18.212 Lecture 3 Feb. 22, 2012

more on the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}\ldots$

$C_n =$

- # Dyck paths with $2n$ steps
- # triangulations of a $(n+2)$-gon
- # parenthesizations of $n+1$ letters
- # plane binary trees with n vertices
- # complete plane binary trees with $n+1$ leaves

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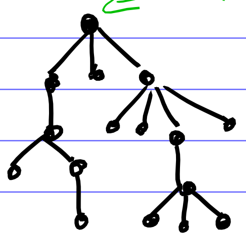
more combinatorial interpretations of C_n:

- # plane trees with $n+1$ vertices

Example

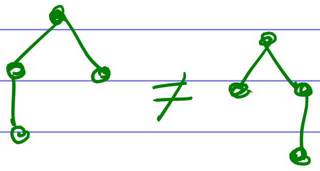
$n = 15$

root



a plane tree with 16 vertices

a drawing of a tree on the plane is part of the structure, e.g.



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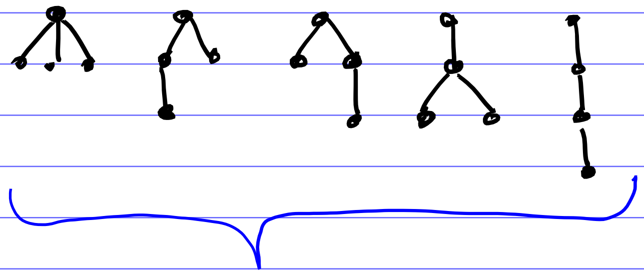
all plane trees for $n=3$:



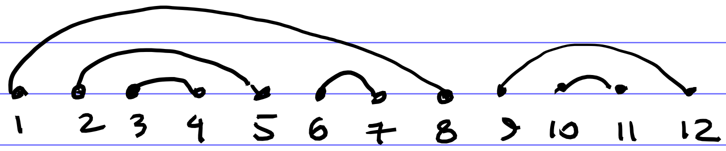
$C_3 = 5$

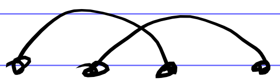
$C_n =$

- # non-crossing arc diagrams (a.k.a. non-crossing matchings) with $2n$ vertices

Example.

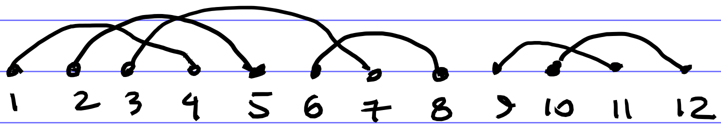
$n = 6$

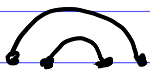


(the arcs are not allowed to cross like )

- # non-nesting arc diagrams with $2n$ vertices

Example



Now the arcs are allowed to cross but not allowed to nest like 

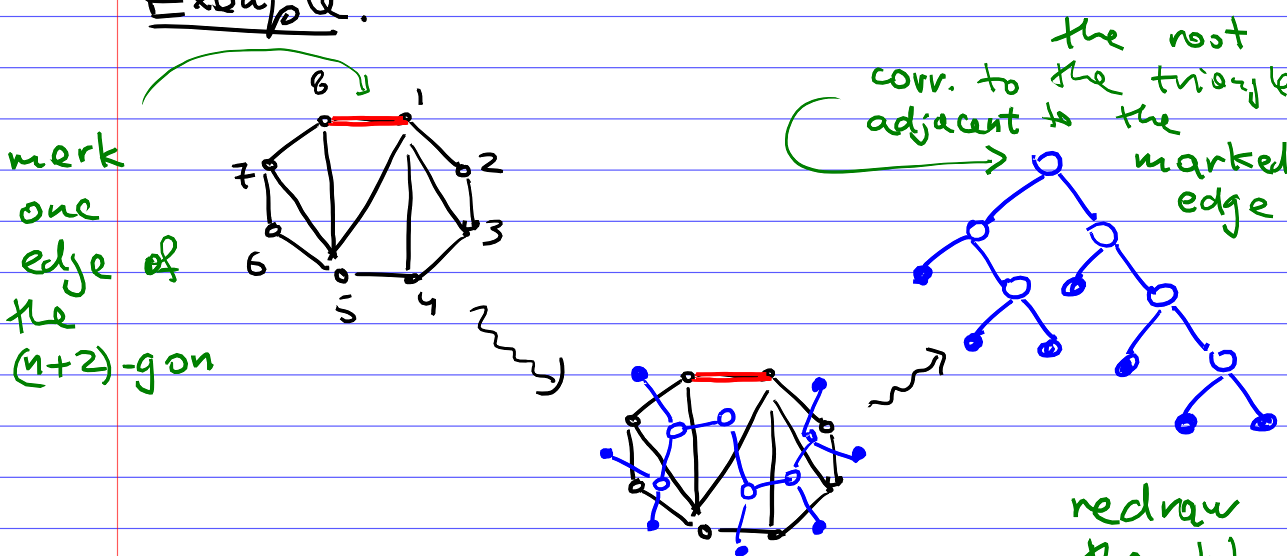
Remark. This "duality" between non-crossing & non-nesting objects is a common phenomenon in the Catalan world.

