18.212 **Lecture 4**, Feb. 24, 2021

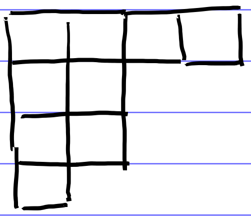
partitions: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a *partition* of n (denoted $\lambda \vdash n$) if

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ integers

$\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$.

Graphically, partitions are represented by *Young diagrams*

Example: $\lambda = (4, 2, 2, 1) \vdash 9$

Young diagram: $\lambda =$ 

4
2
2
1

Def. A *standard Young tableau* (SYT) of *shape* λ is a filling of boxed of the Young diagram λ by $1, 2, \ldots, n$ (without repeated entries) s.t the entries increase in rows & columns of λ.

Example:

1	2	4	7
3	5		
6	9		
8			

an SYT of shape $\lambda = (4, 2, 2, 1)$

Let $f_\lambda :=$ # SYT's of shape λ.

<u>Remark</u>. The numbers f_λ of SYT's is important in representation theory. For example, these numbers give dimensions of irreducible represent. of the symmetric group.

<u>Proposition</u>. $f_{(n,n)} = C_n$ (the Catalan number).

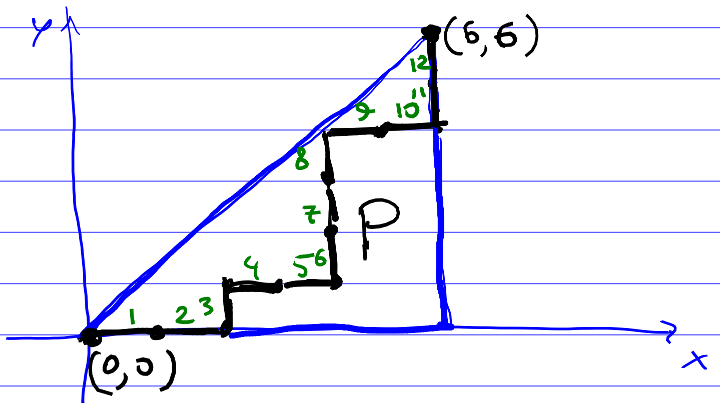
bijection : $\left\{\begin{array}{c}\text{SYT's} \\ \text{of shape} \\ (n,n)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Dyck} \\ \text{paths with} \\ 2n \text{ steps}\end{array}\right\}$

$$\begin{array}{ccc} \cup\!\!\!\mid & & \cup\!\!\!\mid \\ T & \longmapsto & P \end{array}$$

P is the lattice path from $(0,0)$ to (n,n) s.t. if $i \in 1^{st}$ row of T, then the i^{th} step of P is $(1,0)$ & if $j \in 2^{nd}$ row of T, then the j^{th} step of P is $(0,1)$.
(P is a Dyck path drawn in a different orientation.)

<u>Example</u>. $T =$

1	2	4	5	9	10
3	6	7	8	11	12



So SYT's can be thought of as "generalized Dyck paths" and the numbers f_λ are the "generalized Catalan numbers".

Example. SYT's of shape (n,n,n) correspond to "3-dimensional Dyck paths".

T: an SYT of shape (n,n,n)

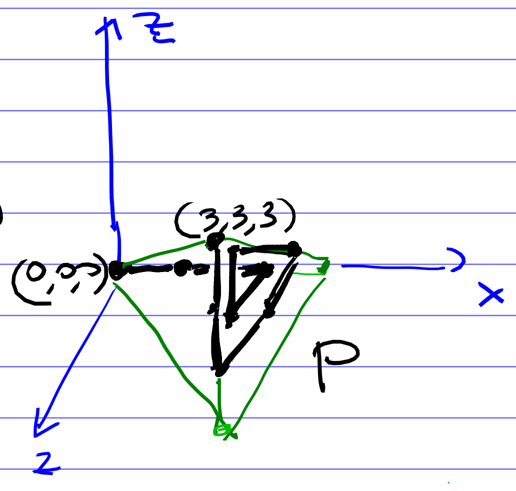
$\rightsquigarrow$

P: lattice path in $\mathbb{R}^3$ from $(0,0,0)$ to (n,n,n) given by:

