18.212 PROBLEM SET 1 (due Monday, March 04, 2018)

Turn in as many problems as you want. (You don't need to turn in all problems to get a perfect grade in the class. Around 6 problems should be enough.)

Problem 1. In class, we sketched a proof of the formula for the Catalan number $C_n = \frac{1}{2n+1}\binom{2n+1}{n}$ using cyclic shifts of sequences of ± 1's. The proof is based on the following two claims. Prove these claims.

Let $(e_1, \ldots, e_{2n+1})$ be a sequence such that such that $e_i \in \{1, -1\}$, $\#\{i \mid e_i = 1\} = n$, and $\#\{i \mid e_i = -1\} = n + 1$.

(1) All $2n+1$ cyclic shifts $(e_i, \ldots, e_{2n+1}, e_1, \ldots, e_{i-1})$, for $i = 1, \ldots, 2n+1$, are different from each other.

(2) Exactly one cyclic shift $(e'_1, \ldots, e'_{2n+1})$ among these $2n + 1$ shifts satisfies $e'_1 + \cdots + e'_j \geq 0$, for $j = 1, \ldots, 2n$.

Problem 2. Consider the random walk of a man on the integer line $\mathbb{Z}$ such that, at each step, that the probability to go from position i to position $i + 1$ is p, and the probability to go from i to $i - 1$ is $1 - p$. The man "falls off the cliff" if he reaches the position 0.

Suppose that the man starts at the initial position $i_0 \geq 1$. Find the probability that he falls off the cliff.

Problem 3. The same setup as in the previous problem. Find the probability that the man starting at position i_0 falls off the cliff after exactly m steps. (Hint: Use the reflection principle.)

Problem 4. Prove that a permutation is queue-sortable if and only if it is 321-avoiding.

Problem 5. Prove that a permutation is stack-sortable if and only if it is 231-avoiding.

Problem 6. Find a bijection between 321-avoiding permutations of size n and 231-avoiding permutations of size n.

Problem 7. Find an expression for the number of permutations w of size n such that w is both 321-avoiding and 3412-avoiding.

(Hint: Calculate the number of such permutations for small values of n, then guess the answer.)

Problem 8. Find an expression for the number of permutations w of size n such that w is both 231-avoiding and 4321-avoiding.

Problem 9. In class, we proved part (1) of Schensted's theorem. Prove part (2) of this theorem:

If the Schensted correspondence maps a permutation w to a pair (P, Q) of standard Young tableaux of the same shape λ, then the size of a largest decreasing subsequence in w equals the number of nonzero parts in partition λ (i.e., the number of rows of its Young diagram).

Problem 10. Fix two positive integers m and n. Let w be a permutation of size $m \cdot n + 1$. Prove that w either has an increasing subsequence of size $m + 1$ or a decreasing subsequence of size $n + 1$.

(Hint: You can use properties of Schented correspondence. There is also a direct proof based on the pigenhole principle.)

Problem 11. Find an explicit expression for the number of permutations w of size $m \cdot n$ such that w does not have an increasing subsequence of size $m + 1$ nor a decreasing subsequence of size $n + 1$.

Problem 12. Prove the "baby hook-length formula":

The number of linear extensions of the poset whose Hasse diagram is a rooted tree T on n vertices equals $n!/\prod_{v \in T} h(v)$, where the product is over all vertices v of the tree, and the "hook-length" $h(v)$ equals the size of the branch of T growing from vertex v.

Problem 13. For positive integers $n_1, \dots, n_m$ and $n = n_1 + \cdots + n_m$, the q-multinomial coefficient is defined as

$$\begin{bmatrix} n \\ n_1, \dots, n_m \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \cdots [n_m]_q!}.$$

Show that

$$\left[\begin{array}{c} n \\ n_1, \ldots, n_m \end{array} \right]_q = \sum_w q^{inv(w)},$$

where the sum is over all permutations w of the multset $\{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$, and $inv(w)$ is the number of inversions in w. Here i^n denotes $i, \ldots, i$ (repeated n times).

Problem 14. Prove the identity for q-binomial coefficients

$$\left[\begin{array}{c} 2n \\ n \end{array} \right]_q = \sum_{k=0}^{n} q^{k^2} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left[\begin{array}{c} n \\ k \end{array} \right]_q$$

(Hint: Use the interpretation of q-binomial coefficients in terms of Young diagrams, and try to subdivide a Young diagram into several pieces to prove the identity.)

Problem 15. Prove the following noncommutative version of binomial theorem.

Let q be a parameter, and let x, y be two noncommuting variables that satisfy the relation

$$yx = qxy.$$

We assume that q commutes with both x and y, i.e., $qx = xq$ and $qy = yq$. Show that

$$(x + y)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k \, y^{n-k}.$$

Bonus Problems

Problem 16. Show that the two statistics $inv(w)$ (the number of inversions) and $maj(w)$ (the major index) on permutations $w \in S_n$ are equidistributed.

Problem 17. An *exceedance* in a permutation $w \in S_n$ is an index $i \in \{1, \ldots, n\}$ such that $w(i) > i$. Similarly, a *weak exceedance* in a permutation $w \in S_n$ is an index $i \in \{1, \ldots, n\}$ such that $w(i) \geq i$. Let $exc(w)$ be the number of exceedances and $wexc(w)$ be the number of weak exceedances in a permutation w. Prove that the statistics $exc(w)$ and $wexc(w) - 1$ on permutations $w \in S_n$ (for $n \geq 1$) are equidistributed.

Problem 18. Prove that the number of set-partitions π of the set $[n] := \{1, \ldots, n\}$ such that, for any $i = 1, \ldots, n-1$, the consecutive numbers i and $i+1$ do not belong to the same block of π equals the number of set-partitions of the set $[n-1]$.

Problem 19. For $1 \le k \le n/2$, find a bijection f between k-element subsets of $\{1, \ldots, n\}$ and $(n-k)$-element subsets of $\{1, \ldots, n\}$ such that $f(I) \supseteq I$, for any k-element subset I.

Problem 20. We say that a pair (i, j), $1 \le i < j \le n$, is an *odd-length inversion* of a permutation $w \in S_n$ if $w_i > w_j$ and $j - i$ is odd. Let $inv(w)$ be the number of all inversions in w and $oinv(w)$ be the number of odd-length invesions in w. Prove the identity

$$\sum_{w \in S_n} (-1)^{inv(w)}\, x^{oinv(w)} = \prod_{i=2}^{n} (1 + (-1)^{i-1}\, x^{\lfloor i/2 \rfloor})$$