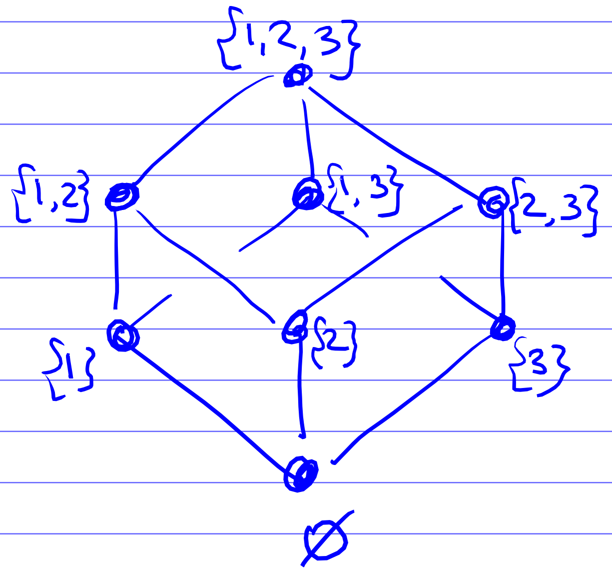
For example, $2^{[3]}$ has 8 elements: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$.

<u>The Boolean lattice</u> B_n is the poset on the set $2^{[n]}$ ordered by inclusion:

For $A, B \subseteq [n]$,

$$A \leq B \iff A \subseteq B.$$

<u>Example</u>: The Hasse diagram of B_3:



The Hasse diagram of B_n is the 1-skeleton of an <u>n-cube</u>

Why is it called the Boolean <u>lattice</u> and not the Boolean <u>poset</u>?

Lattices are a special class of posets, which we will discuss later. We'll see that the poset B_n is actually a lattice.

Clearly, the maximal size of a chain in B_n is $n+1$. For example, we have the chain $\emptyset \lessdot \{1\} \lessdot \{1,2\} \lessdot \dots \lessdot \{1,2,\dots,n\}$

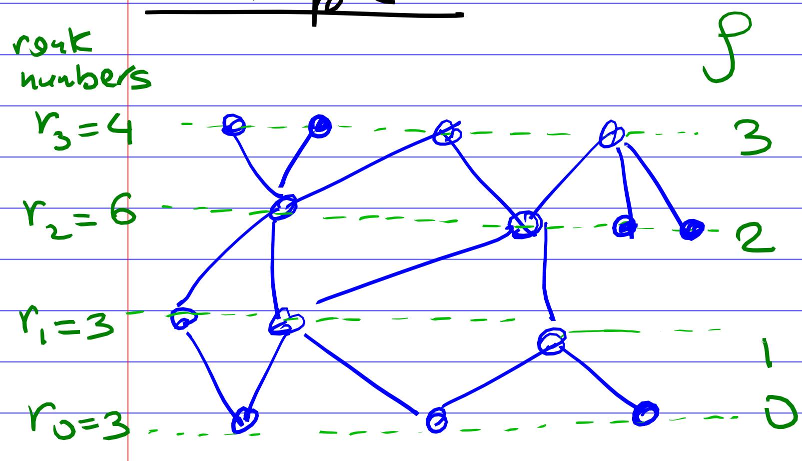
Sperner's thm concerns the maximal size of an antichain in B_3. It says that this size is $\binom{n}{\lfloor n/2 \rfloor}$.

Definition. A poset P is called ranked if there exists a function $\rho : P \to \mathbb{Z}$ (called rank function) such that, for any covering relation $a \lessdot b$, we have

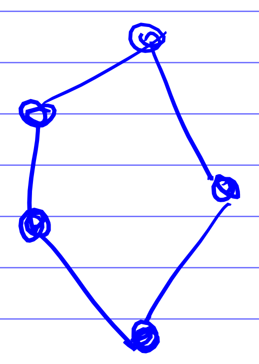
$$\rho(b) = \rho(a) + 1.$$

For finite posets, we'll assume that the minimal value of $\rho(a)$ is 0. This makes the choice of ρ unique.

Example



(the Hasse diagram of) a ranked poset



this is not a ranked poset

For a finite ranked poset P (with the rank function ρ such that $\min_{a \in P} \rho(a) = 0$) the rank numbers are $r_k := \#\{a \in P \mid \rho(a) = k\}$

Def A finite ranked poset P with rank numbers $r_0, r_1, \dots, r_\ell$ is called:

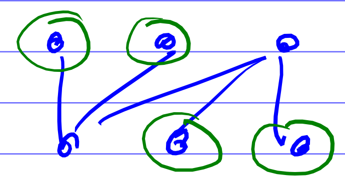
- rank symmetric if
  $r_i = r_{\ell - i}$ for $i = 0, 1, \dots, \ell$
- unimodal if
  $r_0 \le r_1 \le r_2 \le \dots \le r_k \ge r_{k+1} \ge \dots \ge r_\ell$
- Sperner if the maximal size of an antichain in P equals $\max(r_0, r_1, \dots, r_\ell)$.