If $i \in 1^{st} / 2^{nd} / 3^{rd}$ row of T, then the i^{th} step of P is $\vec{e}_1=(1,0,0)$, $\vec{e}_2=(0,1,0)$, $\vec{e}_3=(0,0,1)$, respectively.

Example:

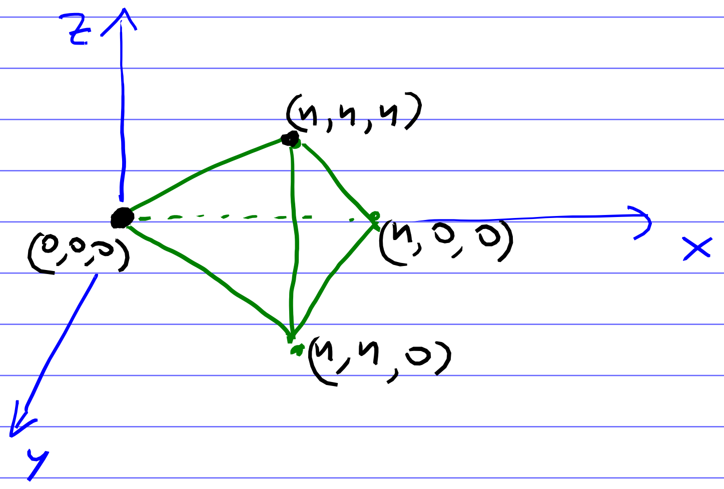
$$T = \begin{array}{|ccc|} 1 & 2 & 5 \\ 3 & 6 & 7 \\ 4 & 8 & 9 \end{array} \rightsquigarrow$$



P is a lattice path in $\mathbb{R}^3$ s.t.

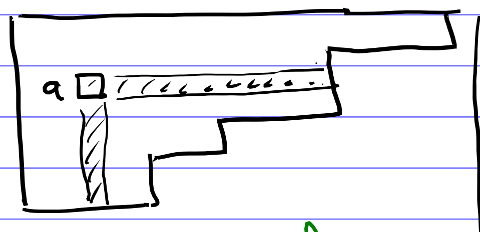
- P is from $(0,0,0)$ to (n,n,n)
- all steps of P are given by the coord. vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$
- P belongs to the tetrahedron

$$\{(x,y,z) \in \mathbb{R}^3 \mid n \geq x \geq y \geq z \geq 0\}$$



Q. Is there an explicit formula for f_λ?

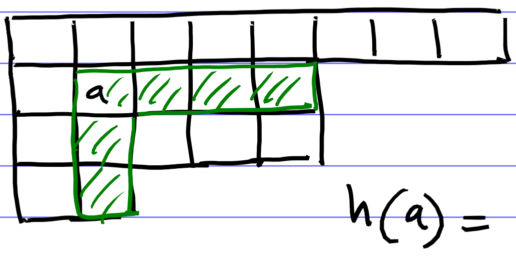
The Hook Length Formula

$\lambda =$ 

For a box a in a Young diagr. λ, the <u>hook</u> of a is the collection of all boxes of λ to the right of a & below of a, including the box a.

The <u>hook length</u> $h(a)$ is the number of boxes in the hook of a.

<u>Example</u>.



$h(a) = 6$

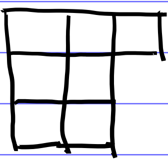
Let $H(\lambda) := \prod_{a \in \lambda} h(a)$

the product over all boxes a of the Young diagr. λ.

<u>Theorem</u> (hook length formula) (Frame-Robinson-Threll '1953).