There are a lot of other combinatorial interpretations of C_n...

Bijections between some of these classes of combinatorial objects are pretty easy (while bijections for others can be more complicated).

$$\left\{\begin{matrix}\text{triangulations} \\ \text{on } (n+2)\text{-gon}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{complete} \\ \text{binary trees} \\ \text{with } n+1 \\ \text{leaves}\end{matrix}\right\}$$

<u>Example.</u>



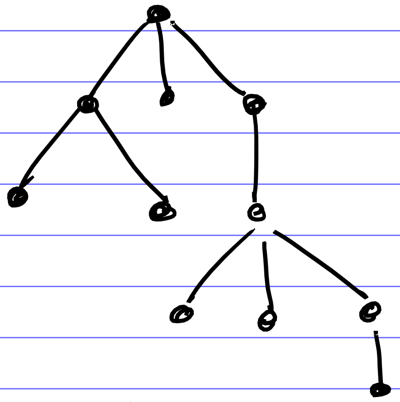
place vertices inside the triangles & just outside of all boundary edges, except the marke edge

redraw the blue tree as a plane binary tree

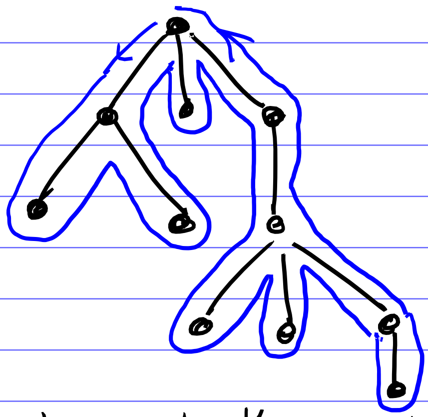
Basically, the plane binary tree is the plane dual graph of the triangulation.

$$\left\{\begin{array}{c}\text{plane trees}\\ \text{with } n+1\\ \text{vertices}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Dyck}\\ \text{paths with}\\ 2n \text{ steps}\end{array}\right\}$$

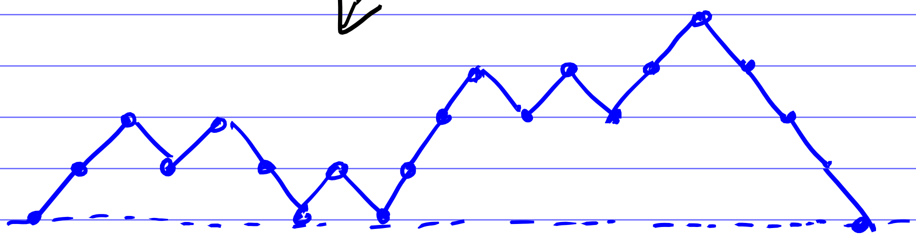
Example



a plane tree with $n+1$ vertices (equiv., n edges)



starting at the root, traverse the tree, as shown (depth-first search) passing through each edge twice in the opposite directions.