Remark Clearly, for $i = 0, \dots, \ell$ $\{a \in P \mid \rho(a) = i\}$ is an antichain.
So any finite ranked poset has an antichain of size $\max(r_0, r_1, \dots, r_\ell)$.

Is the above poset (shown on the left) Sperner?

No. $\max(r_i) = 4$, but the poset has an antichain with 5 elements.

Here is a smaller example of a non-Sperner poset:



$r_0 = r_1 = 3$
but there is an antichain with 4 elements

The Boolean lattice B_n is ranked: $\rho(A) = |A|$ for $A \subseteq [n]$.

Its rank numbers are

$$r_i = r_i(B_n) = \binom{n}{i}$$

So B_n is

- rank-symmetric: $\binom{n}{i} = \binom{n}{n-i}$
- unimodal:

$$\binom{n}{0} \le \binom{n}{1} \le \dots \le \binom{n}{n/2} \ge \dots \ge \binom{n}{n}$$

(Can you find an <u>injective</u> map from $\{A \in 2^{[n]} \mid |A| = i\}$ to

$$\{B \in 2^{[n]} \mid |B| = i+1\},$$

for $i < n/2$?)

<u>Sperner's Theorem</u>.

The Boolean lattice is Sperner.

How do all these general definitions (posets, etc) help us to prove this theorem?

Crucial Definition

Let P be a finite ranked poset with rank function $\rho : P \to \mathbb{Z}$ s.t. $\min_{a \in P} \rho(a) = 0$ and $\max_{b \in P} \rho(b) = \ell$.

A symmetric chain decomposition (SCD) of P is a decomposition of the set of elements of P into a disjoint union of chains

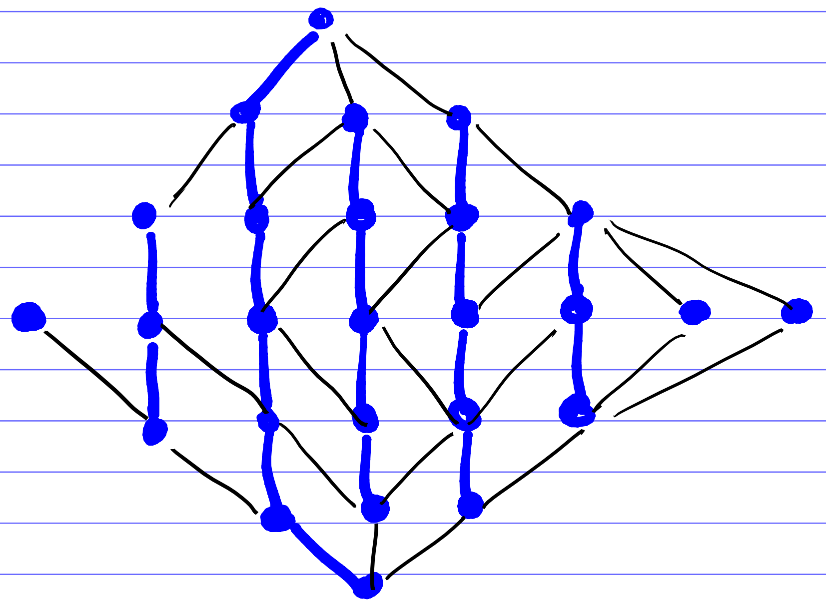
$$P = C_1 \cup C_2 \cup \dots \cup C_k$$

such that each C_i is a saturated chain

$$C_i = a_0 \lessdot a_1 \lessdot \dots \lessdot a_s \quad \text{and}$$

$$\rho(a_s) = \ell - \rho(a_0).$$

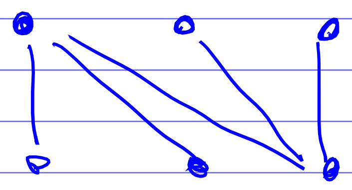
Example.



an SCD of a poset

Not any poset has an SCD.

For example,

this poset $\longrightarrow$ 

is rank-symmetric & unimodal. But it has no SCD.

Lemma. If P has an SCD. Then it is rank-symmetric, unimodal, and Sperner.

Proof. The first 2 properties (rank-symmetricity & unimodality) are trivial, because they hold for each chain C_i in the SCD.

Let's prove that the Sperner property also holds.

For any decomposition of P into a union of chains
$$P = C_1 \cup C_2 \cup \cdots \cup C_k$$
and any antichain A, we have $|C_i \cap A| \leq 1$.