For any partition $\lambda \vdash n$.

$$\boxed{f_\lambda = \frac{n!}{H(\lambda)}}$$

<u>Example</u>. $n=7$, $\lambda = (3,2,2) =$ 

hook lengths $h(a)$ of λ:

5	4	1
3	2	
2	1	

So $f_{(3,2,2)} = \dfrac{7!}{5\cdot 4\cdot 3\cdot 2\cdot 2\cdot 1\cdot 1}$

$= 7\cdot 3 = 21$

There are exactly 21 SYT's of shape $(3,2,2)$.

hook lengths

$n+1$	$\cdots$	$\cdots$	3	2
n	$\cdots$	$\cdots$	2	1

Examples:

$$f_{(n,n)} = \frac{(2n)!}{n!\,(n+1)!} = C_n$$

$$f_{(n,n,n)} = 2\,\frac{(3n)!}{n!\,(n+1)!\,(n+2)!}$$

"3-dim Dyck paths"

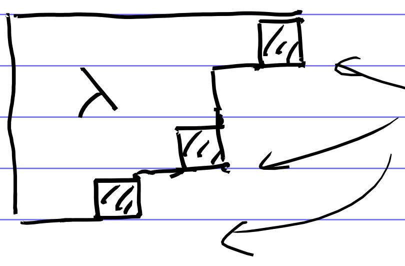
There are several interesting proofs of the hook length formula.

We'll present the probabilistic proof, or the hook walk proof due to Green, Nijenhuis, Wilf, 1979.

f_λ's satisfy the following recurrence relation:

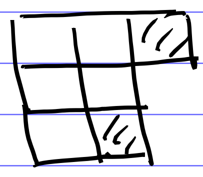
Lemma. $\displaystyle f_\lambda = \sum_{c \text{ is a corner of } \lambda} f_{\lambda - c}$

$f_\varnothing = 1$ (the initial condition)



corner boxes c of Young diagram λ

$\varnothing :=$ the empty partition (or the empty Young diagram)

Example $\lambda =$  $= (3,2,2)$

has 2 corners

$$f_{(3,2,2)} = f_{(2,2,2)} + f_{(3,2,1)}$$

Proof of Lemma

In an SYT of shape $\lambda \vdash n$, the entry "n" is located in a corner box c. If we remove this entry "n" from the tableau, we obtain an SYT of shape $\lambda - c$.

So we get

$$f_\lambda = \sum_{\substack{c \text{ corner} \\ \text{of } \lambda}} f_{\lambda - c}. \qquad \square$$

In order to prove the hook length formula, it is enough to check that the same recurrence relation holds for the RHS $= \frac{n!}{H(\lambda)}$.

The initial condition is easy to check

$$f_\varnothing = 1, \qquad \frac{0!}{H(\varnothing)} = 1$$

the empty product $= 1$.

Then the hook length formula would follow by induction on n:

$$f_\lambda = \sum_{\substack{c \\ \text{corner} \\ \text{of } \lambda}} f_{\lambda - c} = \sum_{c} \frac{(n-1)!}{H(\lambda - c)} = \frac{n!}{H(\lambda)}.$$

by the induction hypothesis

we need to prove this rec. relation for the R.H.S of the hook length formula

We need to prove

<u>Proposition</u>

For any partition λ,

$$\sum_{\substack{c \text{ corner} \\ \text{of } \lambda}} \frac{(n-1)!}{H(\lambda - c)} = \frac{n!}{H(\lambda)}.$$

Equivalently,

$$\sum_{\substack{c \text{ corner} \\ \text{of } \lambda}} \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)} = 1.$$

there are a lot of cancellation in these ratios.

Can these expressions by probabilities of something?

We'll use some simple basic notions from the theory of probability.

<u>Goal:</u> For any λ, construct a certain random process that outputs one corner c of the Young diagram, so that the probability to get a given corner c equals $\frac{1}{n}\frac{H(\lambda)}{H(\lambda-c)}$.