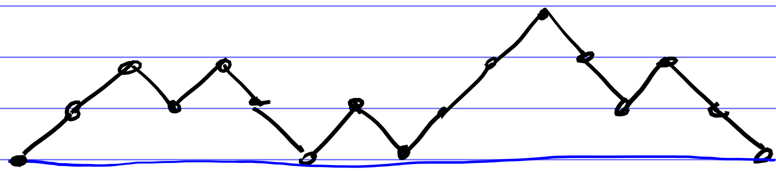
transform the traverse path of the tree into a Dyck path s.t down/up steps in the tree traverse path $\longleftrightarrow$ up/down steps of the Dyck path.

Another easy bijection:

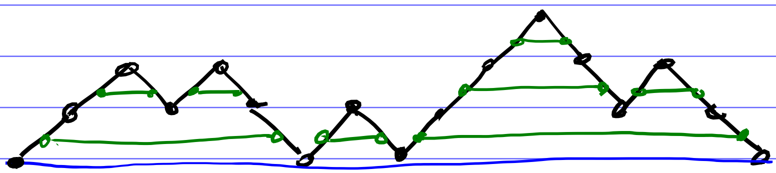
$$\left\{\begin{array}{l}\text{Dyck paths} \\ \text{with } 2n \text{ steps}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{non-crossing} \\ \text{arc diagrams} \\ \text{with } 2n \\ \text{vertices}\end{array}\right\}$$

<u>Example</u>

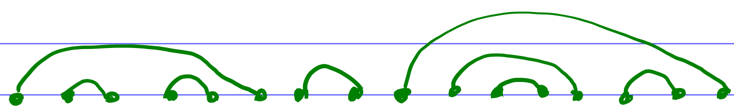
Dyck path



steps of a Dyck path correspond to vertices of the arc diagram



match the up & down steps as shown



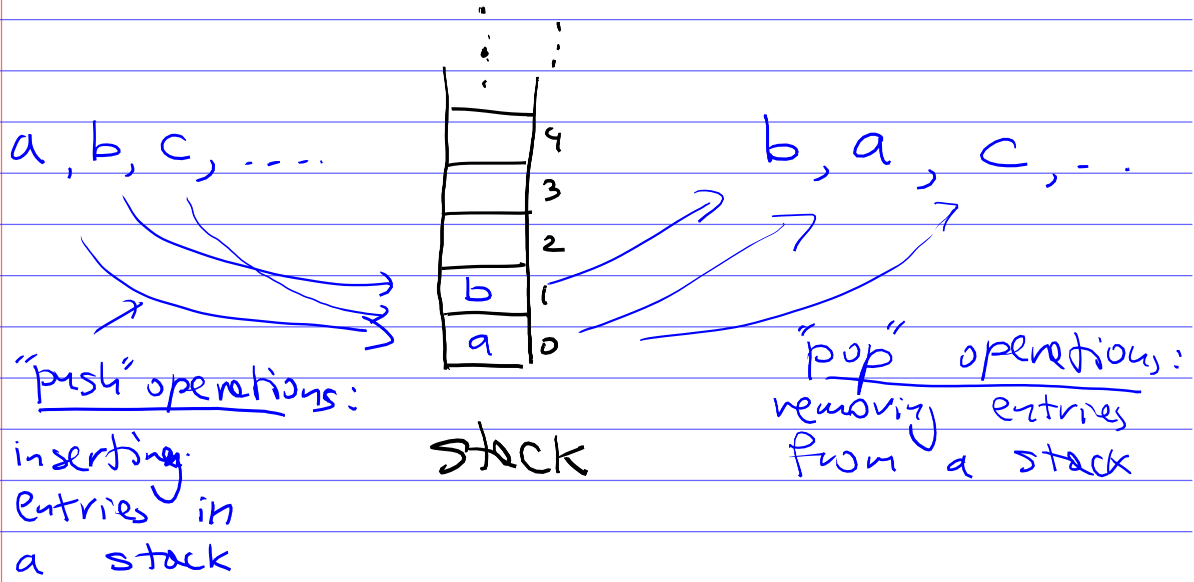
non-crossing arc diagram

and redraw it as an arc diagram

Stack & queue sorting

A litte bit of computer science terminology:

A **stack** is a linear data structure based on the LIFO "Last In First Out" principle: the element inserted *last* is the *first* to come out.