Thus $|A| \leq k$ (# chains).

If $P = C_1 \cup \cdots \cup C_k$ is an SCD, then each chain C_i intersects with the middle level $\{a \in P \mid \rho(a) = \lfloor \ell/2 \rfloor\}$ at exactly one element.

So k = # chains equals the middle rank number $r_{\lfloor \frac{\ell}{2} \rfloor}$.

Thus $|A| \leq r_{\lfloor \frac{\ell}{2} \rfloor} = \max(r_0, r_1, \ldots, r_\ell)$

because P is rank-symmetric & unimodal.

$\square$

It is now enough to show that B_n has an SCD.

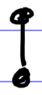
Def. The *product* $P \times Q$ of two posets P & Q is the poset on the set of pairs (a,b) $a \in P$, $b \in Q$ such that

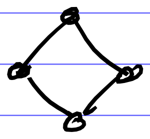
$$(a,b) \leq_{P\times Q} (a',b') \text{ iff}$$

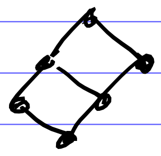
$$a \leq_P a' \text{ and } b \leq_Q b'.$$

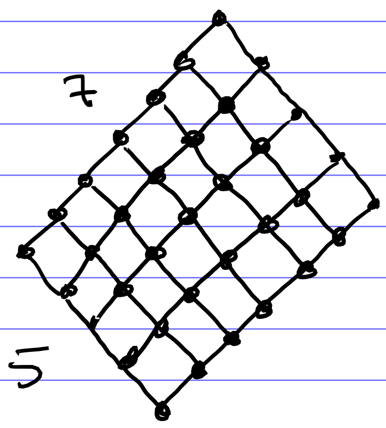
Let $[n]$ denote the poset on $\{1,2,\dots,n\}$ with the total order $1 < 2 < 3 < \dots < n$ (called the n-chain)

Examples

$[2] =$  , $[3] =$  , etc

$[2] \times [2] =$ 

$[2] \times [3] =$ 

$[5] \times [7] =$ 

Similarly, we define products of three or more posets

$$P \times Q \times R := (P \times Q) \times R,$$

etc.

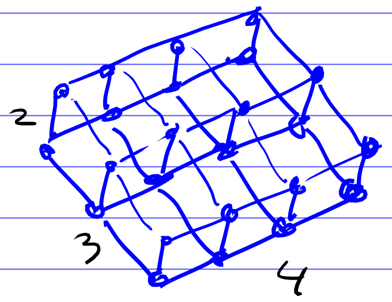
Then $B_n = [2]^n := \underbrace{[2] \times [2] \times \ldots \times [2]}_{n}$

the poset whose Hasse diagram is the 1-skeleton of an n-cube

More generally, $\underbrace{[a] \times [b] \times \ldots \times [c]}_{n}$

is the poset whose Hasse diagram is the $a \times b \times \ldots \times d$ box in the n-dimensional grid.

Example

$[2] \times [3] \times [4] =$ 

Theorem (de Bruijn 1948)

$B_n = [2]^n$ has an SCD.

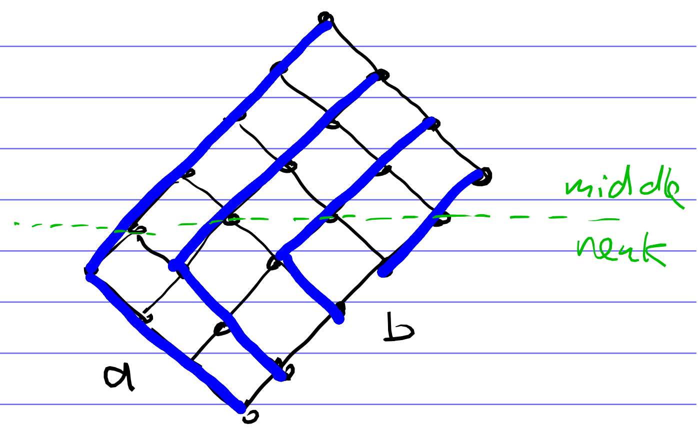
More generally,

Theorem. Any product of chains $[a] \times [b] \times \ldots \times [c]$ has an SCD. In particular, this product is Sperner.

Actually, it is easier to prove this more general theorem.

Lemma $[a]\times[b]$ has an SCD.

Proof. Here is an SCD:

$[a]\times[b] =$ 

a symmetric chain decomposition of the product of 2 chains

Let us now prove the general result.

Proof (The product of any number of chains has an SCD.)

Induction by the number n of chains.

Base: $n=1$ ✓
(Also the case $n=2$ is provided by the previous lemma.)