Then we would deduce that

$$\sum_{\substack{c \text{ corner} \\ \text{of } \lambda}} \frac{1}{n}\frac{H(\lambda)}{H(\lambda-c)} = 1,$$

because the sum of probabilities of all possible outcomes should be 1.

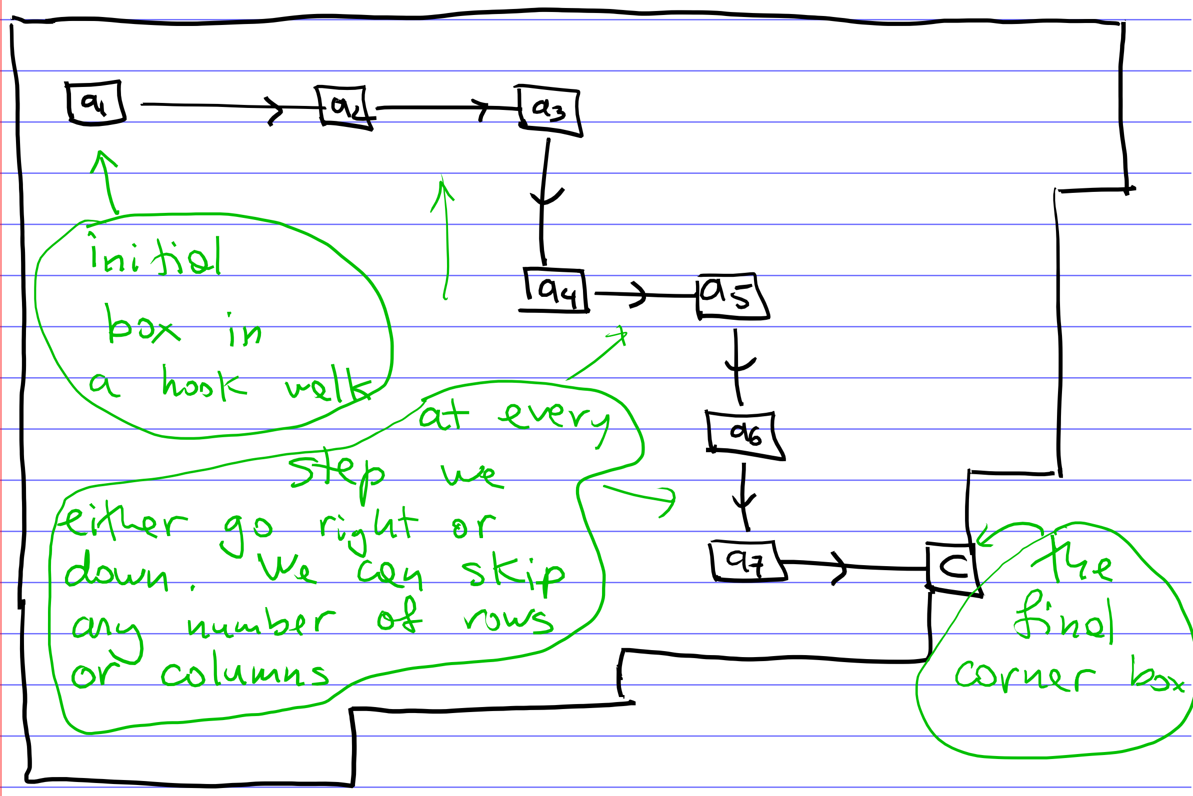
Here is the needed random process:

Hook Walk

Fix a Young diagram λ.

1. Randomly pick any box $a = a_1$ of λ with uniform probability. (The probability of any box of $\lambda = \frac{1}{n}$.)
2. Randomly jump to any other box $a_2 \neq a_1$ that belongs to the hook of a_1 with uniform probability. (The probability to pick any such a_2 is $\frac{1}{h(a_1)-1}$.)
3. Repeat the step 2 to randomly jump to boxes $a_3, a_4, \ldots$

   Stop when we arrive to a corner box $a_k = c$.
4. Output the corner c.