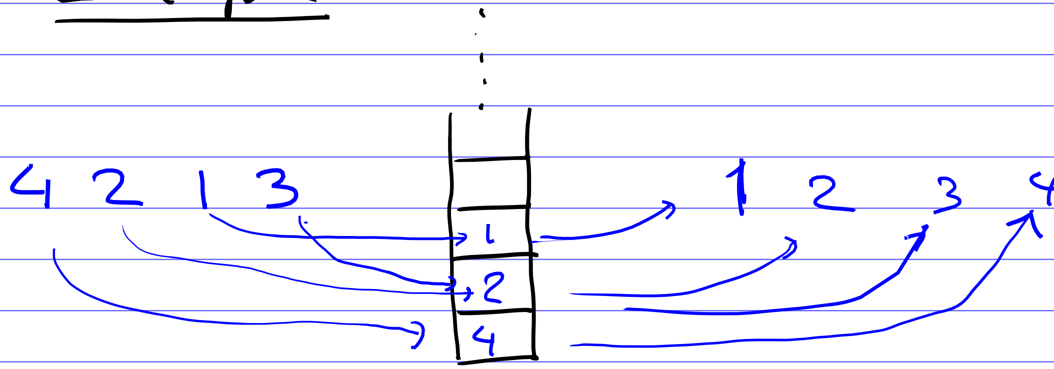
Definition. A *permutation* of size n is an arrangement of the "letters" $1, 2, 3, \ldots, n$

Example $3, 5, 1, 4, 2$

There are $n!$ permutations of size n.

Definition A *stack-sortable* permutation is a permutation that can be sorted to $1, 2, 3, \ldots, n$ using a stack.

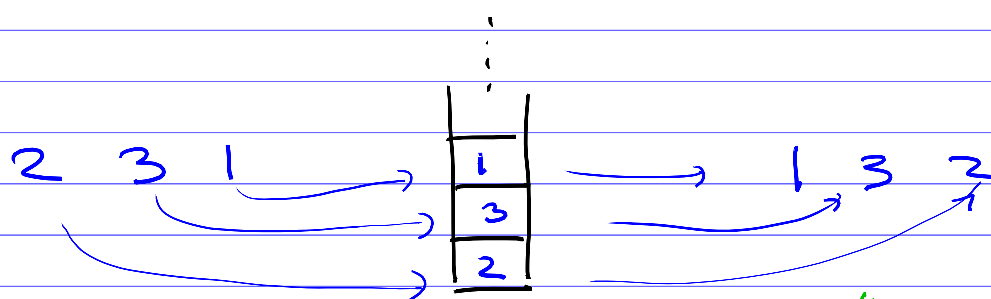
Example



stack

So 4213 is a stack-sortable permutation.

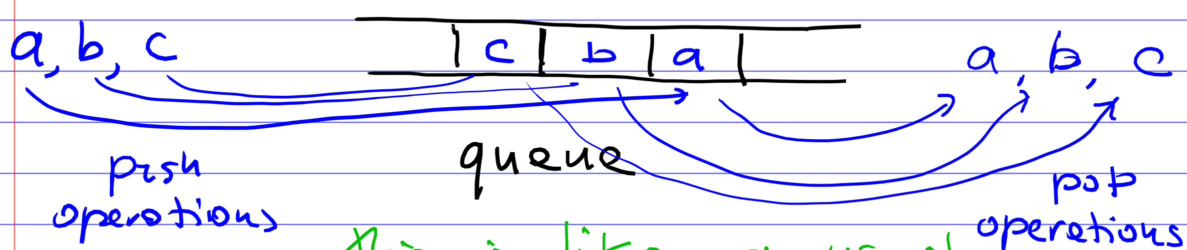
But 231 is <u>not</u> stack-sortable.



stack

Can't sort 231
2 & 3 are still out of order

A *queue* is a linear data structure based on the FIFO "First In First Out" principle: the element inserted first is the first to come out.



this is like a usual line in a store.