Induction step: Let P' be a product on $n-1$ chains and

$P = P' \times [c]$

By induction, P' has an SCD: $P' = C_1 \cup C_2 \cup \cdots \cup C_k$

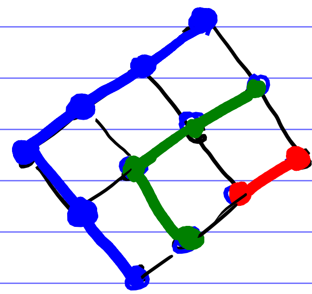
(Each C_i is a chain centered around the mid-level of P'.)

Then $C_i \times [c]$ is the product of two chains. By Lemma, is has an SCD.

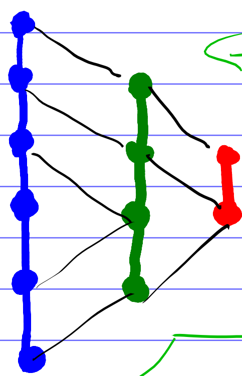
Taking SCD's in all $C_i\times[c]$ we get an SCD in $P = P'\times[c]$. $\square$

Example $[4]\times[3]\times[3]$

$[4]\times[3] =$



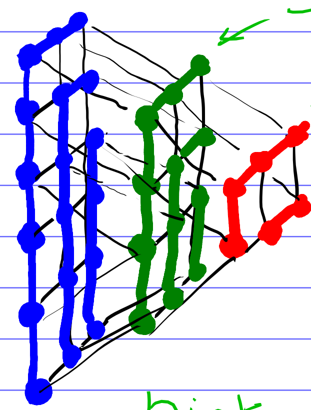
$=$



redraw the Hasse diagr. so that the chains are straight

3 "layers"

$[4]\times[3]\times[3] =$



pick an SCD in each "layer"

Other results about chains and antichains.

Let P be <u>any</u> finite poset

Define

$MA(P)$:= the maximal number of elements in an antichain in P.

$mc(P)$:= the minimal number of chains needed to cover all elements of P

Theorem (Dilworth 1950)

$$MA(P) = mc(P).$$

Here is a dual version:

$MC(P)$:= the maximal number of elements in a ~~anti~~chain in P.

$ma(P)$:= the minimal number of antichains needed to cover all elements of P

Theorem (Minsky, 1971)

$$MC(P) = ma(P).$$

Both Dilworth's and Minsky's theorems are special cases of

Greene's Theorem

For a finite poset P, let $\ell_k := $ the maximal size of a union of k chains;

$m_k :=$ the maximal size of a union of k antichain.

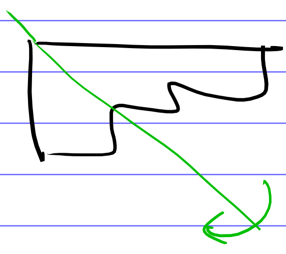
In particular, $\ell_1 = MC(P)$ and $m_1 = MA(P)$.

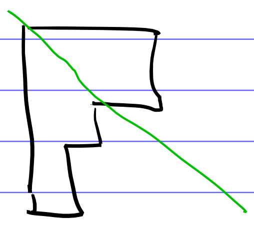
Theorem (Greene, 1976)

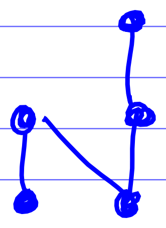
Let $\lambda(P) = (\lambda_1, \lambda_2, \dots) := (\ell_1, \ell_2 - \ell_1, \ell_3 - \ell_2, \dots)$

and $\mu(P) = (\mu_1, \mu_2, \dots) := (m_1, m_2 - m_1, m_3 - m_2, \dots)$

Then $\lambda(P)$ and $\mu(P)$ are partitions: $\lambda_1 \geq \lambda_2 \geq \dots$, $\mu_1 \geq \mu_2 \geq \dots$ and their Young diagrams are conjugate to each other (i.e. obtained by reflection w.r.t. the main diagonal)

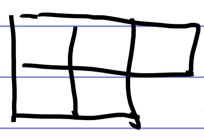
$\lambda(P) =$ 

$\mu(P) =$ 

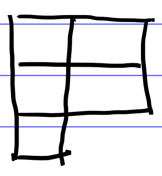
Example. $P =$ 

$(\ell_1, \ell_2, \ell_3, \dots) = (3, 5, 5, \dots)$

$(m_1, m_2, m_3, \dots) = (2, 4, 5, 5, \dots)$

$\lambda(P) = (3, 2) =$ 

conjugate partition

$\mu(P) = (2, 2, 1) =$ 

Remark. Dilworth's Thm: 1st column of $\lambda(P)$ = 1st row of $\mu(P)$.

Minsky's Thm: 1st row of $\lambda(P)$ = 1st column of $\mu(P)$.