$\lambda =$ 

a "hook walk" on boxes of Young diagram starting at box a and ending at corner c.

Let $P(a,c)$ be the probability that a hook walk starting at box a ends at corner c.

We have

$$P(a,c) = \sum_{\substack{\text{hook walks} \\ a_1 = a \to a_2 \to \dots \to a_k = c}}' \frac{1}{h(a_1)-1} \cdot \frac{1}{h(a_2)-1} \cdots \frac{1}{h(a_{k-1})-1}.$$

Let $P(c)$ be the probability that a hook walk (starting at any box a) ends at corner c.

$$P(c) = \sum_{a \in \lambda} \frac{1}{n} P(a,c).$$

the probability to pick an initial box c

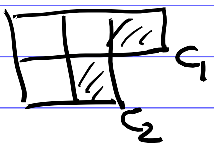
Basic fact from probability theory

$$\sum_c P(c) = 1$$

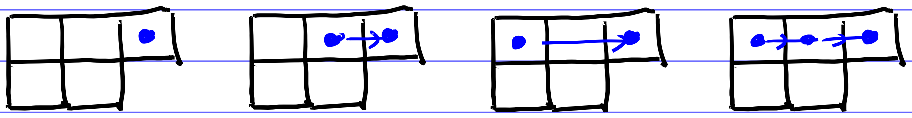
Out goal is to show that

<u>Proposition</u>. $P(c) = \dfrac{1}{n} \dfrac{H(\lambda)}{H(\lambda - c)}$.

Example

$\lambda = (3,2) =$ 

All hook walks ending at corner c_1:

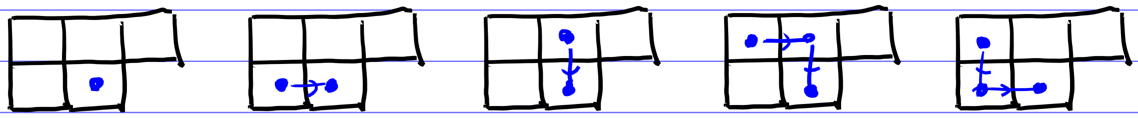


their prob. $\quad \frac{1}{5} \qquad \frac{1}{5}\cdot\frac{1}{2} \qquad \frac{1}{5}\cdot\frac{1}{3} \qquad \frac{1}{5}\cdot\frac{1}{3}\cdot\frac{1}{2}$

So $P(c_1) = \frac{1}{5} + \frac{1}{5}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{3} + \frac{1}{5}\frac{1}{3}\cdot\frac{1}{2}$

$$= \boxed{\frac{2}{5}}$$

All hook walks ending at corner c_2:



prob. $\quad \frac{1}{5} \qquad \frac{1}{5}\cdot\frac{1}{1} \qquad \frac{1}{5}\cdot\frac{1}{2} \qquad \frac{1}{5}\cdot\frac{1}{3}\cdot\frac{1}{2} \qquad \frac{1}{5}\frac{1}{3}\cdot\frac{1}{1}$

$P(c_2) = \frac{1}{5} + \frac{1}{5}\cdot\frac{1}{1} + \frac{1}{5}\cdot\frac{1}{2} + \frac{1}{5}\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{5}\frac{1}{3}\cdot\frac{1}{1}$