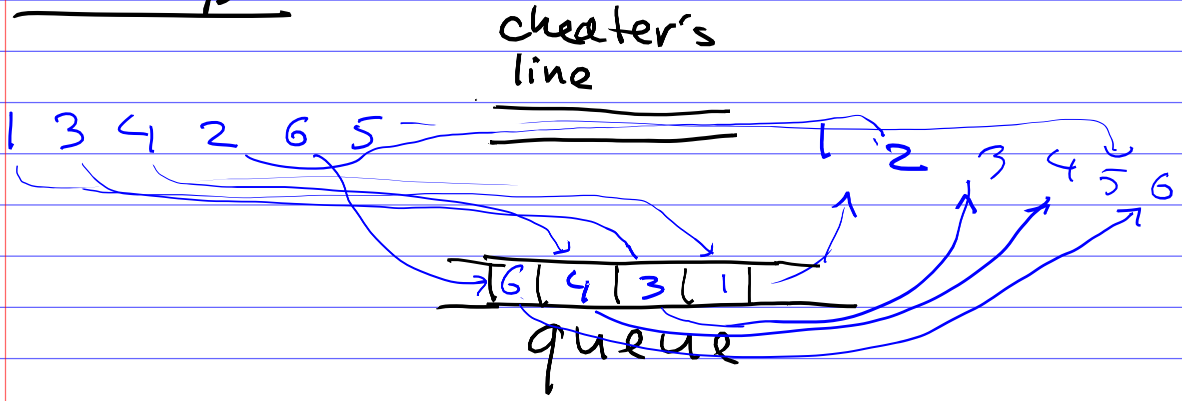
It looks that using a queue we cannot sort any permutation, except 1, 2, 3, .., n, the entries go out of a queue in the same order as they were entered.

Let's define queue - sortable permutations in a slightly different way.

Some entries are "good guys", they go into the queue and go out of the queue in the same order. But some entries are "bad guys"; they just jump in front of everybody who is currently staying in the queue.

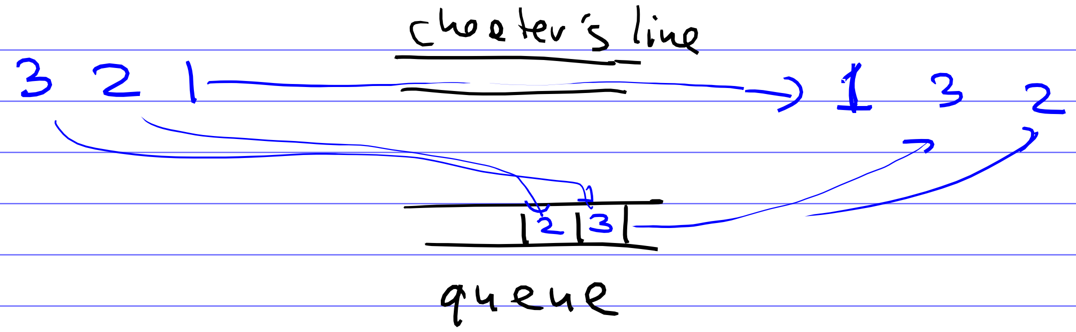
Def. A <u>queue-sortable</u> permutation is a permutation that can be sorted to $1, 2, 3, \ldots, n$ using a queue & a "cheater's line".

Example



So 1 3 4 2 6 5 is queue-sortable permutation.

But 3 2 1 is <u>not</u> queue-sortable



Even if 1 cheats and goes in front of 2 and 3, the entries 2 & 3 are still out of order.

Pattern avoidance in permutation

$w = w_1, w_2, \ldots, w_n$ a permutation of $1, 2, 3, \ldots, n$

$\pi = \pi_1, \ldots, \pi_k$ another permutation call a pattern. (Typically, π is of smaller size).

<u>Def.</u> We say that w <u>contains the pattern π</u> if there exists a subsequence $w_{i_1}, w_{i_2}, \ldots, w_{i_k}$ $(i_1 < i_2 < \ldots < i_k)$ in w (not necessarily adjacent) whose entries are in the same relative order as the entries of π.

<u>Example</u>

$w = 5 \; 7 \; 6 \; \underline{\underline{2}} \; \underline{\underline{4}} \; 1 \; \underline{\underline{3}}$

contains the pattern $\pi = 1\;3\;2$.

<u>Definition</u> A permutation w is called <u>π-avoiding</u> if it does not contain the patten π.

For exemple, w is 123-avoiding if it does not contain 3 entries w_a, w_b, w_c $a < b < c$, such that $w_a < w_b < w_c$.