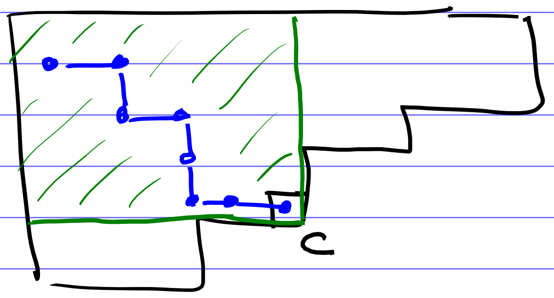
$$= \boxed{\frac{3}{5}}$$

We, indeed, get

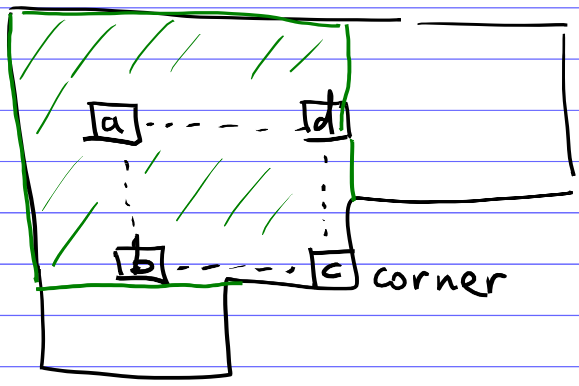
$P(c_1) + P(c_2) = \frac{2}{5} + \frac{3}{5} = 1.$

Observations:

1. If we fix a corner c of λ, then all possible hook walks ending at c happen inside the rectangle:



2.



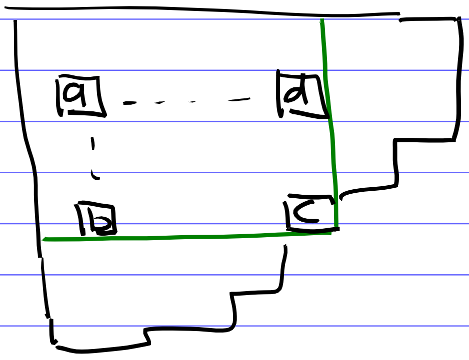
$$h(a) + \overset{1}{\overset{\shortparallel}{h(c)}} = h(b) + h(d)$$

$$(h(a)-1) + \overset{0}{\overset{\shortparallel}{(h(c)-1)}} = (h(b)-1) + (h(d)-1)$$

For any box a of the green rectangle, except the corner c, define the weight

$$wt(a) := \frac{1}{h(a)-1}$$

Then $\forall a$



$$wt(a) = \frac{1}{x+y} \quad \text{where}$$

$$wt(b) = \frac{1}{x} \quad \& \quad wt(d) = \frac{1}{y}$$

Weights of boxes of the green rectangle:

$\frac{1}{x_1+y_1}$	$\frac{1}{x_2+y_1}$	$\cdots$	$\frac{1}{x_k+y_1}$	$\frac{1}{y_1}$
$\frac{1}{x_1+y_2}$	$\frac{1}{x_2+y_2}$	$\cdots$	$\frac{1}{x_k+y_2}$	$\frac{1}{y_2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\frac{1}{x_1+y_\ell}$	$\frac{1}{x_2+y_\ell}$	$\cdots$	$\frac{1}{x_k+y_\ell}$	$\frac{1}{y_\ell}$
$\frac{1}{x_1}$	$\frac{1}{x_2}$	$\cdots$	$\frac{1}{x_k}$	1

corner box c

$x_1, \ldots, x_k, y_1, \ldots, y_\ell$ are the inverses of weights of boxes in the last row & the last column of the green rectangle.

Then the weight of any other box in column i & row j is $\frac{1}{x_i+y_j}$.