<u>Example</u>. The above permutation $w = 5\,7\,6\,2\,4\,1\,3$ is <u>not</u> 132-avoiding but it <u>is</u> 123-avoiding.

Exercise ① Prove that the stack-sortable permutations are exactly the 231-avoiding permutations.

② Prove that the queue-sortable permutations are exactly the 321-avoiding permutations.

Let S_n be the set of all permutations of $1,2,\dots,n$ (called the <u>symmetric group</u>).

Theorem. For any pattern π of size 3, the number of π-avoiding permutations in S_n equals the Catalan number C_n.

So $C_n =$

- # stack-sortable perm. in S_n
- # 231-avoiding perm. in S_n
- # queue-sortable perm. in S_n
- # 321-avoiding perm. in S_n

Exercise. Find a bijection between 231-avoiding permutations and 321-avoiding permutations in S_n.

Exercise. Find the number of permutations in S_n which are both 231-avoiding and 321-avoiding.

Partitions & Young diagrams

Def. A partition of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$.

In other words, a partition of n is a way to decompose n into a sum of some positive integers where we don't care about the order of the summands (so that they can be arranged in a weakly decreasing order $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$).

Note: If we do care about the order of the summands, then corresp. combinatorial objects are called compositions.

For example,

$$3 = 2+1 = 1+2.$$

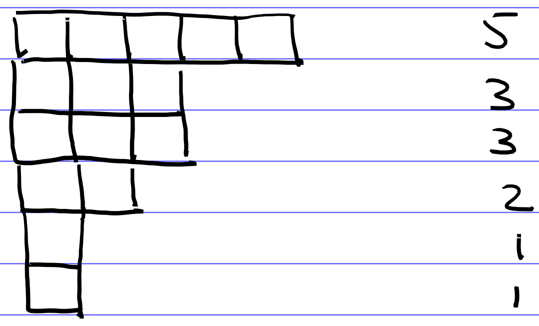
$2+1$ and $1+2$ give the same partition $\lambda = (2,1)$, but they give 2 different compositions $(2,1) \neq (1,2)$.

Graphically, partitions are represented by <u>Young diagrams</u> (which are left justified diagrams with boxes)

<u>Example</u>. $\lambda = (5, 3, 3, 2, 1, 1)$ (a partition of $n = 15$)

The corresponding Young diagram is

$\lambda =$



a Young diagram

Slightly abusing the notation, we'll use the same letter λ for partitions & Young diagr.

<u>Def</u>. A <u>standard Young tableau</u> (SYT) of <u>shape</u> $\lambda = (\lambda_1, \dots, \lambda_k)$ is a filling of boxes of the Young diagram λ by numbers $1, 2, \dots, n$ such that

- each entry $1, 2, \dots, n$ appears exactly once.
- the entries increase in rows & columns of the Young diagram

<u>Note</u>. The plural of "tableau" is "tableaux".

<u>Example</u>.

1	2	4	7
3	5		
6	9		
8			

An SYT with 9 boxes.

One more interpretation of C_n:

C_n = # SYT's of shape (n,n)

the $2 \times n$ rectangle

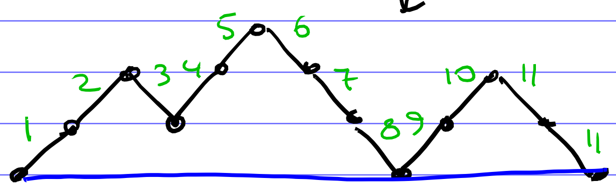
An easy bijection between SYT's of shape (n,n) and Dyck paths:

Example. $n = 6$

$T =$

1	2	4	5	9	10
3	6	7	8	11	12

an SYT of shape (n,n)



the corresponding Dyck path

<u>Rule</u>: The i-th step in the Dyck path is an up (resp. down) step, if the entry i in the tableau T belongs to 1st (resp. 2nd) row.

So SYT's of any shape λ can be viewed as a generalization of Dyck path.

We have the nice formula $\frac{1}{n+1}\binom{2n}{n}$ for C_n.

Q: Is there a "nice" formule for the number of SYT's of any given shape λ?