For any path p,

$$wt(p) := \prod_{\substack{a \text{ is} \\ \text{any box of } p}} wt(a)$$

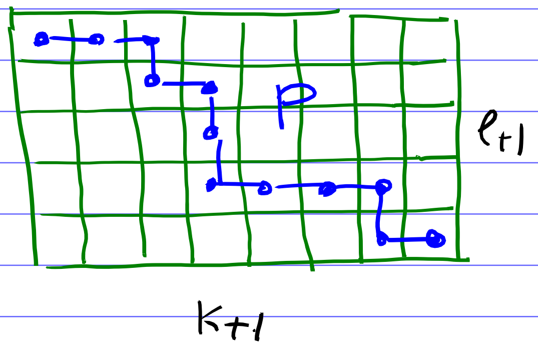
According to our definitions

$$P(a,c) := \sum_{\substack{p \text{ is a} \\ \text{hook walk} \\ \text{from } a \text{ to } c}} wt(p).$$

Let's first analyze the weighted sum over all <u>lattice</u> paths in the rectangle, where (unlike hook walks) we are not allowed to skip rows and columns

<u>Lemma</u>

$$\sum_{\substack{p \text{ is a lattice} \\ \text{path from the} \\ \text{upper left to} \\ \text{the bottom right} \\ \text{corner of the} \\ \text{rectangle}}} wt(p) = \frac{1}{x_1 x_2 \cdots x_k \, y_1 y_2 \cdots y_\ell}$$



$\binom{k+\ell}{k}$ such lattice paths p

<u>Example</u>. $k = \ell = 1$

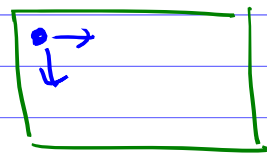
$\frac{1}{x_1+y_1}$	$\frac{1}{y_1}$
$\frac{1}{x_1}$	1

$$\frac{1}{x_1+y_1} \cdot \frac{1}{y_1} + \frac{1}{x_1+y_1} \cdot \frac{1}{x_1}$$

$$= \frac{1}{x_1 \cdot y_1}.$$

Proof Induction on $k+\ell$.

Base: ✓

Induction step: 

$$\sum_{\substack{\text{all lattice} \\ \text{paths } p}} wt(p) = \sum_{\substack{p \text{ starts} \\ \text{with a} \\ \text{horizontal} \\ \text{step}}}{}' + \sum_{\substack{p \text{ starts} \\ \text{with a} \\ \text{vertical} \\ \text{step}}}{}''$$

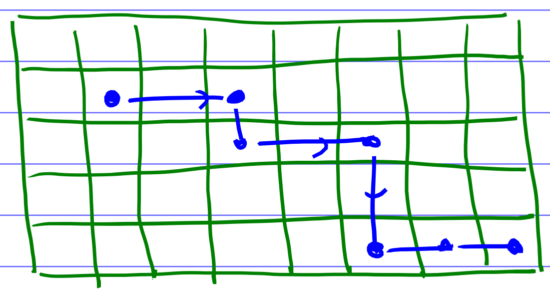
induction hypothesis

$$= \frac{1}{x_1+y_1} \frac{1}{x_2 \cdots x_k y_1 \cdots y_\ell} + \frac{1}{x_1+y_1} \cdot \frac{1}{x_1 \cdots x_k y_2 \cdots y_\ell}$$

$$= \frac{1}{x_1 \cdots x_k y_1 \cdots y_\ell}$$ ✓

$\square$

Let's now return to hook walks (in the green rectangle) what can start at any box and where we are allowed to skip over rows & columns. (But a hook walk should end at the lower right corner (the corner C) of the rectangle.)



a hook walk

Lemma 2. $\displaystyle\sum_{\substack{P \text{ is a} \\ \text{hook walk in} \\ \text{the } (k+1)\times(\ell+1) \\ \text{rectangle}}} wt(P) =$

$$= \left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{x_2}\right)\cdots\left(1+\frac{1}{x_k}\right) \cdot \left(1+\frac{1}{y_1}\right)\left(1+\frac{1}{y_2}\right)\cdots\left(1+\frac{1}{y_\ell}\right).$$

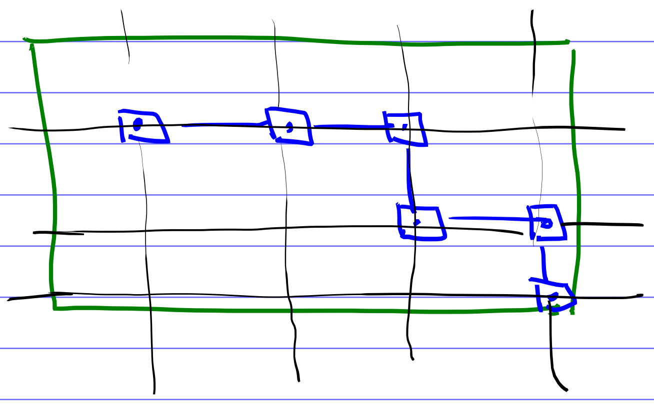
Example $k=\ell=1$

$\frac{1}{x_1+y_1}$	$\frac{1}{y_1}$
$\frac{1}{x_1}$	1

$$\frac{1}{x_1+y_1}\cdot\frac{1}{y_1} + \frac{1}{x_1+y_1}\cdot\frac{1}{x_1} + \frac{1}{x_1} + \frac{1}{y_1} + 1$$

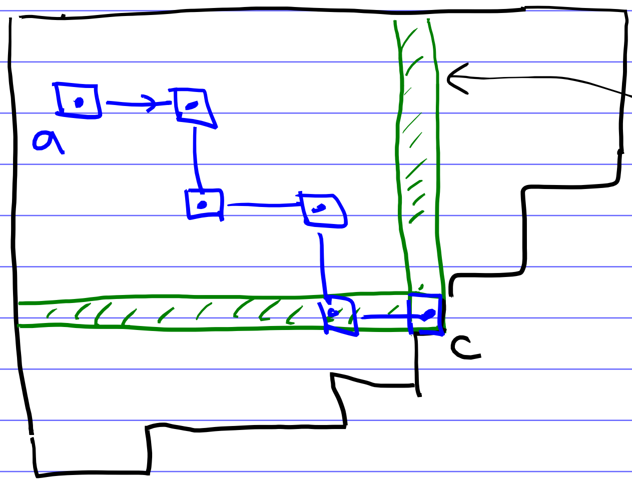
$$= \left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{y_1}\right).$$

Proof. Follows from Lemma 1. Indeed, any hook walk is a lattice path for some sub-rectangle formed by some subsets of rows & columns of the green rectangle. Picking a term in the expansion of the R.H.S. corresponds to picking subsets of rows & columns.



We apply Lemma 1 to the sub-rectangle formed by rows & colums containing the boxes of a hook walk.

back to the probabilities in random hook walks...



the co-hook of c

$$P(c) := \frac{1}{n} \sum_{a} P(a,c)$$

$$\overset{\text{Lemma 2}}{=} \frac{1}{n}\left(1+\frac{1}{x_1}\right)\cdots\left(1+\frac{1}{x_k}\right)\cdot\left(1+\frac{1}{y_1}\right)\cdots\left(1+\frac{1}{y_\ell}\right)$$

$$= \frac{1}{n} \prod_{\substack{b \in \text{cohook}(c) \\ b \neq c}} \left(1+\frac{1}{h(b)-1}\right)$$

$$= \frac{1}{n} \cdot \prod_{\substack{b \in \text{co-hook}(c) \\ b \neq c}} \frac{h(b)}{h(b)-1}$$

$$= \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)}$$

when we remove a corner box c, only the hook lengths for boxes b in the co-hook (c) change (decrease by 1).

Conclusion:

We proved

$$P(c) = \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)}$$

$$\sum_{\substack{c \text{ corner} \\ \text{of } \lambda}} P(c) = 1$$

This implies the needed recurrence relation

$$\frac{n!}{H(\lambda)} = \sum_{\substack{c \text{ corner} \\ \text{of } \lambda}} \frac{(n-1)!}{H(\lambda - c)}$$

for the R.H.S. of the hook length formula.

This proves the hook length formula by induction on n.

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